A note on rigid rings

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Abstract. Let R be an associative ring with identity element and let σ be an endomorphism of R. σ is called rigid if whenever $a \in R$ with $a\sigma(a) = 0$, then $a = 0$. R is called a σ-rigid ring if it has a rigid endomorphism σ. In this article we study some of the elementary properties of rigid rings which are generalizations of some properties of reduced rings. We show, among other things, that the set of all rigid automorphisms of a ring R is a normal subgroup of all automorphisms of R.

Keywords: reduced rings, rigid rings, zero commutative rings, zero insertive rings.

1. Introduction

Throughout this article all rings are associative with multiplicative identity element. The ring of integers is denoted by $\mathbb{Z}$ and the ring of integers modulo a positive integer n is denoted by $\mathbb{Z}_n$. Given a ring $R, R[x]$ denotes the polynomial ring with indeterminate x and with coefficients in R. $E = End(R)$ denotes the ring of all endomorphisms of the ring R. If σ is an endomorphism of R, $R[x, \sigma]$ denotes the skew polynomial ring which is the collection of all polynomials in the indeterminate x and with coefficients in R under usual addition of polynomials while multiplication of polynomials is defined by setting $xr = \sigma(r)x$. The set of all nilpotent elements in R (i.e., the nilradical of R) is denoted by $N(R)$. A ring R is called a reduced ring if $N(R) = (0)$, i.e., if whenever $a^2 = 0$, for $a \in R$, then $a = 0$. A ring R is called a zero commutative ring (zc) if whenever $ab = 0$, then $ba = 0$, and R is called a zero insertive ring (zi) if whenever $ab = 0$, then $arb = 0$ for all $r \in R$. Reduced rings are zc and every zc ring is zi (see, for example, [4] and [9]).

As a sort of generalization of reduced rings Krempa in 1996 [8] introduced what he called rigid rings as we will see the definition in the next section. These rings where studied by many authors (see, for example, [2], [10] and [11]). In this article we will study some properties of rigid rings and see its relations with the other mentioned rings.

2. Rigid rings

Definition 2.1 ([8])**.** *Let R be a ring and let $\sigma \in End(R)$. σ is said to be a rigid endomorphism of the ring R if whenever $a\sigma(a) = 0$, for $a \in R$, then $a = 0$.*

A ring R is called σ-rigid (or rigid) if it has a rigid endomorphism σ.

Clearly, if σ is a rigid endomorphism of a ring R, then $ker\sigma = 0$, and if a ring R is reduced, then the identity endomorphism, id of R is rigid. So reduced rings are id-rigid rings. So rigid rings are in some sense a generalization of reduced rings. However it is not true that every endomorphism of a reduced ring is rigid as the following example shows.

Example 2.1. Let D be an integral domain and let $R = D \oplus D$. Then R is a reduced ring under the usual addition and multiplication. Now the map $\sigma : R \longrightarrow R$ given by $\sigma(a, b) = (b, a)$ is an automorphism of R and $(1, 0)\sigma(1, 0) = (1, 0)(0, 1) = (0, 0)$. So σ is not a rigid endomorphism of R.

However, an inner automorphism of a reduced ring R is rigid. For, let $\sigma : R \longrightarrow R$ be given by $\sigma(a) = uau^{-1}$ for some unit $u \in R$. Suppose that $a\sigma(a) = 0$, then $auau^{-1} = 0$. Hence $aua = 0$. Now $(au)^2 = auau = 0$, so $au = 0$ since R is reduced and hence $a = 0$. This implies that σ is rigid.

On the other hand and as was observed by A. A. Kamal in [5], if R is a σ-rigid ring, then R is a reduced ring. For, let R be a σ-rigid ring and assume $a^2 = 0$ for $a \in R$, then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a^2)\sigma^2(a) = 0$. Thus, since σ is rigid, we have $a\sigma(a) = 0$ and so $a = 0$.

Example 2.2. Let $R = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{Z}_3 \right\}$. Then R is a reduced ring with identity. Let σ be the endomorphism of R given by $\sigma(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. It is easy to show that σ is a non-identity rigid automorphism.

However the ring $T = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{Z}_5 \right\}$ is a reduced ring with identity but the automorphism α of T given by $\alpha(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is not a rigid automorphism.

The following example shows that for certain rings, all its ring endomorphisms, except the identity endomorphism, are not rigid.

Example 2.3. Let $\mathbb{Z}_6$ be the ring of integers modulo 6 and let σ be an endomorphism of $\mathbb{Z}_6$. Since $\mathbb{Z}_6$ is a cyclic group generated by 1 under addition modulo 6, σ, when regarded as a group homomorphism, is completely determine by $\sigma(1)$. Let $\sigma(1) = a$, then $\sigma(x) = xa$ for all $x \in \mathbb{Z}_6$. By Lagrange's Theorem for finite groups the order of a, $|a|$, must be $1, 2, 3,$ or 6. Thus $a = 0, 3, 2, 4, 1,$ or 5. But σ is a ring homomorphism, so we have $a = \sigma(1) = \sigma(1.1) = \sigma(1)^2$. i.e., a is an idempotent element in $\mathbb{Z}_6$. This reduces the values of a to $a = 0, 1, 3, 4$. Therefore $\mathrm{End}(\mathbb{Z}_6)$ has 4 elements: $\sigma_0(x) = 0, \sigma_1(x) = x, \sigma_3(x) = 3x$, and $\sigma_4(x) = 4x$, for all $x \in \mathbb{Z}_6$. Now σ_0, σ_3, and σ_4 are not rigid because $1\sigma_0(1) = 0, 2\sigma_3(2) = 0$, and $3\sigma_4(3) = 0$, while none of the elements, of course, 1, 2 or 3 is zero in $\mathbb{Z}_6$. However σ_1 is rigid since $x\sigma_1(x) = x^2$ for all $x \in \mathbb{Z}_6$. Therefore if $x\sigma_1(x) = 0$, then $x^2 = 0$ in $\mathbb{Z}_6$, hence $x = 0$ since $\mathbb{Z}_6$ is a reduced ring.

Definition 2.2 ([4]). *A ring R is called a zero commutative (abbreviated zc) ring if whenever $ab = 0$ for $a, b \in R$, then $ba = 0$. R is called zero insertive (abbreviated zi) ring if whenever $ab = 0$ for $a, b \in R$, then $arb = 0$ for all $r \in R$.*

In literature the property zero commutative (zero-insertive) is, sometimes, called reversible (semicommutative) (See, for example [10]).

Of course every commutative ring is *zc* and hence *zi*. But the ring $\mathbb{Z}_{n^2}$ of integers modulo n^2 is *zc* but not reduced.

The ring S of all upper triangular matrices of the form $\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}$, where $a, b, c, d \in \mathbb{Z}_3$ is *zi* but not *zc* (see, for example, [9]).

The following Lemma duo to C. Y. Hong, N. K. Kim, and T. K. Kwak, [5]. We restate it here for completeness.

Lemma 2.1 ([5]). *Let R be a σ-rigid ring. (1) If $a, b \in R$ such that $ab = 0$, then $a\sigma^n(b) = 0$ and $b\sigma^n(a) = 0$ for every positive integer n. In particular we have $b\sigma(a) = 0$ and $a\sigma(b) = 0$. (2) If $a\sigma^k(b) = 0$ for some positive integer k, then $ab = 0$.*

Proof. (1) Let $a, b \in R$ be such that $ab = 0$. Then $\sigma(ab) = \sigma(a)\sigma(b) = 0$. Now $(a\sigma^n(b))\sigma(a\sigma^n(b)) = (a\sigma^n(b))\sigma(a)\sigma^{n+1}(b) = a(\sigma^n(b)\sigma(a))\sigma(b)\sigma^n(b) = 0$. Since σ is rigid, we have $a\sigma^n(b) = 0$ for every positive integer n. Similarly we can show that $b\sigma^n(a) = 0$.

(2) Assume $a\sigma^k(b) = 0$ for some positive integer k, by (1) we have $\sigma^k(a)\sigma^k(b) = 0$. Now $\sigma^k(ab) = \sigma^k(a)\sigma^k(b) = 0$. Since σ is a monomorphism, we have $ab = 0$. □

Theorem 2.1. *Let R be a ring.*
(1) *If R is σ-rigid, then R is reduced.*
(2) *If R is a reduced ring, then R is zc.*
(3) *If R is zc, then R is zi.*
(4) *If R is zi, then every idempotent $e = e^2$ in R is central, i.e., $er = re$ for all $r \in R$*

Proof. (1) Let $a \in R$ be such that $a^2 = 0$. Then $(\sigma(a))^2 = \sigma(a^2) = 0$. Consider $(a\sigma(a))(\sigma(a\sigma(a)) = a\sigma(a)(\sigma(a)(\sigma(\sigma(a)) = a(\sigma(a))^2\sigma(\sigma(a)) = 0$. Now since σ is rigid we have $a\sigma(a) = 0$ and hence $a = 0$.

(2) Let $a, b \in R$ be such that $ab = 0$. Then $(ba)^2 = (ba)(ba) = b(ab)b = 0$. Since R is reduced, we have $ba = 0$.
(3) Let $a, b \in R$ be such that $ab = 0$. Since R is zc we have $ba = 0$. Then $bax = 0$ for all $x \in R$. Again since R is zc, we have $axb = 0$ for all $x \in R$.
(4) Since $e \in R$ is an idempotent we $e(1 - e) = 0$. Since R is zi, we have $ex(1 - e) = 0$. Hence $ex = exe$. Similarly $exe = xe$. Therefore $ex = xe$ for all $x \in R$. □

Lemma 2.2. *Let σ and τ be rigid endomorphisms of a ring R. Then $\sigma\tau$ is a rigid endomorphism of R.*

Proof. Let σ and τ be rigid endomorphisms of R and let $a \in R$ be such that

$$a((\sigma\tau)(a)) = 0. \qquad (*)$$

Now $(a\tau(a))\sigma(a\tau(a)) = (a\tau(a))\sigma(a)\sigma(\tau(a)) = a.[\tau(a)\sigma(a)].(\sigma\tau)(a) = 0$. This last equation is true because R is zi by Theorem 2.1 and by applying the zero insertivity of R to $(*)$ above. So we have $(a\tau(a))\sigma(a\tau(a)) = 0$. Now since σ is rigid we have $a\tau(a) = 0$ and since τ is rigid we have $a = 0$. This shows $\sigma\tau$ is a rigid endomorphism of R. □

Lemma 2.3. *Let R be a $\sigma -$ rigid ring. Then the identity endomorphism of R is a rigid endomorphism of R.*

Proof. Let R be a σ-rigid ring and let $id : R \longrightarrow R$ be the identity map. Suppose $a(id(a)) = 0$. Then $a^2 = 0$. But since R is a reduced ring by Theorem 2.1 we have $a = 0$. □

Remark 2.1. Let S be the set of all rigid endomorphisms of a ring R. Then by Lemma 2.2 and Lemma 2.3 above, S is a monoid under composition of maps.

Lemma 2.4. *Let R be ring. If σ is a rigid automorphism of R, then σ^{-1} is a rigid automorphism of R.*

Proof. Let σ be a rigid automorphism of R and let $a \in R$ be such that $a\sigma^{-1}(a) = 0$. Then $\sigma(a\sigma^{-1}(a)) = \sigma(0) = 0$. So $\sigma(a)(\sigma\sigma^{-1}(a)) = \sigma(a)a = 0$. Since R is zc by Theorem 2.1 we have $a\sigma(a) = 0$ and by the rigidity of σ we have $a = 0$. □

Theorem 2.2. *Let R be a ring and let N be the set of all rigid automorphisms of R. Then N is a group under composition of maps.*

Proof. Clear from the above two lemmas. □

Now, let $G = Aut(R)$ be the group of all automorphisms of R. We have the following

Theorem 2.3. *N is a normal subgroup of G.*

Proof. Clearly N is a subgroup of G. Now let $\sigma \in N, \alpha \in G$. We show that $\alpha\sigma\alpha^{-1} \in N$. For, let $a \in R$ be such that $a(\alpha\sigma\alpha^{-1}(a)) = 0$. Apply α^{-1} on this equation we get $\alpha^{-1}(a)\alpha^{-1}(\alpha\sigma\alpha^{-1}(a)) = \alpha^{-1}(0) = 0$. Hence we have $(\alpha^{-1}(a))\sigma(\alpha^{-1}(a)) = 0$. By the rigidity of σ we have $\alpha^{-1}(a) = 0$, which implies that $a = 0$. Hence $\alpha\sigma\alpha^{-1}$ is a rigid automorphism of R and then N is a normal subgroup of G. □

Example 2.4. The group $End(\mathbb{Z} \oplus \mathbb{Z})$ has nine elements each is of the form $\sigma(x, y) = (ax + by, cx + dy)$, where $(a, b) = (0, 0), (1, 0)$ and $(0, 1)$; and $(c, d) = (0, 0), (1, 0)$ and $(0, 1)$. The group $G = Aut(\mathbb{Z} \oplus \mathbb{Z})$ consists of two elements α and β, where $\alpha(x, y) = (x, y)$, the identity endomorphism and $\beta(x, y) = (y, x)$. Clearly α is a rigid automorphism and β is not a rigid automorphism. Hence the group N of rigid automorphisms of $\mathbb{Z} \oplus \mathbb{Z}$ is the trivial group.

Let σ be an endomorphism of a ring R and let S be a σ-invariant subring of R, in the sense that $\sigma(s) \in S$ for all $s \in S$. So the restriction of σ to S gives an endomorphism of S. Clearly if R is σ-rigid, then S is σ-rigid.

Let R be a ring and let σ be an endomorphism of R. σ can be extended to an endomorphism $\bar{\sigma}$ of $R[x]$ in a usual way as follows (see, [1]):

$$\bar{\sigma}(a_0 + a_1x + \cdots + a_nx^n) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n.$$

The following Theorem states that the notion of rigidness can be extended from the ring R to the ring of polynomials $R[x]$

Theorem 2.4 ([1])**.** *Let R be a ring and let σ be an endomorphism of R. Then $\bar{\sigma}$ is a rigid endomorphism of $R[x]$ if and only if σ is a rigid endomorphism of R.*

Remark 2.2. Let R be a rigid ring, then $R[x]$ is rigid by the above Lemma and hence $R[x_1, x_2, \cdots, x_n]$ is rigid for any commuting indeterminates $x_1, x_2, \cdots, x_n$. In particular if R is a reduced ring, then $R[x_1, x_2, \cdots, x_n]$ is a reduced ring.

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