Contragredient hom-Lie superalgebras

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Abstract. In this paper we introduce the notions of $\mathbb{Z}$-graded hom-Lie superalgebras and contragredient hom-Lie superalgebras. We introduce the invariant bilinear forms on a $\mathbb{Z}$-graded hom-Lie superalgebra and we prove that a consistent supersymmetric α-invariant form on the local part can be extended uniquely to a bilinear form with the same property on the whole $\mathbb{Z}$-graded hom-Lie superalgebra. Furthermore, we check the condition in which hom-Lie superalgebra is simple.
Keywords: hom-Lie superalgebra, contragredient hom-Lie superalgebra, $\mathbb{Z}$-graded hom-Lie superalgebra.

1. Introduction

The notion of hom-Lie algebras was introduced by J. T. Hartwig, D. Larsson and S. Silvestrov in [16] described the structures on certain deformations of Witt algebras and Virasoro algebras, which are widely utilized in the theoretical physics; such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory [1, 10, 11, 12, 24]. The hom-Lie algebras attract more and more attention. New structures on hom-Lie algebras which have great importance and utility, have been defined and discussed by mathematicians that we express them: The hom-Lie algebras were discussed by D. Larsson and S. Silvestrov in [20, 21, 22]. The authors studied quadratic hom-Lie algebras in [9]; representation theory, cohomology and homology theory in [2, 27, 29]. In [19, 21, 23, 28], S. Silvestrov et al. introduced the general quasi-Lie algebras and including as special cases the color hom-Lie algebras

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[5, 7, 8] and in particular hom-Lie superalgebras. Recently, different features of hom-Lie superalgebras has been studied by authors in [3, 4, 6, 26, 30]. On the other hand, we have the graded structure on Lie algebras which were discussed by authors. The first basic example of graded Lie algebras was provided by Nijenhuis and then by Frolicher and Nijenhuis in [14]. In [18], Kac introduced some features of graded Lie algebras and then generalized it to Lie superalgebras in [17]. Now, we want to construct a special graded structure ($\mathbb{Z}$-graded), on hom-Lie superalgebras. In the first section of this paper, hom-Lie algebras, hom-Lie superalgebras and some of their useful related definitions are presented. In section two, we present the notion of $\mathbb{Z}$-graded hom-Lie superalgebras, local hom-Lie superalgebras and contragredient hom-Lie superalgebras. Morever we prove that we can extended uniquely a consistent supersymmetric α-invariant bilinear form on the local part to the whole $\mathbb{Z}$-graded hom-Lie superalgebra and then we prove that if contragredient hom-Lie superalgebra $\mathfrak{g}(A,\tau,\alpha)$ be finite dimensional and Cartan matrix A have some properties, then there exists a consistent supersymmetric α-invariant bilinear form (.,.) on the $\mathfrak{g}(A,\tau,\alpha)$.

Throughout this paper we let $\mathbb{K}$ be an algebraically closed field of characteristic 0. All $\mathbb{Z}_2$-graded vector spaces are considered over $\mathbb{K}$ and linear maps are $\mathbb{K}$-linear maps. Each element in the hom-Lie superalgebra is supposed to be homogeneous and degree of x is denoted by $|x|$, for all $x \in \mathfrak{g}$.

We recall the definition of hom-Lie algebras and Lie superalgebras from [17, 27]. Also, we present notions of a hom-Lie superalgebra as a generalization of a Lie superalgebra [26].

Definition 1.1 ([27]). *A hom-Lie algebra is a triple* $(\mathfrak{g},[.,.],\alpha)$*, where* $\mathfrak{g}$ *is a vector space equipped with a skew-symmetric bilinear map* $[.,.]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *and a linear map* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *such that;*

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,$$

for all x*,* y*,* z *in* $\mathfrak{g}$*.*

- *A hom-Lie algebra is called a multiplicative hom-Lie algebra, if* α *is an algebraic morphism, i.e. for any* $x,y\in\mathfrak{g}$*,*

$$\alpha([x,y])=[\alpha(x),\alpha(y)].$$

- *We call a hom-Lie algebra regular, if* α *is an automorphism.*

Definition 1.2 ([17]). *A Lie superalgebra is a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$*, together with a graded Lie bracket* $[.,.]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *of degree zero, i.e.* $[.,.]$ *is a bilinear map with*

$$[\mathfrak{g}_i,\mathfrak{g}_j]\subset\mathfrak{g}_{i+j(mod2)},$$

such that for homogeneous elements $x,y,z\in\mathfrak{g}$*, the following identities hold:*

- $[x,y]=-(-1)^{|x||y|}[y,x]$,

- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

Now, analogous to the definition of hom-Lie algebras, a hom-Lie superalgebra is defined in a way that makes it a generalization of a Lie superalgebra.

Definition 1.3 ([26]). *A hom-Lie superalgebra is a triple* $(\mathfrak{g}, [.,.], \alpha)$ *consisting of a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$*, an even linear map (bracket)* $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *and an even homomorphism* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *satisfying the following supersymmetry and hom-Jacobi identity, i.e.*

- $[x, y] = -(-1)^{|x||y|}[y, x]$,
- $(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|y||x|}[\alpha(y), [z, x]] + (-1)^{|z||y|}[\alpha(z), [x, y]] = 0$, *where* x, y *and* z *are homogeneous elements in* $\mathfrak{g}$.
- *A hom-Lie superalgebra is called multiplicative hom-Lie superalgebra, if* α *is an algebraic morphism, i.e. for any* $x, y \in \mathfrak{g}$ *we have*

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

- *A hom-Lie superalgebra is called regular hom-Lie superalgebra, if* α *is an algebraic automorphism.*

Remark 1.4 ([3]). Setting $\alpha = id$ in Definition 1.3 we obtain the definition of a Lie superalgebra. Hence hom-Lie superalgebras include Lie superalgebras as a subcategory, thereby motivating the name "hom-Lie superalgebras" as a deformation of Lie superalgebras by an endomorphism.

Example 1.5 ([3]). (Affine hom-Lie superalgebra) Let $V = V_0 \oplus V_1$ be a 3-dimensional superspace where V_0 is generated by e_1, e_2 and V_1 is generated by e_3. The triple $(V, [.,.], \alpha)$ is a hom-Lie superalgebra defined by $[e_1, e_2] = e_1$, $[e_1, e_3] = [e_2, e_3] = [e_3, 3_3] = 0$ and α is any homomorphism.

Let $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ be two hom-Lie superalgebras. An even homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ is said to be a homomorphism of hom-Lie superalgebras, if

$\phi[u, v] = [\phi(u), \phi(v))]'$,
$\phi o \alpha = \beta o \phi$.

The hom-Lie superalgebras $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ are isomorphic, if there is a hom-Lie superalgebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that ϕ be bijective [26].

A sub-vector space $I \subseteq \mathfrak{g}$ is a hom-subalgebra of $(\mathfrak{g}, [.,.], \alpha)$, if $\alpha(I) \subseteq I$ and I is closed under the bracket operation $[.,.]$, i.e. $[I, I] \subseteq I$. Also, hom-subalgebra I is called a hom-ideal of $\mathfrak{g}$, if $[I, \mathfrak{g}] \subseteq I$. Moreover, if $[I, I] = 0$, then I is Abelian [15].

Definition 1.6 ([25]). *The center of a hom-Lie superalgebra* $\mathfrak{g}$*, denoted by* $Z(\mathfrak{g})$*, is the set of elements* $x \in \mathfrak{g}$ *satisfying* $[x, \mathfrak{g}] = 0$.

Remark 1.7. When $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a surjective endomorphism, then one can easily check that $(Z(\mathfrak{g}), \alpha)$ is an Abelian hom-Lie superalgebra and an hom-ideal of $(\mathfrak{g}, [.,.], \alpha)$.

Lemma 1.8. *The quotient* $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ *is an Abelian hom-Lie superalgebra. Moreover,* $[\mathfrak{g}, \mathfrak{g}]$ *is the smallest hom-ideal with this property: if* $\mathfrak{g}/I$ *is Abelian for some hom-ideal* $I \subset \mathfrak{g}$*, then* $[\mathfrak{g}, \mathfrak{g}] \subset I$.

Proof. The proof is straightforward. □

We are going to need the following definition throughout the rest of the paper.

Definition 1.9 ([4])**.** *A representation of the hom-Lie superalgebra* $(\mathfrak{g}, [.,.], \alpha)$ *on a* $\mathbb{Z}_2$*-graded vector space* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *with respect to* $\beta \in gl(V)_{\bar{0}}$ *is an even linear map* $\rho : \mathfrak{g} \to gl(V)$*, such that for all* $x, y \in \mathfrak{g}$*, the following equalities are satisfied:*

$$\rho(\alpha(x)) o \beta = \beta o \rho(x);$$

$$\rho([x, y]) o \beta = \rho(\alpha(x)) o \rho(y) - (-1)^{|x||y|} \rho(\alpha(y)) o \rho(x).$$

A representation V *of* $\mathfrak{g}$ *is called irreducible or simple, if it has no nontrivial subrepresentations. Otherwise* V *is called reducible.*

Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra. We consider $\mathfrak{g}$ as a representation on itself via the bracket and with respect to the morphism α.

Example 1.10 ([4])**.** The α^s-adjoint representation of the hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$, which we denote by ad_s, is defined by

$$ad_s(a)(x) = [\alpha^s(a), x], \quad \text{for all} \quad a, x \in \mathfrak{g}.$$

2. Local and contragredient hom-Lie superalgebras

In [17], Kac introduced the notion of a $\mathbb{Z}$-graded Lie superalgebra and in this section we introduce concept of a $\mathbb{Z}$-graded hom-Lie superalgebra and state some results about it.

Definition 2.1. *Let* $\mathfrak{g}$ *be a hom-Lie superalgebra. It is called a* $\mathbb{Z}$*-graded hom-Lie superalgebra if it is decomposision of itself into a direct sum of finite-demensional* $\mathbb{Z}_2$*-graded subspaces* $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$*, for which* $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

A $\mathbb{Z}$*-graded hom-Lie superalgebra is said to be consistent, if* $\mathfrak{g}_{\bar{0}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{2i}$ *and* $\mathfrak{g}_{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{2i+1}$.

Remark 2.2. One can see that if $\mathfrak{g}$ is a $\mathbb{Z}$-graded hom-Lie superalgebra, then $\mathfrak{g}_0$ is a hom-Lie subalgebra and $[\mathfrak{g}_0, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$; therefore the restriction of the adjoint representation to $\mathfrak{g}_0$ induces linear representation of it on the subspaces $\mathfrak{g}_j$.

Definition 2.3. *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}$ is called simple, if it does not have any nontrivial graded hom-ideal and* $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$.

Remark 2.4. From Definition 2.3, one can consider the followings:

- A left or right graded hom-ideal of $\mathfrak{g}$ is automatically a two sided hom-ideal.
- The condition $[\mathfrak{g},\mathfrak{g}] \neq \{0\}$ serves to eliminate the zero-dimensional and two one-dimensional hom-Lie superalgebras. It follows that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Now, we consider an important notion of hom-Lie superalgebra.

Definition 2.5. *Let $\hat{\mathfrak{g}}$ be a $\mathbb{Z}_2$-graded space, decomposed into a direct sum of $\mathbb{Z}_2$-graded subspaces, $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Suppose that whenever $|i+j| \leq 1$, a bilinear operation is defined $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$, such that $((x,y) \mapsto [x,y])$, satisfying the axiom of anticommutativity and the hom-Jacobi identity for hom-Lie superalgebras, each time the three terms of the identity are defined. Then $\hat{\mathfrak{g}}$ is called a local hom-Lie superalgebra.*

From the above definition we have the following remark.

Remark 2.6. For a $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, there corresponds a local hom-Lie superalgebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which we call the local part of $\mathfrak{g}$.

Homomorphisms, transivity and bitransivity, for a local hom-Lie superalgebra are defined as for a $\mathbb{Z}$-graded hom-Lie superalgebra. In this section we consider only $\mathbb{Z}$-graded hom-Lie superalgebras $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ in which the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$.

Definition 2.7. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local hom-Lie superalgebra.*

- *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with local part $\hat{\mathfrak{g}}$ is said to be maximal ($\mathfrak{g}_{max}(\hat{\mathfrak{g}})$) if for any other $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local part $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.*
- *A $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with local part $\hat{\mathfrak{g}}$ is said to be minimal ($\mathfrak{g}_{min}(\hat{\mathfrak{g}})$) if for any other $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}'$, an isomorphism of the local part $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ extends to an epimorphism of $\mathfrak{g}'$ onto $\mathfrak{g}$.*

Theorem 2.8 ([13])**.** *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a multiplicative local hom-Lie superalgebra such that $\alpha^2 = \alpha$. Then there exists a maximal and a minimal $\mathbb{Z}$-graded hom-Lie superalgebra whose local parts are isomorphic to $\hat{\mathfrak{g}}$.*

Corollary 2.9. *Let $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a local hom-Lie superalgebra.*

i. *Any $\mathbb{Z}$-graded hom-Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$, is a quotient of maximal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}_{max}(\hat{\mathfrak{g}})$.*

ii. *Minimal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}_{min}(\hat{\mathfrak{g}})$ is a quotient of any $\mathbb{Z}$-graded hom-Lie superalgebra whose local part is isomorphic to $\hat{\mathfrak{g}}$.*

iii. *$\mathfrak{g}_{max}(\hat{\mathfrak{g}})$ has a unique maximal graded hom-ideal J_{max} such that*

$$J_{max} \cap \hat{\mathfrak{g}} = \{0\} \quad \text{and} \quad \mathfrak{g}_{max}(\hat{\mathfrak{g}})/J_{max} = \mathfrak{g}_{min}(\hat{\mathfrak{g}}).$$

Proof. The proof is obvious by Theorem 2.8. □

Proposition 2.10 ([13]). *We have the following properties*

i. *A bitransitive $\mathbb{Z}$-graded hom-Lie superalgebra is minimal.*

ii. *A minimal $\mathbb{Z}$-graded hom-Lie superalgebra with bitransitive local part is bitransitive.*

iii. *Two bitransitive $\mathbb{Z}$-graded hom-Lie superalgebra are isomorphic, if and only if their local parts are isomorphic.*

Definition 2.11 ([9]). *Let $(\mathfrak{g}, [.,.], \alpha)$ be a hom-Lie superalgebras. Let f be a bilinear form on $\mathfrak{g}$.*

The form f is said to be consistent, if

$$f(x, y) = 0 \quad \text{for all} \quad x \in g_0, y \in \mathfrak{g}_1;$$

f is said to be nondegenerate, if

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} | f(x, y) = 0, \quad \text{for all} \quad y \in \mathfrak{g}\};$$

f is said to be invariant, if

$$f([x, y], z) = f(x, [y, z]) \quad \text{for all} \quad x, y, z \in \mathfrak{g};$$

f is said to be supersymmetric, if

$$f(x, y) = -(-1)^{|x||y|} f(y, x) \quad \text{for all} \quad x, y, z \in \mathfrak{g}.$$

An invariant bilinear form f is said to be α-invariant, if

$$f(\alpha(x), y) = f(x, \alpha(y)) \quad \text{for all} \quad x, y, z \in \mathfrak{g}.$$

It is obvious that the kernel of an invariant form f on $\mathfrak{g}$ (that is, the set of $x \in \mathfrak{g}$ for which $f(x, \mathfrak{g}) = 0$) is an hom-ideal in $\mathfrak{g}$.

Hence, we have the following proposition:

Proposition 2.12. *Let $\mathfrak{g}$ be a simple hom-Lie superalgebra. Then every nonzero invariant form on $\mathfrak{g}$ is nondegenerate and any two invariant forms on $\mathfrak{g}$ are proportional.*

Proposition 2.13. *Let $(.,.)$ be a consistent supersymmetric α-invariant bilinear form on the local part of a $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ for which $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$, where $i + j \neq 0$. If $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ generates $\mathfrak{g}$, then the form can be extended uniquely to a consistent supersymmetric α-invariant bilinear form with the same property on the whole $\mathfrak{g}$.*

Proof. Let $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$, where $i + j \neq 0$. We prove by induction on k such that for $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_{-i}$, $|i| \leq k$, we can define (x, y) so that the bilinear form $(.,.)$ be α-invariant so long as all the elements in this equation lie in the space $\bigoplus_{i=-k}^{k} \mathfrak{g}_i$. When $k = 1$, the assertion holds by hypothesis, since $(.,.)$ is defined on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For $i > 0$, let x_i and y_i be elements in $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ respectively. Suppose that

$$A = (-1)^{|s-k||s|}(-1)^{|s-k||k|}([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s).$$

By the induction hypothesis we have the following equality $(0 < s < k)$:

$$\begin{aligned}
A &= (-1)^{|s-k||s|}(-1)^{|s-k||k|}([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s) \\
&= (-1)^{|k-s||s-k|}([\alpha(x_{k-s}), [x_s, y_{k-s}]], y_s) + (-1)^{|k-s||s|}([\alpha(x_s), [y_{k-s}, x_{k-s}]], y_s) \\
&= (-1)^{|k-s||s-k|}([\alpha(x_{k-s}), [x_s, y_{k-s}]], y_s) - (-1)^{|k-s||s|}([[y_{k-s}, x_{k-s}], \alpha(x_s)], y_s) \\
&= -(-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}([[x_s, y_{k-s}], \alpha(x_{k-s})], y_s) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||s-k|}([[x_{k-s}, y_{k-s}], \alpha(x_s)], y_s) \\
&= -(-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}([x_s, y_{k-s}], [\alpha(x_{k-s}), y_s]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [\alpha(x_s), y_s]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}([x_s, y_{k-s}], [y_s, \alpha(x_{k-s})]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [x_s, \alpha(y_s)]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}([x_s, y_{k-s}], [y_s, \alpha(x_{k-s})]) \\
&\quad + (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_{k-s}, y_{k-s}], [x_s, \alpha(y_s)]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\quad - (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}([x_s, \alpha(y_s)], [x_{k-s}, y_{k-s}]) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(-1)^{|k-s||s|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\quad - (-1)^{|k-s||s|}(-1)^{|k-s||k-s|}(x_s, [\alpha(y_s), [x_{k-s}, y_{k-s}]]),
\end{aligned}$$

therefore, we have

$$\begin{aligned}
(-1)^{|s-k||k|}([[x_{k-s}, x_s], &\alpha(y_{k-s})], y_s) \\
&= (-1)^{|k-s||s-k|}(-1)^{|k-s||2s-k|}(x_s, [y_{k-s}, [y_s, \alpha(x_{k-s})]]) \\
&\quad - (-1)^{|k-s||k-s|}(x_s, [\alpha(y_s), [x_{k-s}, y_{k-s}]],
\end{aligned}$$

finally

$$([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s) = (x_s, [[\alpha(y_{k-s}), y_s], x_{k-s}]).$$

If we set

$$(\alpha[x_{k-s}, x_s], [y_{k-s}, y_s]) = ([\alpha[x_{k-s}, x_s], y_{k-s}], y_s) = ([[x_{k-s}, x_s], \alpha(y_{k-s})], y_s),$$

then this definition will be well-defined and the form will satisfy the induction hypothesis. □

Definition 2.14. *Let $A = (a_{ij})$ be an $(n \times n)$-matrix with elements from a field $\mathbb{K}$. Let τ be a subset of $I = \{1, 2, ..., n\}$ that tells us which of the generators is odd. Let $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be vector spaces with bases $\{f_i\}$, $\{h_i\}$ and $\{e_i\}$, $i \in I$, respectively. It is east to see that the following relations together with linear map α such that is closed on $\mathfrak{g}_i$ determine the structure of a local hom-Lie superalgebra $\hat{\mathfrak{g}}(A, \tau, \alpha)$ on the spaces $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$:*

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \\ &[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \\ &deg(h_i) = \bar{0}, \\ &deg(e_i) = deg(f_i) = \bar{0}, \qquad i \notin \tau, \\ &deg(e_i) = deg(f_i) = \bar{1}, \qquad i \in \tau. \end{aligned} \tag{1}$$

According to Theorem 2.8, there exists a minimal $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}(A, \tau, \alpha)$ with local part $\hat{\mathfrak{g}}(A, \tau, \alpha)$. We call $\mathfrak{g}(A, \tau, \alpha)$ contragredient hom-Lie superalgebra, A its Cartan matrix and n its rank.

Remark 2.15. By setting $\alpha = id$, we have contragredient Lie superalgebra, also if setting $\tau = \emptyset$, then we have contragredient Lie algebra.

Proposition 2.16. *Let Z be center of $\mathbb{Z}$-graded hom-Lie superalgebra $\mathfrak{g}(A, \tau, \alpha)$.*

i. Z consist of elements of the form $\sum_{i=1}^{n} b_i h_i$, where $\sum_{i=1}^{n} a_{ij} b_i = 0$.

ii. The factor $\mathfrak{g}'(A, \tau, \alpha) = \frac{\mathfrak{g}(A,\tau,\alpha)}{Z}$, with the induced gradation, is transitive.

Proof. i. By relations (1) we have

1. If $\gamma_j \in \mathfrak{g}_0$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = 0.$$

2. If $\gamma_j \in \mathfrak{g}_{-1}$, i.e, $\gamma_j = f_j$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = -(\sum_{i=1}^{n} b_i a_{ij}) f_j = -(\sum_{i=1}^{n} a_{ij} b_i) f_j = 0.$$

3. If $\gamma_j \in \mathfrak{g}_1$, i.e, $\gamma_j = e_j$, then

$$[\sum_{i=1}^{n} b_i h_i, \gamma_j] = \sum_{i=1}^{n} b_i [h_i, \gamma_j] = (\sum_{i=1}^{n} b_i a_{ij}) e_j = (\sum_{i=1}^{n} a_{ij} b_i) e_j = 0.$$

ii. The proof is obvious by Proposition 2.10.

□

If $c \in \mathbb{K}$ and replacing h_i by ch_i and f_i by cf_i, then the $i-th$ row of A is multiplied by c. Therefore, we can assume that a_{ii} is either 0 or 2, for any i.

If the matrix A can be obtained from the matrix $\tilde{A}$ by multiplying any rows by nonzero numbers and by renumbering the indices, then the matrix A and $\tilde{A}$ will be called equivalent. Contragredient hom-Lie superalgebras with equivalent Cartan matrices are isomorphic.

If the Cartan matrix A can be decomposed, that is, it takes the block diagonal form under a suitable permutation of indices

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

then $\mathfrak{g}(A, \tau, \alpha)$ is direct sum of the contragredient hom-Lie superalgebras with the Cartan matrices A_1 and A_2.

In this paper, in many cases we will need some assumptions for the matrix A. For convenience we present them in one place:

(M1) $a_{ii} = 2$ for all i;

(M2) $a_{ij} = 0$ implies that $a_{ji} = 0$;

(M3) for any set of natural numbers $i_1, i_2, ..., i_k \in I$, such that $k \leq n$; we have

$$a_{i_1 i_2} a_{i_2 i_3} ... a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} ... a_{i_1 i_k};$$

(M4) for any $i, j \in I$, there exist $i_1, i_2, ... i_k \in I$ for which

$$a_{i i_1} a_{i_1 i_2} ... a_{i_k j} \neq 0.$$

Proposition 2.17. *Let* $\mathfrak{g}(A, \tau, \alpha) = \bigoplus_{i=-m}^{m} \mathfrak{g}_i$ *be finite dimensional and* Z *be its center. Then* $\mathfrak{g}'(A, \tau, \alpha) = \frac{\mathfrak{g}(A,\tau,\alpha)}{Z}$ *is simple if and only if condition (M4) is satisfied for the Cartan matrix A.*

Proof. Let J be a nonzero hom-ideal of $\mathfrak{g}'(A, \tau, \alpha)$ and $g = \sum_{i \geq k} g_i$ be decomposition of a nonzero element in J and k be the largest integer for all nonzero elements in J. Then $[g_k, \mathfrak{g}_1] = 0$. By using definition of α^s-adjoint representation,

$$\bigoplus_{i,j \geq 0} (ad_s(\mathfrak{g}_{-1}))^i (ad_s(\mathfrak{g}_0))^j (g_k)$$

is a nonzero hom-ideal of $\mathfrak{g}'(A,\tau,\alpha)$. Thus $k=m$ and g is a homogeneous element in J and hom-ideal generated by g_k is contained in J and is homogeneous. Therefore $J\cap(\mathfrak{g}_{-1}\oplus\mathfrak{g}_1)\neq 0$. So, for some $i\in I$, e_i or f_i is in J. It would be easily seen that condition (M4) implies that for all $i\in I$, $e_i, f_i\in J$. Thus $J=\mathfrak{g}'(A,\tau,\alpha)$. Therefore $\mathfrak{g}'(A,\tau,\alpha)$ is simple. Conversly, let condition (M4) is not satisfied for some $i,j\in I$. So it would be easily seen that hom-ideal generated by e_i does not contain e_j; therefore it is contradicted that $\mathfrak{g}'(A,\tau,\alpha)$ is simple. □

Proposition 2.18. *Let $\mathfrak{g}(A,\tau,\alpha)$ is finite dimensional and Cartan matrix A have properties (M2) and (M3). Then there exists a consistent supersymmetric α-invariant bilinear form (.,.) on the $\mathfrak{g}(A,\tau,\alpha)$.*

Proof. Let $(\mathfrak{g}_i,\mathfrak{g}_j)=0$, where $i\neq j$. According to Proposition 2.13, it is sufficient to define bilinear form (.,.) on the $\hat{\mathfrak{g}}(A,\tau,\alpha)$ of $\mathfrak{g}(A,\tau,\alpha)$. Let

$$
\begin{aligned}
&(e_i, f_j) = 0, \quad i\neq j,\\
&(e_1, f_1) = 1.
\end{aligned}
\tag{2}
$$

The form $(.,.)$ will be invariant on $\hat{\mathfrak{g}}(A,\tau,\alpha)$ if $([e_i,f_i],h_j)=(e_i,[f_i,h_j])$ or from 1 $(h_i,h_j)=(e_i,a_{ji}f_i)$ or

$$(h_i,h_j)=a_{ji}(e_i,f_i). \tag{3}$$

Now, it is possible to define the form $(.,.)$ on $\hat{\mathfrak{g}}(A,\tau,\alpha)$ with (M2), (M3), (2) and (3). □

The α-invariant bilinear form $(.,.)$ constructed in the Proposition 2.18 will be called the canonical α-invariant bilinear form.

Using the above Proposition, the following corollary can be obtained.

Corollary 2.19. *For any canonical α-invariant bilinear form with property (M1), we have*

$$a_{ji}=\frac{2(h_i,h_j)}{(h_i,h_i)}.$$

Proof. If we set $i=j$ in (3), then $(h_i,h_i)=a_{ii}(e_i,f_i)$. Since $a_{ii}=2$, $(h_i,h_i)=2(e_i,f_i)\Rightarrow(e_i,f_i)=\frac{(h_i,h_i)}{2}$. Therefore, $a_{ii}=\frac{(h_i,h_j)}{(e_i,f_i)}=\frac{2(h_i,h_j)}{(h_i,h_i)}$. □

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References

[1] N. Aizawa, H. Sato, *q-deformation of the Virasoro algebra with central extension*, Phys. Lett B., 256 (1991), 185-190.

[2] F. Ammar, Z. Ejbehi, A. Makhlouf, *Cohomology and deformations of hom-algebras*, J. Lie Theory, 21 (2010), 813-836.

[3] F. Ammar, A. Makhlouf, *Hom-Lie superalgebras and hom-Lie admissible superalgebras*, J. Algebra, 324 (2010), 1513-1528.

[4] F. Ammar, A. Makhlouf, N. Saadaoui, *Cohomology of hom-Lie superalgebras and q-deformed Witt superalgebra*, Czech Math. J., 63 (2013), 721-761.

[5] A. Armakan, M. R. Farhangdoost, *Extensions of hom-Lie algebras in terms of cohomology*, Czech Math. J., 67 (2017), 317-328.

[6] A. Armakan, M. R. Farhangdoost, *Geometric aspects of extensions of hom-Lie superalgebras*, Int. J. Geom. Meth. Mod. Phys., 14 (2017), 1750085.

[7] A. Armakan, S. Silvestrov, M. R. Farhangdoost, *Enveloping algebras of color hom-Lie algebras*, Turk. J. Math., 43 (2019), 316-339.

[8] A. Armakan, S. Silvestrov, M. R. Farhangdoost, *Extensions of hom-Lie color algebras*, Georgian Math. J., 28 (2021), 15-27.

[9] S. Benayadi, A. Makhlouf, *Hom-Lie algebras with symmetric invariant non-degenerate bilinear forms*, J. Geom. Phys., 76 (2014), 38-60.

[10] M. Chaichian, D. Ellinas, Z. Popowicz, *Quantum conformal algebra with central extension*, Phys. Lett. B., 248 (1990), 95-99.

[11] M. Chaichian, A. Isaev, J. Lukierski, Z. Popowicz, P. Preŝnajder, *q-deformations of Virasoro algebra and conformal dimensions*, Phys. Lett. B., 262 (1991), 32-38.

[12] C. Daskaloyannis, *Generalized deformed Virasoro algebras*, Mod. Phys. Lett. A., 7 (1992), 809-816.

[13] M. R. Farhangdoost, A. R. Attari Polsangi, *$\mathbb{Z}$-graded Hom-Lie superalgebras*, Publ. Math. Debrecen, (2022), in print.

[14] A. Frölicher, A. Nijenhuis, *A theorem on stability of complex structures*, Proc. Natl. Acad. Sci. U.S.A., 43 (1957), 239-241.

[15] B. Guan, L. Chen, B. Sun, *On hom-Lie superalgebras*, Adv. Appl. Clifford Al., 29 (2019).

[16] J. T. Hartwig, D. Larsson, S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra, 295 (2006), 314-361.

[17] V. Kac, *Lie superalgebras*, Adv. Math., 26 (1977), 8-96.

[18] V. Kac, *Simple irreducible graded Lie algebras of finite growth*, Math. USSR. Izv., 2 (1968), 1271-1311.

[19] D. Larsson, S. Silvestrov, *Graded quasi-Lie algebras*, Czech J. Phys., 55 (2005), 1473-1478.

[20] D. Larsson, S. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra, 288 (2005), 321-344.

[21] D. Larsson, S. Silvestrov, *Quasi-Lie algebras*, In: Fuchs, editor. Noncommutative Geometry and Representation Theory in Mathematical Physics Contemp Math 391, Providence, RI, USA: Amer Math Soc., (2005), 241-248.

[22] D. Larsson, S. Silvestrov, *Quasi-deformations of* $sl_2(\mathbb{F})$ *using twisted derivations*, Commun. Algebra, 35 (2007), 4303-4318.

[23] D. Larsson, S. Silvestrov, *On generalized N-complexes coming from twisted derivations*, Generalized Lie Theory in Mathematics, Physics and Beyond, 2009, 81-88.

[24] K. Liu, *Characterizations of the quantum Witt algebra*, Lett. Math. Phys., 24 (1992), 257-265.

[25] Y. Liu, L. Chen, Y, Ma, *Hom-Nijienhuis operators and* T^**-extensions of hom-Lie superalgebras*, Linear Algebra Appl., 439 (2013), 2131-2144.

[26] A. Makhlouf, *Paradigm of nonassociative hom-algebras and hom-superalgebras*, Proceedings of Jordan structures in algebra and analysis meeting: tribute to El Amin Kaidi for his 60th birthday, Almería, 2009 (J. Carmona Tapia et al., ed.). Univ. de Almería, Departamento de Álgebra y Análisis Matemático, Almería, (2010), 143-177.

[27] Y. Sheng, *Representations of hom-Lie algebras*, Algebras Represent Theory, 15 (2012), 1081-1098.

[28] G. Sigurdsson, S. Silvestrov, *Graded quasi-Lie algebras of Witt type*, Czech J. Phys., 56 (2006), 1287-1291.

[29] D. Yau, *Hom-algebras and homology*, J. Lie Theory, 19 (2009), 409-421.

[30] J. Zhou, L. Chen, Y. Ma, *Generalized derivations of hom-Lie superalgebras*, Acta Math. Sinica (Chin Ser), 58 (2014), 3737-3751.

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