Geometric study on modified Koebe functions imposed by confluent hypergeometric functions

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Abstract. The current endeavor, employing confluent hypergeometric functions and convolution tool, introduces a new normalized modified Koebe function in the complex open unit disk. Certain sufficient stipulations on parameters of the modified Koebe function to be starlike, convex and close-to-convex are discussed and presented. Moreover, close-to-convexity of modified Koebe functions with respect to specific functions is discussed. In addition, the stipulations on the modified Koebe function to be included in the Hardy space are also studied.
Keywords: regular function, Koebe functions, hypergeometric functions, convolution product.

1. Introduction

The theory of special functions (SFT) is a classical field in mathematics consisting of regular functions defined via series. It originated in the 19th century and evolved in the 20th century. Among the important researchers include Gauss, Jacobi, Klein and others. Recently, SFT is a crucial part of an ongoing investigation in numerous areas of researches in engineering, physical, and mathematical sciences. In complex analysis, in 1985, De Branges [7] is a mathematician at Purdue who employed a class of special functions called hypergeometric functions to solve the major problem in the geometric theory of regular functions (GTR), namely, Bieberbach conjecture. He has renewed attention in studying this class of functions. Prior to proving conjecture, there were only a few papers

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dealing with the relation between GTR and hypergeometric functions. Indeed, GTR studies the relationship between the regular features of a given function and the geometric features of its image domain. This relevance between geometrical and regularity is a pivotal feature in GTR. Accordingly, it has sparked interest in studying several geometric features of various special functions in a complex domain. In other words, diverse authors have discussed sufficient stipulations of univalence, starlikeness, convexity and close-to-convexity on the parameters for various special functions. In 1959 and 1960, Kreyszig and Todd examined the univalence feature of the error function [17], $\exp(z^2)$ function [16], and Bessel function [18]. Later, in 1961, Merkes and Scott [19] discussed the starlikeness feature of Gaussian hypergeometric functions. Beside that, in 1984, Carlson and Shaffer [8] introduced a linear operator including an incomplete beta function. The first sentence of this paper is remarkable in view of de Branges' proof a year later, as it reads "Connections between GTR and SFT have not got much attention". Afterward, in 1986, Ruscheweyh and Singh [33] considered the order of the starlikeness feature of hypergeometric functions. In 1990, Miller and Mocanu [20] derived stipulations for univalence, convexity and starlikeness of Gaussian and confluent hypergeometric functions. In 1996 and 1998, Ponnusamy studied the Hardy spaces of the above interesting functions [28], Close-to-convexity features of confluent hypergeometric functions [26]. Following this, in 2006, Baricz [36] utilized the Ponnusamy's idea and acquired the Hardy spaces of Bessel-type functions. Pursuing this line of study, a number of geometric results have recently been introduced to the literature concerning the Bessel function ([5], [6]), Struve functions ([38]), Wright functions ([29], [31]), Mittag-Leffler functions [3], Lommel functions ([37], [21]).

On the other hand, the convolution product is a significant tool introduced by Hadamard [11] in 1899. It remains one of the most effective and efficient tools to investigate various normalized functions ([21], [1]) and complex operators, ([1], [2], [13] and [14]). Consequently, this study imposes a new normalized modified Koebe function based on the confluent hypergeometric function and convolution technique. Certain sufficient stipulations are examined such that modified Koebe function are starlike, convex and close-to-convex. Furthermore, the Hardy space of modified Koebe function are also acquired.

2. Preliminaries

Let $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ identify the open unit disc on the complex domain $\mathbb{C}$. The class of regular functions in $\mathfrak{D}$ is indicated by Ω. Consider the subclass Λ of Ω consisting of regular functions ϑ of the formula:

$$\vartheta(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{\mathfrak{n}}\, z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}), \tag{1}$$

which are normalized by $\vartheta(0) = \vartheta'(0) - 1 = 0$. Denoted by Ξ the subclass of Λ including univalent functions in $\mathfrak{D}$. The Koebe function κ defined by

$$\kappa(z) = \frac{z}{(1-z)^2} = \sum_{\mathfrak{n}=1}^{\infty} \mathfrak{n}\, z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}), \tag{2}$$

is a remarkable instance of a univalent function in Ξ. Actually, Koebe function and its rotations $e^{v}\kappa\left(e^{iv}z\right)$ are the only extremal functions for numerous problems in GTR. The known subclasses of Ω are classes of convex, starlike, close-to-convex functions, which are determined by regular and geometric stipulations. Study [34] first introduced in 1913, the class $\mathcal{C}_{\mathcal{O}}$ of convex functions which is formulated by regular stipulation: $\mathcal{C}_{\mathcal{O}} = \{\vartheta \in \Lambda : \ 0 < \Re(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}), \ z \in \mathfrak{D}\}$. Geometrically, a function $\vartheta \in \Lambda$ is called convex if the image $\vartheta(\mathfrak{D})$ is a convex domain. That is $\varrho z_1 + (1-\varrho)z_2 \in \vartheta(\mathfrak{D})$ whenever $z_1, z_2 \in \vartheta(\mathfrak{D})$ and $\varrho \in [0,1]$. Nevanlinna [22] presented in 1921, the class $\mathcal{S}^*$ of starlike functions which is defined by regular stipulation: $\mathcal{S}^* = \{\vartheta \in \Lambda : \ 0 < \Re(\frac{z\vartheta'(z)}{\vartheta(z)}), \ z \in \mathfrak{D}\}$. Geometrically, a function $\vartheta \in \Lambda$ is called starlike if the image $\vartheta(\mathfrak{D})$ is a starlike domain with respect to the origin. That is, if $\varrho z \in \vartheta(\mathfrak{D})$ whenever $z \in \vartheta(\mathfrak{D})$ and $\varrho \in [0,1]$. Robertson [32] imposed in 1936, the classes $\mathcal{C}_{\mathcal{O}}(\zeta)$ and $\mathcal{S}^*(\zeta)$ of convex and starlike functions of order $\zeta \in [0,1)$ which are given by

$$\mathcal{C}_{\mathcal{O}}(\zeta) = \left\{\vartheta \in \Lambda : \ \zeta < \Re\left(1 + \frac{z\vartheta''(z)}{\vartheta'(z)}\right), \ z \in \mathfrak{D}\right\}, \tag{3}$$

and

$$\mathcal{S}^*(\zeta) = \left\{\vartheta \in \Lambda : \ \zeta < \Re\left(\frac{z\vartheta'(z)}{\vartheta(z)}\right), \ z \in \mathfrak{D}\right\}. \tag{4}$$

Clearly, $\mathcal{C}_{\mathcal{O}}(0) = \mathcal{C}_{\mathcal{O}}$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. Kaplan [15] considered in 1952, the class $\mathcal{C}_{\mathcal{V}}$ of close-to convex functions which are formed by $\mathcal{C}_{\mathcal{V}} = \{\vartheta \in \Lambda : \ 0 < \Re(\frac{z\vartheta'(z)}{\mathcal{G}(z)}), \ \mathcal{G} \in \mathcal{S}^*, \ z \in \mathfrak{D}\}$. In other words, a function $\vartheta \in \Lambda$ is called close-to convex if the image $\vartheta(\mathfrak{D})$ is a close-to convex domain, meaning that, the complement of $\vartheta(\mathfrak{D})$ can be expressed as the union of non-intersecting half-lines. More generally, Noonan [23] investigated in 1973, the class $\mathcal{C}_{\mathcal{V}}(\zeta)$ of close-to convex functions of order $\zeta \in [0,1)$ which are given by

$$\mathcal{C}_{\mathcal{V}}(\zeta) = \left\{\vartheta \in \Lambda : \ \zeta < \Re\left(\frac{z\vartheta'(z)}{\mathcal{G}(z)}\right), \ \mathcal{G} \in \mathcal{S}^*(\zeta), \ z \in \mathfrak{D}\right\}. \tag{5}$$

Evidently, $\mathcal{C}_{\mathcal{V}}(0) = \mathcal{C}_{\mathcal{V}}$. Baricz [4] proposed in 2006, two interesting classes of order $\zeta \in [0,1)$ which are defined by

$$\mathcal{P}_{\mu}(\zeta) = \left\{\mathsf{p} \in \Omega : \mathsf{p}(0) = 1, \ \ \zeta < \Re\left(e^{i\mu}\mathsf{p}(z)\right), \ \mu \in \mathbb{R}, \ z \in \mathfrak{D}\right\}, \tag{6}$$

and

$$\mathcal{R}_{\mu}(\zeta) = \left\{\vartheta \in \Lambda : \zeta < \Re\left(e^{i\mu}\vartheta'(z)\right), \ \mu \in \mathbb{R}, \ z \in \mathfrak{D}\right\}. \tag{7}$$

Note that for $\mu = 0$, the classes (6) and (7) reduces to $\mathcal{P}(\zeta)$ and $\mathcal{R}(\zeta)$, respectively. Further, for $\mu = 0$ and $\zeta = 0$, the classes $\mathcal{P}$ and $\mathcal{R}$ are obtained.

Hadamard [11] presented in 1899, the important mathematical operation on any two regular functions, called convolution product and denoted by $*$, to yield a third regular function. It is formulated as: for $\vartheta_1,\ \vartheta_2 \in \Lambda$ given by $\vartheta_1(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{1\mathfrak{n}}\ z^{\mathfrak{n}}$ and $\vartheta_2(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{2\mathfrak{n}}\ z^{\mathfrak{n}}$, then

$$\vartheta_1(z) * \vartheta_2(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \varkappa_{1\mathfrak{n}}\ \varkappa_{2\mathfrak{n}}\ z^{\mathfrak{n}}, \qquad (z \in \mathfrak{D}). \tag{8}$$

The scientist Wallis first utilized the term "hypergeometric series" in 1655. Then, systematic studies were conducted by Gauss (1813), Kummer (1836), and Riemann (1857). The hypergeometric function ${}_0\mathcal{F}_1(\tau, z)$ is stated as the following:

$$ {}_0\mathcal{F}_1(\tau, z) = \sum_{\mathfrak{n}=0}^{\infty} \frac{1}{(\tau)_{\mathfrak{n}}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}!}, \qquad (\tau \in \mathbb{C} \setminus \{0, -1, \ldots\},\ z \in \mathbb{C}), \tag{9}$$

where $(\tau)_{\mathfrak{n}} = \frac{\Gamma(\tau+\mathfrak{n})}{\Gamma(\tau)} = \tau(\tau+1)...(\tau+\mathfrak{n}-1)$, denotes the (Pochhammer) symbol. This function, called a confluent hypergeometric limit function, achieves the differential equation $z\vartheta''(z) + \tau\vartheta'(z) = \vartheta(z)$, [9].

Moreover, consider the space Ω^{∞} of bounded functions ϑ on Ω. Let $\vartheta \in \Omega$, set

$$\mathfrak{M}_{\mathsf{p}}(\mathsf{r}, \vartheta) = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left| \vartheta \left(\mathsf{r} e^{i\tau} \right) \right|^{\mathsf{p}} d\tau \right)^{1/\mathsf{p}}, & 0 < \mathsf{p} < \infty, \\ \sup\{ |\vartheta(z)| : |z| \le \mathsf{r} \}, & \mathsf{p} = \infty. \end{cases} \tag{10}$$

The function $\vartheta \in \Omega$ belongs to the hardy space $\Omega^p (0 < p \le \infty)$. If $\mathfrak{M}_{\mathsf{p}}(\mathsf{r}, \vartheta)$ is bounded for $\mathsf{r} \in [0, 1)$, then $\vartheta \in \Omega^{\mathsf{p}}$. Notice that $\Omega^{\infty} \subset \Omega^{\mathsf{q}} \subset \Omega^{\mathsf{p}}$, $0 < \mathsf{q} < \mathsf{p} < \infty$, [10]. Also for, in [30], $\Re\left(\vartheta'(z)\right) > 0$ in $\mathfrak{D}$, then

$$\begin{cases} \vartheta' \in \Omega^{\mathsf{q}}, & \mathsf{q} < 1, \\ \vartheta \in \Omega^{\mathsf{q}/(1-\mathsf{q})}, & 0 < \mathsf{q} < 1. \end{cases} \tag{11}$$

The following is a description of some results that will be utilized to gain the main results.

Lemma 2.1 ([24]). *If* $\vartheta \in \Lambda$ *achieves the inequality*

$$\left| z\vartheta''(z) \right| < \frac{1-\zeta}{4}, \qquad (z \in \mathfrak{D},\ 0 \le \zeta < 1), \tag{12}$$

then

$$\Re z\vartheta'(z) > \frac{1+\zeta}{4}, \qquad (z \in \mathfrak{D},\ 0 \le \zeta < 1). \tag{13}$$

Lemma 2.2 ([25]). *Let $\vartheta \in \Lambda$. If*

$$1 \geq 2\varkappa_2 \geq ... \geq \mathfrak{n}\varkappa_\mathfrak{n} \geq (\mathfrak{n}+1)\varkappa_\mathfrak{n} \geq ... \geq 0, \quad \text{or } 1 \leq 2\varkappa_2 \leq ... \leq \mathfrak{n}\varkappa_\mathfrak{n} \geq (\mathfrak{n}+1)\varkappa_\mathfrak{n} \leq ... \leq 2, \tag{14}$$

then ϑ is close-to-convex with respect $-\log(1-z)$. Moreover, if the odd function $\psi(z) = z + \ell_3 z^3 + ... + \ell_{2\mathfrak{n}+1} z^{2\mathfrak{n}+1} + ...$ is regular in $\mathfrak{D}$ and if

$$1 \geq 3\ell_3 \geq ... \geq (2\mathfrak{n}+1)\ell_{2\mathfrak{n}+1} \geq (\mathfrak{n}+1)\varkappa_\mathfrak{n} \geq ... \geq 0, \quad \text{or } 1 \leq 3\ell_3 \leq ... \leq \mathfrak{n}\varkappa_\mathfrak{n} \geq (2\mathfrak{n}+1)\ell_{2\mathfrak{n}+1} \leq ... \leq 2, \tag{15}$$

then ψ is univalent in $\mathfrak{D}$.

Lemma 2.3 ([35]). *$\mathcal{P}_0(\zeta) * \mathcal{P}_0(\xi) \subset \mathcal{P}_0(\varpi)$, where $\varpi = 1 - 2(1-\zeta)(1-\xi)$ with $\zeta, \xi < 1$ and the value of ϖ is best possible.*

Lemma 2.4 ([27]). *For $\zeta < 1, \xi < 1$ and $\varpi = 1 - 2(1-\zeta)(1-\xi)$, we have $\mathcal{R}_0(\zeta) * \mathcal{R}_0(\xi) \subset \mathcal{R}_0(\varpi)$ or equivalently $\mathcal{P}_0(\zeta) * \mathcal{P}_0(\xi) \subset \mathcal{P}_0(\varpi)$.*

Lemma 2.5 ([12]). *If $\vartheta \in \mathcal{C}_\mathcal{V}(\zeta)$, where $\zeta \in [0,1)$ is not of the formula*

$$\vartheta(z) = \begin{cases} \eta + \delta z \left(1 - ze^{i\rho}\right)^{2\zeta - 1}, & \zeta \neq \frac{1}{2}, \\ \eta + \delta \log\left(1 - ze^{i\rho}\right), & \zeta = \frac{1}{2}. \end{cases} \tag{16}$$

for η, $\delta \in \mathcal{C}$ and $\rho \in \mathbb{R}$, then the following statements are true:

1. *There is $\sigma = \sigma(\vartheta) > 0$ such that $\vartheta' \in \Omega^{\sigma + 1/[2(1-\zeta)]}$.*
2. *If $\zeta \in [0, 1/2)$, then there is $\omega = \omega(\vartheta) > 0$ such that $\vartheta \in \Omega^{\omega + 1/(1-2\zeta)}$.*
3. *If $\zeta \leq 1/2$, then $\vartheta \in \Omega^\infty$.*

3. Modified Koebe functions

This section introduces, for $\varrho \in \Omega_\mathfrak{X}$, a new modified Koebe function based on confluent hypergeometric limit function.

Corresponding to (2), we impose the following normalized function:

$$\begin{aligned} \mathcal{E}_\varrho(z) &= \frac{z}{1-z} + z\ \varrho\left[\kappa'(z) - 1\right] \qquad\qquad (0 \leq \varrho) \\ &= z + \sum_{\mathfrak{n}=1}^{\infty} z^\mathfrak{n} + \sum_{\mathfrak{n}=1}^{\infty} \varrho\ \mathfrak{n}^2\ z^\mathfrak{n} \\ &= z + \sum_{\mathfrak{n}=2}^{\infty} \left(1 + \varrho\ \mathfrak{n}^2\right) z^\mathfrak{n}. \end{aligned} \tag{17}$$

In view of the confluent hypergeometric function ${}_0\mathcal{F}_1(\tau, z)$ given by (7) and normalized function $\mathcal{E}_\varrho(z)$ defined by (15), we introduce a new modified Koebe function by using convolution product.

For $\vartheta \in \Lambda$, $z \in \mathbb{C}$, $0 \leq \varrho$, $\tau \in \mathbb{C} \setminus \{0, -1, \ldots\}$, we consider that

$$\mathcal{E}_{\varrho}(z) * \mathcal{K}_{\varrho,\tau}(z) = z\,{}_0\mathcal{F}_1(\tau, z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{z^{\mathfrak{n}}}{(\tau)_{\mathfrak{n}-1}\ (\mathfrak{n}-1)!}. \tag{18}$$

Therefore, the calculation gives the following modified Koebe function:

$$\begin{aligned} \mathcal{K}_{\varrho,\tau}(z) &= z + \sum_{\mathfrak{n}=2}^{\infty} \frac{z^{\mathfrak{n}}}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} \\ &= z + \sum_{\mathfrak{n}=1}^{\infty} \frac{z^{\mathfrak{n}+1}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}. \end{aligned} \tag{19}$$

4. Geometric features of modified Koebe functions

This section investigates certain appropriate stipulations on parameters of the modified Koebe function to order be starlike, convex and close-to-convex in $\mathfrak{D}$.

Theorem 4.1. *Let $\varrho \in [0, \infty)$ and $\zeta \in [0, 1)$. If*

$$\frac{(2-\zeta) + \sqrt{5\zeta^2 - 16\zeta + 12 - 48\zeta\varrho + 16\zeta^2\varrho + 32\varrho}}{2(1-\zeta)(1+4\varrho)} < \tau, \tag{20}$$

then $\mathcal{K}_{\varrho,\tau} \in \mathcal{S}^*(\zeta)$ in $\mathfrak{D}$.

Proof. To demonstrate the starlikeness of order ζ for the function $\mathcal{K}_{\varrho,\tau}(z)$, it is adequate to verify that

$$\left| \frac{z\mathcal{K}'_{\varrho,\tau}(z)}{\mathcal{K}_{\varrho,\tau}(z)} - 1 \right| < 1 - \zeta. \tag{21}$$

Consider

$$\mathcal{K}'_{\varrho,\tau}(z) = 1 + \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\ z^{\mathfrak{n}}\ , \tag{22}$$

and

$$\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} = 1 + \sum_{\mathfrak{n}=1}^{\infty} \frac{z^{\mathfrak{n}}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}. \tag{23}$$

Furthermore, for $\mathfrak{n} \in \mathbb{N}$ leads to

$$\mathfrak{n} \leq \mathfrak{n}!, \quad 2 \leq \mathfrak{n}+1, \quad \text{and} \quad \tau(\tau+1)^{\mathfrak{n}-1} \leq \tau(\tau+1)_{\mathfrak{n}-1} = (\tau)_{\mathfrak{n}}\,. \tag{24}$$

In light of the famed triangle inequality and (22), (23), (24), it acquires

$$\begin{aligned} \left|\mathcal{K}'_{\varrho,\tau}(z) - \frac{\mathcal{K}_{\varrho,\tau}(z)}{z}\right| &\leq \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\ \mathfrak{n}!} \\ &\leq \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ \tau\ (\tau+1)^{\mathfrak{n}-1}} \\ &\leq \frac{1}{\tau\,(1+4\varrho)} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1} = \frac{\tau+1}{\tau^2\,(1+4\varrho)}, \quad (\tau > 0). \end{aligned} \tag{25}$$

Using (23), (24) and the reverse triangle inequality implies that

$$\begin{aligned} \left|\frac{\mathcal{K}_{\varrho,\tau}(z)}{z}\right| &\geq 1 - \sum_{\mathfrak{n}=1}^{\infty} \frac{1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\ \mathfrak{n}!} \\ &\geq 1 - \frac{1}{\tau\,(1+4\varrho)} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1} \\ &= 1 - \frac{\tau+1}{\tau^2\,(1+4\varrho)} = \frac{\tau^2\,(1+4\varrho) - \tau - 1}{\tau^2\,(1+4\varrho)}, \quad (\tau > 0). \end{aligned} \tag{26}$$

Hence, by combining (25) and (26), then

$$\left|\frac{z\mathcal{K}'_{\varrho,\tau}(z)}{\mathcal{K}_{\varrho,\tau}(z)} - 1\right| \leq \frac{\tau+1}{\tau^2\,(1+4\varrho) - \tau - 1} < 1 - \zeta. \tag{27}$$

Therefore, $\mathcal{K}_{\varrho,\tau}$ is starlike function of order ζ, where $0 \leq \zeta < 1 - \frac{\tau+1}{\tau^2(1+4\varrho)-\tau-1}$.

Theorem 4.2. *Let* $\varrho \in [0, \infty)$ *and* $\zeta \in [0, 1)$. *If*

$$\frac{(5-3\zeta) + 4(1-\zeta) + \sqrt{(5-3\zeta)^2 + 8(1-\zeta)\left[2 - \zeta + 13\varrho - 7\zeta\varrho + 2\varrho^2(1-\zeta)\right]}}{2(1-\zeta)(1+4\varrho)} < \tau, \tag{28}$$

then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{O}}(\zeta)$ *in* $\mathfrak{D}$.

Proof. In order to examine the convexity of order ζ for the function $\mathcal{K}_{\varrho,\tau}(z)$, it is sufficient to prove that

$$\left|\frac{z\mathcal{K}''_{\varrho,\tau}(z)}{\mathcal{K}'_{\varrho,\tau}(z)}\right| < 1 - \zeta. \tag{29}$$

Consider

$$\mathcal{K}''_{\varrho,\tau}(z) = \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}(\mathfrak{n}+1)}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\ \mathfrak{n}!}\ z^{\mathfrak{n}}. \tag{30}$$

By utilizing (30), (24) and the following inequality: for all $\mathfrak{n} \in \mathbb{N}$

$$\frac{\mathfrak{n}(\mathfrak{n}+1)}{2^{\mathfrak{n}}} \leq \mathfrak{n}! \;, \tag{31}$$

we attain

$$\begin{aligned} \left|z\mathcal{K}''_{\varrho,\tau}(z)\right| &= \left|\sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}(\mathfrak{n}+1)}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\, \mathfrak{n}!}\ z^{\mathfrak{n}}\right| \leq \frac{1}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}(\mathfrak{n}+1)}{(\tau+1)^{\mathfrak{n}-1}\ \mathfrak{n}!} \\ &\leq \frac{2}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{2}{\tau+1}\right)^{\mathfrak{n}-1} = \frac{2\,(\tau+1)}{(1+4\varrho)\,\tau\,(\tau-1)}, \quad (\tau>1). \end{aligned} \tag{32}$$

Moreover, by using (22) and the following inequalities: for $\mathfrak{n} \in \mathbb{N}$

$$2(\mathfrak{n}+1) \leq 4^{\mathfrak{n}}, \quad \text{and} \quad \mathfrak{n}! \geq 2^{\mathfrak{n}-1}, \tag{33}$$

we achieve

$$\begin{aligned} \left|\mathcal{K}'_{\varrho,\tau}(z)\right| &= \left|1+\sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\, \mathfrak{n}!}\ z^{\mathfrak{n}}\right| \\ &\geq 1-\frac{1}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \frac{\mathfrak{n}+1}{(\tau+1)^{\mathfrak{n}-1}\ \mathfrak{n}!} \\ &\geq 1-\frac{2}{(1+4\varrho)\,\tau} \sum_{\mathfrak{n}=1}^{\infty} \left(\frac{2}{\tau+1}\right)^{\mathfrak{n}-1} \\ &= \frac{(1+4\varrho)\,\tau^2-(3+4\varrho)\,\tau-2}{\tau(\tau-1)\,(1+4\varrho)}, \quad (\tau>1). \end{aligned} \tag{34}$$

By combining (32) and (34), we yield

$$\left|\frac{z\mathcal{K}''_{\varrho,\tau}(z)}{\mathcal{K}'_{\varrho,\tau}(z)}\right| \leq \frac{2\,(\tau+1)}{(1+4\varrho)\,\tau^2-(3+4\varrho)\,\tau-2} < 1-\zeta. \tag{35}$$

Thus, $\mathcal{K}_{\varrho,\tau}$ is convex function of order ζ, where $0 \leq \zeta < 1-\frac{2(\tau+1)}{(1+4\varrho)\tau^2-(3+4\varrho)\tau-2}$.

Theorem 4.3. *Let $\varrho \in [0,\infty)$ and $\zeta \in [0,1)$. If*

$$\frac{8+(1-\zeta)(1+4\varrho) + \sqrt{\zeta^2-50\zeta+113+4\varrho(1-\zeta)(4\varrho(1-\zeta)+50-2\zeta)}}{2(1-\zeta)(1+4\varrho)} < \tau. \tag{36}$$

then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}(\frac{1+\zeta}{2})$ in $\mathfrak{D}$.

Proof. Utilizing inequality (32) and Lemma 2.1, we gain

$$\left|z\mathcal{K}''_{\varrho,\tau}(z)\right| \leq \frac{2(\tau+1)}{(1+4\varrho)\tau(\tau-1)} \leq \frac{1-\zeta}{4}. \tag{37}$$

where $0 \leq \zeta < 1 - \frac{8(\tau+1)}{(1+4\varrho)\tau(\tau-1)}$ and

$$\frac{8+(1-\zeta)(1+4\varrho)+\sqrt{\zeta^2-50\zeta+113+4\varrho(1-\zeta)(4\varrho(1-\zeta)+50-2\zeta)}}{2(1-\zeta)(1+4\varrho)} < \tau.$$

This shows that $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}(\frac{1+\zeta}{2})$. Hence, $\Re\left(\mathcal{K}'_{\varrho,\tau}(z)\right) > \frac{1+\zeta}{2}$.

Theorem 4.4. *Let $\varrho \in [0,\infty)$ and $\zeta \in [0,1)$. If*

$$\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau. \tag{38}$$

then $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$ in $\mathfrak{D}$.

Proof. To show that $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$, it is sufficient to verify that $|\mathfrak{A}(z)-1| < 1$ where $\frac{(\mathfrak{A}(z)/z)-\zeta}{1-\zeta}$. By utilizing the inequalities $\tau(\tau+1)^{\mathfrak{n}-1} \leq \tau(\tau+1)_{\mathfrak{n}-1} = (\tau)_{\mathfrak{n}}$ and $1 \leq \mathfrak{n}!$, for $\mathfrak{n} \in \mathbb{N}$. we gain

$$\begin{aligned} |\mathfrak{A}(z)-1| &= \left|\frac{1}{1-\zeta}\sum_{\mathfrak{n}=1}^{\infty}\frac{z^{\mathfrak{n}}}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,\mathfrak{n}!}\right| \\ &\leq \frac{1}{(1-\zeta)(1+4\varrho)\ \tau}\sum_{\mathfrak{n}=1}^{\infty}\frac{1}{(\tau+1)_{\mathfrak{n}-1}} \\ &\leq \frac{1}{(1-\zeta)(1+4\varrho)\ \tau}\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{1}{\tau+1}\right)^{\mathfrak{n}-1} = \frac{\tau+1}{\tau^2(1-\zeta)(1+4\varrho)}. \end{aligned} \tag{39}$$

Therefore, $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}(\zeta)$ for $0 < \zeta < 1 - \frac{\tau+1}{\tau^2(1+4\varrho)}$.

Setting $\zeta = 0$ in the above Theorems 4.1, 4.2, 4.3 and 4.4 , respectively, we yield the required outcome.

Corollary 4.1. *Let $\varrho \in [0,\infty)$ and $z \in \mathfrak{D}$, then*

1. *If $\frac{1+\sqrt{3+8\varrho}}{1+4\varrho} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{S}^*$.*
2. *If $\frac{9+\sqrt{41+\varrho(13+2\varrho)}}{2(1+4\varrho)} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{O}}$.*
3. *If $\frac{9+4\varrho+\sqrt{113+4\varrho(4\varrho+50)}}{2(1+4\varrho)} < \tau$, then $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{V}}\left(\frac{1}{2}\right)$.*
4. *If $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$, then $\frac{\mathcal{K}_{\varrho,\tau}}{z} \in \mathcal{P}$.*

5. Close-to-convexity of modified Koebe functions with respect to specific functions

This sections discusses some stipulations on the parameters ϱ and τ under which the modified Koebe functions are assured to be close-to-convex with respect to the functions $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$, respectively.

Theorem 5.1. *Let $1 \le \varrho$ and $1 \le \tau$, then $z \to \mathcal{K}_{\varrho,\tau}(z)$ is in $\mathcal{C}_{\mathcal{V}}$ with respect to the function $-\log(1-z)$.*

Proof. Set

$$\Xi(z) = \mathcal{K}_{\varrho,\tau}(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \wp_{\mathfrak{n}-1} z^{\mathfrak{n}}. \tag{40}$$

We yield $0 < \wp_{\mathfrak{n}-1}$ for all $2 \le \mathfrak{n}$ and $\wp_1 = \frac{1}{(1+4\varrho)\tau} \le 1$. To show this outcome, we utilize Lemma 2.2. Thus, we have to show that $\{\mathfrak{n}\wp_{\mathfrak{n}-1}\}_{2\le\mathfrak{n}}$ is decreasing. Now

$$\begin{aligned} &\mathfrak{n}\,\wp_{\mathfrak{n}-1} - (\mathfrak{n}+1)\,\wp_{\mathfrak{n}} \\ &= \frac{\mathfrak{n}}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} - \frac{\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,(\mathfrak{n})!} \\ &= \frac{1}{(\mathfrak{n}-1)!}\left[\frac{\mathfrak{n}^2\ (\tau+\mathfrak{n}-1)\ \left(1+\varrho(\mathfrak{n}+1)^2\right) - (\mathfrak{n}+1)\left(1+\varrho\ \mathfrak{n}^2\right)}{\mathfrak{n}\ (\tau)_{\mathfrak{n}}\ (1+\varrho\ \mathfrak{n}^2)\ (1+\varrho\ (\mathfrak{n}+1)^2)}\right] > 0. \end{aligned} \tag{41}$$

By utilizing the stipulations on parameters, we notice that $\mathfrak{n}\,\wp_{\mathfrak{n}-1} - (\mathfrak{n}+1)\,\wp_{\mathfrak{n}} > 0$ for all $2 \le \mathfrak{n}$. Therefore $\{\mathfrak{n}\wp_{\mathfrak{n}-1}\}_{2\le\mathfrak{n}}$ is a decreasing sequence. Hence the outcome.

Theorem 5.2. *Let $1 \le \varrho$ and $1 \le \tau$, then $z \to z\mathcal{K}_{\varrho,\tau}(z^2)$ is in $\mathcal{C}_{\mathcal{V}}$ with respect to the function $\frac{1}{2}\log\frac{1+z}{1-z}$.*

Proof. Set

$$\Xi(z) = z\mathcal{K}_{\varrho,\tau}(z^2) = z + \sum_{\mathfrak{n}=2}^{\infty} \Lambda_{2\mathfrak{n}-1}\, z^{2\mathfrak{n}-1}. \tag{42}$$

Here, $\Lambda_{2\mathfrak{n}-1} = \wp_{\mathfrak{n}-1} = \frac{1}{(1+\varrho\ \mathfrak{n}^2)\,(\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!}$ for all $2 \le \mathfrak{n}$ and hence we gain $\wp_1 = \frac{1}{\tau\,(1+4\varrho)}$ and $\Lambda_{2\mathfrak{n}-1} > 0$ for all $2 \le \mathfrak{n}$.

To show this outcome, we apply Lemma 2.2. Thus, we have to show that $\{(2\mathfrak{n}-1)\wp_{\mathfrak{n}-1}\}_{2\le\mathfrak{n}}$ is decreasing. Now,

$$\begin{aligned} &(2\mathfrak{n}-1)\,\wp_{\mathfrak{n}-1} - (2\mathfrak{n}+1)\,\wp_{\mathfrak{n}} \\ &= \frac{2\mathfrak{n}-1}{(1+\varrho\ \mathfrak{n}^2)\ (\tau)_{\mathfrak{n}-1}\,(\mathfrak{n}-1)!} - \frac{2\mathfrak{n}+1}{(1+\varrho\ (\mathfrak{n}+1)^2)\ (\tau)_{\mathfrak{n}}\,(\mathfrak{n})!} \\ &= \frac{1}{(\mathfrak{n}-1)!}\left[\frac{(2\mathfrak{n}^2-\mathfrak{n})(\tau+\mathfrak{n}-1)(1+\varrho(\mathfrak{n}+1)^2) - (2\mathfrak{n}+1)(1+\varrho\ \mathfrak{n}^2)}{\mathfrak{n}(\tau)_{\mathfrak{n}}(1+\varrho\ \mathfrak{n}^2)(1+\varrho\ (\mathfrak{n}+1)^2)}\right] > 0. \end{aligned} \tag{43}$$

By employing the stipulations on parameters, we notice that $(2\mathfrak{n}-1)\wp_{\mathfrak{n}-1} - (2\mathfrak{n}+1)\wp_{\mathfrak{n}} > 0$ for all $2 \leq \mathfrak{n}$. Hence $\{(2\mathfrak{n}-1)\wp_{\mathfrak{n}-1}\}_{2\leq\mathfrak{n}}$ is a decreasing sequence. Therefore the outcome.

6. Hardy spaces of modified Koebe functions

This section examines stipulations for modified Koebe functions to belong to the Hardy space.

Theorem 6.1. *Let* $\zeta \in [0,1)$ *and*

$$\frac{(5-3\zeta)+4(1-\zeta) + \sqrt{(5-3\zeta)^2+8(1-\zeta)\left[2-\zeta+13\varrho-7\zeta\varrho+2\varrho^2(1-\zeta)\right]}}{2(1-\zeta)(1+4\varrho)} < \tau.$$

Then

1. $\mathcal{K}_{\varrho,\tau} \in \Omega^{1/(1-2\zeta)}$ *for* $\zeta \in \left[0,\frac{1}{2}\right)$.
2. $\mathcal{K}_{\varrho,\tau} \in \Omega^{\infty}$ *for* $\zeta \geq \frac{1}{2}$.

Proof. By applying the following concept of hypergeometric function:

$$(44)\qquad {}_2\mathcal{F}_1(\mathfrak{a},\mathfrak{b},\mathfrak{c};z) = \sum_{\mathfrak{n}=0}^{\infty} \frac{(\mathfrak{a})_{\mathfrak{n}}(\mathfrak{b})_{\mathfrak{n}}}{(\mathfrak{c})_{\mathfrak{n}}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}!},$$

we yield

$$(45)\qquad \begin{aligned} \eta + \frac{dz}{z\left(1-ze^{i\rho}\right)^{1-2\zeta}} &= \eta + dz\ {}_2\mathcal{F}_1\left(1,1-2\zeta,1;ze^{i\rho}\right) \\ &= \eta + d\sum_{\mathfrak{n}=0}^{\infty} \frac{(1-2\zeta)_{\mathfrak{n}+1}}{\mathfrak{n}!} e^{i\rho\mathfrak{n}} z^{\mathfrak{n}}, \end{aligned}$$

for η, $\delta \in \mathcal{C}$, $\zeta \neq \frac{1}{2}$ and $\rho \in \mathbb{R}$. In addition, we obtain

$$(46)\qquad \begin{aligned} \eta + d\log\left(1-ze^{i\rho}\right) &= \eta - dz\ {}_2\mathcal{F}_1\left(1,1,z;ze^{i\rho}\right) \\ &= \eta - d\sum_{\mathfrak{n}=0}^{\infty} \frac{1}{\mathfrak{n}+1} e^{i\rho\mathfrak{n}} z^{\mathfrak{n}}. \end{aligned}$$

Consequently, $\mathcal{K}_{\varrho,\tau}$ is not of the forms $\eta + dz\left(1-z^{i\rho}\right)^{2\zeta-1}$ for $\zeta \neq \frac{1}{2}$ and $\eta + d\log\left(1-ze^{i\rho}\right)$ for $\zeta = \frac{1}{2}$, respectively. Furthermore, by Theorem 4.2, $\mathcal{K}_{\varrho,\tau} \in \mathcal{C}_{\mathcal{O}}(\zeta)$. Thus, by utilizing Lemma 2.5, we achieve the requited outcome.

Theorem 6.2. *Let* $1 \leq \varrho$, $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$ *and* $\vartheta \in \mathcal{R}$. *Then the convolution* $\mathcal{K}_{\varrho,\tau} * \vartheta$ *is in* $\Omega^{\infty} \cap \mathbb{R}$.

Proof. Let $\digamma(z) = \mathcal{K}_{\varrho,\tau}(z) * \vartheta(z)$. Then, $\digamma'(z) = \frac{\mathcal{K}_{\varrho,\tau}(z)}{z} * \vartheta'(z)$. Utilizing Corollary 4.1(4), we yield $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} \in \mathcal{P}$. Since $\vartheta \in \mathcal{R}$, thus by utilizing Lemma 2.3, we attain $\digamma \in \mathcal{R}$. It is obvious that $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z}$ is an entire function and hence ϑ is entire. This leads to ϑ is bounded. Therefore, we have the required outcome.

Theorem 6.3. *Let $\varrho \in [0,\infty)$, $\zeta \in [0,1)$, $\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau$ and $z \in \mathfrak{D}$. If $\vartheta \in \mathcal{R}(\xi)$ with $\xi < 1$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(\varpi)$, where $\varpi = 1 - 2(1-\zeta)(1-\xi)$.*

Proof. Let $\digamma(z) = \mathcal{K}_{\varrho,\tau}(z) * \vartheta(z)$. Then, $\digamma'(z) = \frac{\mathcal{K}_{\varrho,\tau}(z)}{z} * \vartheta'(z)$. By employing Theorem 4.4, we acquire $\frac{\mathcal{K}_{\varrho,\tau}(z)}{z} \in \mathcal{P}(\zeta)$. By utilizing Lemma 2.4 and considering $\digamma'(z) \in \mathcal{P}(\xi)$, we achieve $\digamma'(z) \in \mathcal{P}(\varpi)$, where $\varpi = 1-2(1-\zeta)(1-\xi)$. Therefore, we obtain $\digamma \in \Re(\varpi)$.

Corollary 6.1. *Let $\varrho \in [0,\infty)$, $\zeta \in [0,1)$, $\frac{1+\sqrt{1+4(1-\zeta)(1+4\varrho)}}{2(1-\zeta)(1+4\varrho)} < \tau$. If $\vartheta \in \mathcal{R}(\xi)$, $\xi = (1-2\zeta)/(2-2\zeta)$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(0)$.*

Corollary 6.2. *Let $\varrho \in [0,\infty)$, $\frac{1+\sqrt{1+4(1+4\varrho)}}{2(1+4\varrho)} < \tau$. If $\vartheta \in \mathcal{R}(1/2)$, then $\mathcal{K}_{\varrho,\tau} * \vartheta \in \mathcal{R}(0)$.*

7. Conclusions

Conducting the study of the confluent hypergeometric functions in a complex domain, we have defined a new normalized modified Koebe-type function. This Koebe-type function may be utilized to yield new classes of normalized regular functions in the open unit disk. Moreover, we have imposed sufficient stipulations on parameters of this function to be starlike, convex and close-to-convex. We have additionally investigated the Hardy spaces of the modified Koebe function. Normalized regular functions are suggested in this work by employing the confluent hypergeometric functions. These functions have been done to reveal some of the connections of the special functions.

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