# Investigation the order elements 3 in certain twisted groups of lie type

**Halah Qahtan Hamdi**
*University of Baghdad*
*College of Science*
*Department of Remote Sensing and GIS*
*Iraq*
*halahamdi73@gmail.com*

**Abstract.** Suppose $G$ is a finite group and that $X$ is a subgroup of $G$. The commuting graph is denoted by $C(G, X)$ which if $x \neq y$ and $xy = yx$, has a set of vertices $Xx, y \in X$ joining. Assume that $X$ for elements of order 3 is a $G$-conjugacy class. In this article, to evaluate $X$, the commuting graphs for unique Twisted Lie form groups of function two over $X$ are used.

**Keywords:** exceptional groups, commuting graph, diameter, cliques.

## 1. Introduction

One of the modern methods of analyzing the structure of the group which has been recently proven to be as effective is studying the action of a group on a graph. Similar approaches are also distinctively used to study the algebraic properties of rings [1]. In addition to the central function of involution in the finite group, the elements of order 3, as well as the elements of order 3, are considered significant in the analysis of finite group algebraic properties. For example, a class of elements of order 3 can be generated by the Conway group, which is the group of auto morphisms of the 24-dimensiona Leech lattice, so that both of them either commute or produce the alternating groups $A_4, A_5$ or $SL_2(3), SL_2(5)$. In John Thompson's Quadratic pairs of prime 3, such a class also has an essential case. See several of the studies collected in this sense [2, 3, 4, 5]. Suppose $G$ is a finite group and $X$ is a subset of $G; C(G, X)$ denotes the graph for commuting as $X$ as its vertex set with two vertices $x, y \times X$ is adjacent if $x \neq y$ and$x, y$ are commuting. In the seminal paper [6], For a given type of isomorphism of an involution centralizer, where there are finitely many non-abelian groups able to contain it up to isomorphism, Fowler and Brauer first noticed the commuting graph they were outstanding for showing. For the works of the Margulis - Platanov conjecture, these graphs are of considerable importance (see, [7] as they appeared in [6] had $X = G\ \{1\}$). The commuting graph is called the commuting involution graph, since $G$ is a finite group, and $X$ is a conjugate involution class. In the commuting graph, Rowley, Hart (number Perkins), Bates, and Bundy have some similar results in finite groups, including, for example, diameter estimation and disc structure analysis (see, [6, 8, 9, 11]). The commuting graph $C(G, X)$ was studied in [12] by Nawawi and Rowley when G is either a sporadic simple group $McL$ or a symmetric group $Sn$ and $X$ is a$G$-conjugacy class of elements order 3. The commuting graphs for $G$-classes of elements of order 3 in Mathieu groups is analyzed by Azeez and Aubad [13]. Everett and Rowley point out the various assorted commuting graphs examined in [14]. $X$ is considered to be a $G$-conjugacy class of order 3 elements from now on. This analysis aims to analyze some features of the commuting graph $C(G, X)$ when $G$ is one of the two characteristic Twisted classes of Lie form. In truth, we assume that G is one of the basic groups, ${}^3D_4(2)$ or ${}^2F_4(2)$'. The research includes studying the configurations of the disks and measuring the diameters, girths, and number of cliques for these diagrams. We will assume, from now on, that G is one of the groups described above. In addition, we can assume that $t$ is an element of order 3 in $G$ and $X = t^G$. For $x \in X$, the $i^{\text{th}}$ disc of $x$, which denote by $\Delta_i(x), (i \in N)$ define to be

$$\Delta_i(x) = \{y \in X | d(x, y) = i\},$$

where $d(,)$ is defined as the standard distance metric on a graph $C(G, X)$. One can immediately see that $G$, operating by conjugation on $X$, immerses $G$ in the group of graph auto morphisms of $C(G, X)$. In addition, it is transitive to Cleary, $G$ on the vertices of $C(G, X)$. Let $t - X$ now be a fixed class $X$ variable. The main objective is to research the disc structure of vertex t in $C(G, X)$. The diameter of the commuting graph $C(G, X)$. is denoted by $DiamC(G, X)$ and is defined as:

$$DiamC(G, X) = \max_{x \in X} \{i | \Delta_i(x) \neq \phi \text{ and } \Delta_{i+1}(x) = \phi\}.$$

A graph's girth is defined as the length of the shortest cycle in the graph. Sometimes called a clique, a complete graph is called. The clique number of G is called the size of the largest clique that can be made up of edges and vertices of $G$. Finally, we can focus on the Atlas [15] for F or the names of the $G$ conjugacy groups. Our primary outcome is as follows.

**Theorem 1.1.** *Let $G$ be isomorphic to one of ${}^3D_4(2)$ or ${}^2F_4(2)$ then:*

1. *The sizes of the discs $\Delta_i(t)$ are listed in Table 1 and the $G$-conjugacy classes of $tx$ for $x \in \Delta_i(t); i \in N$ are given in Table 2.*
2. *If $(G, X) = {}^3D_4(2), 3A)$ then $DimC(G, X) = 5$;*
3. *If $(G, X) = {}^3D_4(2), 3B)$ then $DimC(G, X) = 4$;*
4. *If $(G, X) = {}^2F_4(2), 3A)$ then $DimC(G, X) = 7$.*

With the assistance of Magma [16], Gap [17] and the Online Atlas [17], analytical methods have been used to achieve the above results. We allocated

| G | $X = t^G$ | $\|\Delta_1(t)\|$ | $\|\Delta_2(t)\|$ | $\|\Delta_3(t)\|$ | $\|\Delta_4(t)\|$ | $\|\Delta_5(t)\|$ | $\|\Delta_6(t)\|$ | $\|\Delta_7(t)\|$ |
|---|---|---|---|---|---|---|---|---|
| ${}^3D_4(2)$ | 3A | 113 | 6048 | 75276 | 57834 | 504 | | |
| | 3B | 145 | 9936 | 261792 | 54270 | | | |
| ${}^2F_4(2)$ | 3A | 25 | 216 | 1836 | 15552 | 76464 | 71442 | 864 |

Table 1: The Discs for $C(G, X)$., $G\ {}^3D_4(2),\ {}^2F_4(2)$‘.

the equations to the $C_G(t)$-orbits on $X$ (where $C_G(t)$ is acting by conjugation). In Section 2, we also provide details on the action of $C_G(t)$ on $X$. Specially , we collate the $C_G(t)$-orbit sizes on each $X_C = \phi$, where $X_C$ being defined as below. For a $G$-Class $C$ define the set:

$$X_C = \{x \in X | tx \in X\}.$$

One can see that if $X_C \neq \phi$ then it is equal to a union of certain $C_G(t)$orbits of $X$. The way of $X_C$ breaks into $C_G(t)$orbits it will essential to determinate which discs of $t$ contain the vertices in $X_C$. Also knowing the is the size of $X_C$ can be beneficial which leads us to class structure constants. Class structure constants are the sizes of sets:

$$\{(g_1, g_2) \in C_1 \times C_2\ |g_1 g_2 = g\},$$

where $C_1, C_2, C_3$ are $G$conjugacy classes and $g$ is a fixed element of $C_3$. Now these constants can be calculated directly from the complex character table of $G$ which are recorded in the Atlas and are available electronically in the standard libraries of the computer algebra package Gap. If we take $C_1 = C, C_2 = X = C_3$ and $g = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)|\,|C_G(h)|} \sum_{i=1}^{s} \frac{\chi_i(h)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(i)}.$$

Where $h$ is a representative from $C$ and $\chi_1, \chi_2, ..., \chi_s$ the complex irreducible characters of $G$. Here, we include some detail on how to get the tables, although we have made some changes, as mentioned, we use the class names in Atlas. We suppress the notation ”slave” first. So for example, for the classes we write $7A, 7B7B^{*2}, 7C^{*4}$ of ${}^3D_4(2)$, respectively. Secondly, to clarify, as we mean to unite these classes and their characters are in alphabetical order, we compact the letter component of the class name.

As an example, in Table 2, for $G \cong {}^3D_4(2)$ and $X = 3B, 9AC$ is short-hand for $9A \cup 9B \cup 9C$.

| G | $X = t^G$ | $\Delta_1(t)$ | $\Delta_2(t)$ | $\Delta_3(t)$ | $\Delta_4(t)$ | $\Delta_5(t)$ | $\Delta_6(t)$ | $\Delta_7(t)$ |
|---|---|---|---|---|---|---|---|---|
| $^3D_4(2)$ | 3A | 1A, 3A, 3B | 7AC, 9AC $(504^2)$ , 21AC (504) | 3A(378), 4A(216), 4B, 7D(216, $1512^8$ 8B $(1512^6)$, 9AC $1512^2$ 12A $1512^2$ 13AC $(1512^3)$ 21AC $(1512^4)$ 28AC $(1512^2)$ | 2A, $3A(27, 216^2, 378)$, $216^2$, 378 4A(216 ,378) 4C,6A 6B 7AC 7D $(216^2)$ 8B $(216^2)$, , 9AC 1512 12A $(1512^2)$ 13AC $(1512^6)$ ,28AC $(1512^2)$ | 3A 3A | | |
| $^3D_4(2)$ | 3B | 1A, 3B $(72^2)$ | 3A,3B $(216^2)$ , 4B $(216^6)$ , 9AC 216 12A $(216^{12}$ ), 18AC $(216^6)$ | 4C,6A $(648^6)$, 6B,7AC $(324^2)$ 7D,8A 9AC $(324^8$, $648^2)$, 12A $(648^{12})$, 13AC $(648^{36})$, 14AC $(324^8$, $648^8)$, 18AC $(324^6$, $(324^8$, $(648^{14}$, 21AC $(648^{24})$, 28AC $(324^4$, $(648^{12})$ | 2B,3B( $81^3), 4A$ , $4B(81^3)$ 6A $4B(81^4)$, 7AC (108), 8B $(648^{12})$, 9AC 4B(108, 324), 13AC $648^{13})$, 14AC (324) 13AC 18AC $(648^4)$, 21AC (648), 28AC $(648^4)$ | | | |
| $^2F_4(2)$‘ | 3A | 1A, 3A( $12^2)$ | 3A $(36^2)$ | $4C(27^2$, $108^8)$, $6A(27^2$, $(108^8)$, | 3A $(108^4)$, 4C $(108^4)$, 6A $(108^{16})$, 8C $(108^{16})$, 12AB $(108^6)$, 13AB $(108^{26})$ 16AD $(108^6)$ | $2B, 3A(54^2, 108^4)$, $4AB(108^4)$, $(108^2), 6A$ $(108^{52}), 8AB$ $(108^{38}), 8CD$ $(108^{42}), 10A$ $(108^{64}), 12AB$ $(108^{56}), 13AB$ $(108^{68}), 16AD$ $(108^{40})$, | 2A,3A(27),4AB $(108^4), 4C(27^2$, $54^6, 108^{18}), 5A, 6A$ $(27^2, 546, 108^{52})$, $8AB(108^{20}), 8CD$ $(108^{38}), 10A$ $(108^{56})$, $12AB(108^{58})$, $13AB(108^{27})$, $16AD(108^{50})$ | 8AB $(108^2)$, 13AB $(108^2)$, |

Table 2: $G$- Class of Goods $tx$ for $x \in \Delta_i(t)$.

The difference between $t$ and $x$ in $C(G, X)$ cannot always be defined by the $G$-class to which $tx$ belongs, a very remarkable finding of the present analysis. The cases where $tx$ is included in the interval between $t$ and $x$ defined by the $G$-class are $G \cong {}^3D_4(2)$; $X = 3A$, $tx \in \{1A, 2A, 3A, 4B, 4C, 6A, 6B, 7A, 7B, 7C\}, G \cong {}^3D_4(2)$; $X = 3B$, $tx \in \{1A, 2B, 3A, 4A, 4C, 6B, 7D, 8A\}$ and $G \cong {}^2F_4(2)$; $X =$

$3A,\ tx \in \{1A, 2A, 2B, 5A\}$. The following additional cases where the distance between $t$ and $x$ in $C(G, X)$ is not defined by the $G$-class comprising $tx$. For instance,, when $G \cong {}^3D_4(2);\ X = 3A,\ tx \in \{9A, 9B, 9C\}$ each of $9A,\ 9B$ and $9C$ breaks into five $C_G(t)$-orbits, 2 of size 504 being in $2(t)$ ,2 of size 1512 being in $\Delta_3(t)$ and 1 of size 1512 being in $3(t)$. For more details, see Table 2.

## 2. $C_G(t)$-orbits on X

The sizes of the $C_G(t)$-orbits are given in their action on $X_C$ in this section, as we described above., when $X_C$ for $C$ is a $G$-conjugacy class. In the succeeding table's exponential notation are used to point the multiplicity of a specific size. For instance, in the $C_G(t)$-orbits table for $C({}^3D_4(2), 3B)$ the entry $648^6,\ 81^4$ next to 6A is show that $X_{6A}$ It is a union of ten $C_G(t)$-orbits, six of which are 648 in size and two of which are 81 in size. In the $C_G(t)$-orbits table, we will see another example of this. for $C({}^2F_4(2), 3A)$ the entry 10896 next to $16AD$ is points that each of $X_{16A}$ ,$X_{16B}$, $X_{16C}$ and $X_{16D}$ is the union of ninety six $C_G(t)$-orbits, all of them have size 108. In the next table, we include class information, including the class size, structure of the $C_G(t)$-orbits and permutation rank. For the repressions for ${}^3D_4(2)$ and ${}^2F_4(2)$ẁe applied the

| Group | $X = t^G$ | $\|X\|$ | Permutation Rank | $C_G(t) structures$ |
|---|---|---|---|---|
| ${}^3D_4(2)$ | 3A | 139776 | 118 | $Z_3xPSL(2,8)$ |
| | 3B | 326144 | 600 | $(((Z_3xZ_3) : Z_3) : Q_8) : Z_3$ |
| ${}^2F_4(2)$ | 3A | 166400 | 1564 | $((Z_3xZ_3) : Z_3) : Z_4$ |

Table 3: The Classes Information

permutation representations with 819 and 1600 points respectively (see, [18]).

### 2.1 $C_G(t)$-orbits of commuting graph C (3D4(2),3A)

In the following table, we provide complete details on the scale of the $C_G(t)$-orbits in the case of $G\ {}^3D_4(2);\ X = 3A$. Also, the way of $X_C$ breaks into $C_G(t)$-orbits are given: The $C_G$(t)-orbits of the commuting graph $C({}^3D_4(2), 3A)$ in this case distribute in five discs each with different size.

### 2.2 $C_G(t)$-orbits of commuting graph C (3D4(2),3B)

In the next table we give a full details about the $C_G(t)$-orbits sizes in case $G \cong {}^3D_4(2);\ X = 3B$. Also, the way of $X_C$ breaks into $C_G(t)$-orbits are presented:

The $C_G(t)$-orbits of the commuting graph $C({}^3D_4(2), 3B)$ in this case disseminate in four discs each with different size.

| Class name | $C_G(t)$-Orbit size |
|---|---|
| 1A | 1 |
| 2A | $27^2$ |
| 3A | $27, 56, 216^2, 378^2, 504$ |
| 3B | 56 |
| 4A | $216^2, 378$ |
| 4B | 378 |
| 4C | $756^2$ |
| 6A | $1512^4$ |
| 6B | $756^2$ |
| 7AC | 504 |
| 7D | $216^3, 1512^8$ |
| 8B | $1512^8$ |
| 9AC | $504^5$ |
| 12A | $1512^4$ |
| 13AC | $1512^9$ |
| 21AC | $1512^5$ |
| 28AC | $1512^4$ |

Table 4: $C_G(t)$-Orbits sizes of $C(^3D_4(2), 3A)$

### 2.3 $C_G(t)$-orbits of commuting graph $C(^2F_4(2)`, 3B)$

In the following table we give a full information on the $C_G(t)$-orbits sizes in case $G\ ^2F_4(2)`$; $X = 3A$. Also, the form of $X_C$ breaks into $C_G(t)$-orbits are given: The $C_G(t)$-orbits of the commuting graph $C(^2F_4(2)`, 3B)$ in this case disseminate in seven discs each with different size.

## 3. Girths and cliques number

The strategy we going to use to calculate the cliques number for the commuting graph $C(G, X)$, mainly depend on the fact that G, acting by conjugation, induces graph auto morphisms of $C(G, X)$, and is transitive on its vertices. Also easy computational check inside $\Delta_1(t)$ one can find $x, y \in \Delta_1(t)$ such that $x$ and $y$ are commute. Thus, the girth of the commuting graph is 3.The following table the cliques number for the commuting graph $C(G, X)$, The above table telling us that the girth $C(G, X).,G \cong {}^3D_4(2),\ {}^2F_4(2)$ìs 3 while clique number for $C(G, X).,\ G \cong {}^3D_4(2), {}^2F_4(2)$ and $X = 3A$, equal 3 and $C(G, X).,\ G \cong {}^3D_4(2),\ X = 3B$ is 8. Also the maximum elementary abelian subgroup of the elements of $X$ is of the form $Z_3 \times Z_3$.

| Class name | $C_G(t)$-Orbit size |
|---|---|
| 1A | 1 |
| 2B | $81^3$ |
| 3A | $216^5$ |
| 3B | $72^2, 81^3, 216^2$ |
| 4A | 81 |
| 4B | $81^3, 216^6$ |
| 4C | $648^4$ |
| 6A | $81^4, 648^6$ |
| 6B | $756^2$ |
| 6B | $648^{10}$ |
| 7ABC | $108, 324^2$ |
| 7D | $648^9$ |
| 8A | $648^{12}$ |
| 8B | $648^{12}$ |
| 9ABC | $108, 216, 324^9, 648^9$ |
| 12A | $216^{12}, 648^{12}$ |
| 13ABC | $648^{49}$ |
| 14ABC | $324^9, 648^8$ |
| 18ABC | $216^6, 324^6, 648^{18}$ |
| 21ABC | $648^{25}$ |
| 28ABC | $324^4, 648^{16}$ |

Table 5: $C_G(t)$-Orbits sizes of $C(^2F_4(2)`, 3A)$

| Group | $X = t^G$ | Clique number | Girth |
|---|---|---|---|
| $^3D_4(2)$ | 3A | 8 | 3 |
| | 3B | 6 | 3 |
| $^2F_4(2)`$ | 3A | 8 | 3 |

Table 6: Girth and Clique Number for $C(G, X)$., $G$ $^3D_4(2)$, $^2F_4(2)`$.

## Conclusions

This paper aims to study the algebraic structure of $G$-classes of elements of order 3 in exceptional groups of Lie type $^3D_4(2)$, $^2F_4(2)`$. For that purpose we employed the commuting graph on the G-classes of elements of order 3 of such groups. A very important results have been gained for instance, $C_G(t)$-orbits sizes, girth, clique number of the graph. Also the maximum elementary abelian subgroup of the elements of $G$-classes of elements of order 3.

**References**


[1] A. Aubad, S. Kadem, A.H. Majeed, *The non-zero divisor graph of a ring*, IJPAM, 43 (2020), 975-983.

[2] M. Aschbacher, M. Hall, *Groups generated by a class of elements of order 3*, J. Algebra, 24 (1973), 591-612.

[3] A. Zhurtov, *Frobenius groups generated by two elements of order 3*, Siberian Math. J., 42 (2001), 450-454.

[4] V. Mazurov, *A characterization of alternating groups*, Algebra Logika, 44 (2005), 54-69, Algebra and Logic, 44 (2005), 31-39.

[5] A. Maksimenko, A. Mamontov, *The local finiteness of some groups generated by a conjugacy class of order 3 elements*, Siberian Math. J., 48 (2007), 508-518.

[6] R.Brauer, K. A .Fowler, *On groups of even order*, Ann. of Math., 62 (1955), 565-583.

[7] S. Rapinchuk, Y. Segev, M. Seitz, *Finite quotients of the multiplicative group of a finite dimensional division algebra are solvable*, J. Amer. Math. Soc., 15 (2002), 929-978.

[8] C. Bates, D. Bundy, S. Hart, P. Rowley, *Commuting involution graphs for sporadic simple groups*, J. Algebra, 316 (2007), 849-868.

[9] C. Bates, D. Bundy, S. Hart, P. Rowley, *Commuting involution graphs for finite Coxeter groups*, J. Group Theory, 6 (2003), 461-476.

[10] C. Bates, D. Bundy, S. Hart, P. Rowley, *Commuting involution graphs for symmetric groups*, J. Algebra, 266 (2003), 133-153.

[11] C. Bates, D. Bundy, S. Hart, P. Rowley, *Commuting involution graphs in special linear groups*, Comm. Algebra, 32 (2004), 4179-4196.

[12] A. Nawawi, P. Rowley, *On commuting graphs for elements of order 3 in symmetric groups*, PhD thesis, University of Manchester, 2012.

[13] D. Azeez, A. Aubad, *Analysing the commuting graphs for elements of order 3 in Mathieu groups,* International Journal of Psychosocial Rehabilitation, 24 (2020), 1475-7192.

[14] A. Everett, P. Rowley, *Commuting involution graphs for 4-dimensional projective symplectic groups*, Graphs Combin, 36 (2020), 959-1000.

[15] H. Conway, R.T. Curtis, S.P. Norton, R. A. Parker, *ATLAS of finite groups: maximal subgroups and ordinary characters for simple groups*, Oxford, Clarendon Press, 1985.

[16] J. Brinkman, W. Bosma, J. Cannon, C. Playoust, *The magma algebra system I. The user language*, J. Symbolic Comput., 24 (1997), 235-265.

[17] The GAP Group., *GAP groups, algorithms, and programming*, Version 4.11.0, http://www.gap-system.org, 2020.

[18] J. Tripp, I. Suleiman, S. Rogers R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, A. Wilson, P. Walsh, *A world wide web atlas of group representations*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`

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