On the multivariate extension of an increasing, form order-modular on a continuous 0-lattice

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Abstract. In the paper [17], we show a theorem, which generalizes the two methods of extensions, namely the theorem of Carathéodory and Daniell. We should be able to directly obtain an extension of modular-forms ("Valuation" under Birkhoff [2]) order-continuous increasing on 0-lattice, the two others that would flow naturally as special cases.

In this paper, we established a Multivariate case of our work intituled extension of an increasing form order-modular on a continuous 0-lattice (see, [17]).

Keywords: multivariate extension, lattice, order theory, σ-lattice, modular-form, valuation, multivariate extension of Carathéodory, extension of Daniell, measure, integral, linearity, complementary.

1. Introduction

The classic theory of integration based on extension procedures (measure or integral) is applied on sets or family of more irregular functions as those on which these concepts were originally defined.

The two most used methods are the one hand, the extension of a measure defined on a ring, which is established by Carathéodory [14]; and secondly, the extension is established by Daniell [15] as abstract integral on a vector space defined by a positive linear order-continuous form. The tools used are different views because in the first case, the Boolean structure is heavily used, and that the linearity is always true in the second case.

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2. Result

2.1 Definition and notation

A lattice as defined in [12] is a set E with an order relation satisfying, for all elements $(a_1, ..., a_k) := (a^i)_{i=1}^k$ and $(b_1, ..., b_k) := (b^i)_{i=1}^k$ of $E \subseteq \mathbb{R}^k$, there exists an upper bound and a lower bound on the $(a^i)_{i=1}^k$ and $(b^i)_{i=1}^k$.

E for providing an algebra lattice structure, denoted by $\vee$ the upper bound and $\wedge$ the lower bound defined by the following internal laws

- $(a^i)_{i=1}^k \vee (b^i)_{i=1}^k = ((a_1 \vee b_1), ..., (a_k \vee b_k)) := (\sup(a_1, b_1), ..., \sup(a_k, b_k))$.
- $(a^i)_{i=1}^k \wedge (b^i)_{i=1}^k = ((a_1 \wedge b_1), ..., (a_k \wedge b_k)) := (\inf(a_1, b_1), ..., \inf(a_k, b_k))$.

in the following, we consider $T \subseteq \mathbb{R}^k$ be a lattice with the smallest element 0_T, which will be denoted by 0-lattice [16]. We introduce the following definition.

Definition 2.1. *A sequence $((x_n^i)_{i=1}^k)_n$ is order-convergent to $(x^i)_{i=1}^k$ when there is an increasing sequence $((y_n^i)_{i=1}^k)_n$ and an decreasing sequence $((z_n^i)_{i=1}^k)_n$ such as,*

$$\vee_n^{\uparrow}(y_n^i)_{i=1}^k = (x^i)_{i=1}^k = \wedge_n^{\downarrow}(z_n^i)_{i=1}^k \quad \textit{and} \quad (y_n^i)_{i=1}^k \leq (x^i)_{i=1}^k \leq (z_n^i)_{i=1}^k.$$

Where, in the following, we denote by $\uparrow$ increasing superior (respectively, by $\downarrow$ decreasing inferior).

Definition 2.2. *Let H a real function $H : T \subseteq \mathbb{R}^k \dashrightarrow \overline{\mathbb{R}}$,*

1. *We say that H is semi-continuous order inferiorly (i.e. S.C.I.) (respectively, semi-continuous order superiorly (i.e. S.C.S.), if for every sequence $((x_n^i)_{i=1}^k)_n$ converges to $(x^i)_{i=1}^k$ order, we have*

$$H((x^i)_{i=1}^k) \leq \liminf_{n \longrightarrow +\infty} H((x_n^i)_{i=1}^k) := \underline{\lim}_n H((x_n^i)_{i=1}^k)$$

   (respectively, $H((x^i)_{i=1}^k) \geq \limsup_{n \longrightarrow +\infty} H((x_n^i)_{i=1}^k) := \overline{\lim}_n H((x_n^i)_{i=1}^k)$.

2. *We say that H is order-s.c.i. (respectively, order-s.c.s.) if H is verified s.c.i. (respectively, s.c.s.) for every $(x_n^i)_{i=1}^k$ in T.*

3. *We say that H is continuous order if H is order-s.c.i. and order-s.c.s..*

Remark 2.1. 1. H is increasing order-s.c.i. (respectively, order-s.c.s.), is equivalent to, $H((x_n^i)_{i=1}^k) = \sup_n^{\uparrow} H((x_n^i)_{i=1}^k)$ when $(x_n^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k)$ (respectively, $H((x_n^i)_{i=1}^k) = \inf_n^{\downarrow} H((x_n^i)_{i=1}^k)$ when $(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}((x_n^i)_{i=1}^k)$. As on [3], we introduce the following definition.

2. If H is a form modular when $H(0_T) = 0$ where $0_T := (0^i)_{i=1}^k := \underbrace{(0, ..., 0)}_{k \text{ times}}$.

Definition 2.3. *We say that H is form sub-modular (respectively, form inf-modular), if, for all $((x^i)_{i=1}^k,(y^i)_{i=1}^k)$ of T^2*

$$H((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + H((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \leq H((x^i)_{i=1}^k) + H((y^i)_{i=1}^k)$$

(respectively, $H((x^i)_{i=1}^k \vee (y^i)_{i=1}^k)+H((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \geq H((x^i)_{i=1}^k)+H((y^i)_{i=1}^k)$). In the following, let $T \subseteq \mathbb{R}^k$ a σ-lattice (see, [10]) (i.e. stable by countable $\wedge$ and $\vee$).

Definition 2.4. *We say that $\vee$ (respectively, $\wedge$) is order-continuous on $T \times T \subseteq \mathbb{R}^k \times \mathbb{R}^k$, if, for all increasing sequences $((x_n^i)_{i=1}^k)_n$ and $((y_n^i)_{i=1}^k)_n$ of $T \subseteq R^k$, we have*

$$(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) \wedge (\vee_n^{\uparrow}(y_n^i)_{i=1}^k) = \vee_n^{\uparrow}((x_n^i)_{i=1}^k) \wedge (y_n^i)_{i=1}^k),$$

(respectively, for all decreasing sequences $((x_n^i)_{i=1}^k)_n$ and $((y_n^i)_{i=1}^k)_n$ of $T \subseteq R^k$, we have $(\wedge_n^{\downarrow}(x_n^i)_{i=1}^k) \vee (\wedge_n^{\downarrow}(y_n^i)_{i=1}^k) = \wedge_n^{\downarrow}((x_n^i)_{i=1}^k) \vee (y_n^i)_{i=1}^k)$.

In the following, we denote F, by the function defined from τ to $\mathbb{R}^+$ such that F is an increasing form order-modular and continuous, when τ is sub-lattice of $T \subseteq \mathbb{R}^k$ such that 0_τ is his smallest element.

In the following, we will gave the extension of F.

2.2 Construction of the "half σ-lattice" generated by τ

Consider the following sets

$$\tau^{\uparrow} = \{\vee_n^{\uparrow}(x_n^i)_{i=1}^k / ((x_n^i)_{i=1}^k)_n \in \tau^{\mathbb{N}}\}$$

and

$$\tau^{\downarrow} = \{\wedge_n^{\downarrow}(x_n^i)_{i=1}^k / ((x_n^i)_{i=1}^k)_n \in \tau^{\mathbb{N}}\}.$$

Remark 2.2. . Note that τ is a lattice of $T \subseteq \mathbb{R}^k$, then all $\vee$-countable (respectively, $\wedge$-countable) can be written as $\vee$-countable of an increasing sequence (respectively, $\wedge$-countable of a decreasing sequence). Therefore, it is clear that we have the following set

$$\tau^{\uparrow} = \{\vee_{(x^i)_{i=1}^k \in \mathbb{D}}(x^i)_{i=1}^k / \mathbb{D} \text{ is a countable part of } \tau\} \tag{2.1}$$

and

$$\tau^{\downarrow} = \{\wedge_{(x^i)_{i=1}^k \in \mathbb{D}}(x^i)_{i=1}^k / \mathbb{D} \text{ is a countable part of } \tau\}. \tag{2.2}$$

From the order continuity of $\wedge$ and $\vee$, it is easy to establish that $\tau^{\uparrow}$ (respectively, $\tau^{\downarrow}$) is a lattice and is stable by $\vee$-countable (respectively, $\wedge$-countable).

We have the following lemma.

Lemma 2.1. *Let $\tau^{\uparrow}$ and $\tau^{\downarrow}$ as defined in (2.1) and (2.2) then, we have*

$$\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F((y_n^i)_{i=1}^k) \quad \text{if} \quad \vee_n^{\uparrow}(x_n^i)_{i=1}^k = \vee_n^{\uparrow}(y_n^i)_{i=1}^k \in \tau$$

and

$$\inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((y_n^i)_{i=1}^k) \quad \textit{if} \quad \wedge_n^{\downarrow}(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}(y_n^i)_{i=1}^k \in \tau.$$

So, we can define the extensions of F *denoted by* $F^{\uparrow}$ *to* $\tau^{\uparrow}$ *and* $F^{\downarrow}$ *to* $\tau^{\downarrow}$ *by*

$$F^{\uparrow}((x^i)_{i=1}^k) = \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k) \quad \textit{where} \quad (x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \in \tau^{\uparrow}$$

and

$$F^{\downarrow}((x^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) \quad \textit{where} \quad (x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k \in \tau^{\downarrow}.$$

Proof of Lemma 2.1. If $\vee_n^{\uparrow}(x_n^i)_{i=1}^k = \vee_n^{\uparrow}(y_n^i)_{i=1}^k$ where, $((x_n^i)_{i=1}^k, (y_n^i)_{i=1}^k)_n$ is a sequence of $(\tau^{\uparrow})^2$ then, we have $(x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k \leq (x_n^i)_{i=1}^k$, for all n, m integer and since F is increasing, we have $F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) \leq F((x_n^i)_{i=1}^k)$, for all n, m integer.

On the other hand $\wedge$ is order-continuous on τ, then $\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = (y_m^i)_{i=1}^k \in \tau$, for all m integer and F is order - s.c.i. on τ, wish satisfies $\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = F(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k)) = F((y_m^i)_{i=1}^k)$, for all integer m. So, we have $\sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k) = F(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_m^i)_{i=1}^k)) \leq \sup_n{}^{\uparrow} F((x_m^i)_{i=1}^k)$, for all integer m and

$$\sup_m{}^{\uparrow} F((y_m^i)_{i=1}^k) \leq \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k),$$

by swapping the role of the sequences, we deduce the sought equality.

In the same way, similarly, it follows the equality inf-decreasing,

$$\text{(i.e. } \inf_n{}^{\downarrow} F((x_n^i)_{i=1}^k) = \inf_n{}^{\downarrow} F((y_n^i)_{i=1}^k) \quad \text{if} \quad \wedge_n^{\downarrow}(x_n^i)_{i=1}^k = \wedge_n^{\downarrow}(y_n^i)_{i=1}^k).$$

It is concluded that the functions $F^{\uparrow}$ and $F^{\downarrow}$ are well defined.

The next propositions below gave the properties and extensions built.

Proposition 2.1. *Let* τ *is a 0-lattice and* $F : \tau \subseteq T \subseteq \mathbb{R}^k \dashrightarrow \mathbb{R}^+$ *is an increasing-order modular form and continues on* $\tau^{\uparrow}$*, then we have*

1. $F^{\uparrow}$ *is increasing on* $\tau^{\uparrow}$*;*
2. $F^{\uparrow}$ *is a modular form on* $\tau^{\uparrow}$*;*
3. $F^{\uparrow}$ *is order-s.c.i. on* $\tau^{\uparrow}$*.*

Proposition 2.2. *Let* τ *is a 0-lattice and* $F : \tau \subseteq T \subseteq \mathbb{R}^k \dashrightarrow \mathbb{R}^+$ *is an increasing-order modular form and continues on* $\tau^{\downarrow}$*, then we have*

1. $F^{\downarrow}$ *is increasing on* $\tau^{\downarrow}$*;*
2. $F^{\downarrow}$ *is a modular form on* $\tau^{\downarrow}$ *;*

3. $F^{\downarrow}$ *is order-s.c.s. on* $\tau^{\downarrow}$.

Lemma 2.2. *Let* $((a_n^i)_{i=1}^k)_n$ *and* $((b_n^i)_k^{i=1})_n$ *are real positive increasing sequences (respectively, real positive decreasing sequences) of* $\mathbb{R}^k$ *then, we have*

$$\sup_n{}^{\uparrow}((a_n^i)_{i=1}^k + (b_n^i)_{i=1}^k) = \sup_n{}^{\uparrow}(a_n^i)_{i=1}^k + \sup_n{}^{\uparrow}(b_n^i)_k^{i=1}$$

(*respectively,* $\inf_n{}^{\downarrow}((a_n^i)_{i=1}^k + (b_n^i)_{i=1}^k) = \inf_n{}^{\downarrow}(a_n^i)_{i=1}^k + \inf_n{}^{\downarrow}(b_n^i)_k^{i=1}$).

Proof of Proposition 2.1. 1. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \leq \vee_n^{\uparrow}(y_n^i)_{i=1}^k = (y^i)_{i=1}^k$, where $((x_n^i)_{i=1}^k, (x_n^i)_{i=1}^k)_n$ is a sequence of $\tau^2 \subseteq T \times T \subset \mathbb{R}^k \times \mathbb{R}^k$ so, we have $(x^i)_{i=1}^k = (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k) \in \tau^{\uparrow}$, by Lemma 2.1, we have

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) = F^{\uparrow}((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \\ &:= \sup_n{}^{\uparrow} F((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq \sup_n{}^{\uparrow} F((y^i)_{i=1}^k) := F^{\uparrow}(\vee_n^{\uparrow}(y_n^i)_{i=1}^k) \\ &= F^{\uparrow}((y^i)_{i=1}^k). \end{aligned}$$

then, we deducted that $F^{\uparrow}$ is increasing on $\tau^{\uparrow}$.

2. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and $\vee_n^{\uparrow}(y_n^i)_{i=1}^k = (y^i)_{i=1}^k$ are elements of $\tau^{\uparrow}$ since F is a modular-form on τ, then, for all integer n, we have

$$F(((x^i)_{i=1}^k) \vee ((y^i)_{i=1}^k)) + F((x^i)_{i=1}^k) \wedge (y^i)_{i=1}^k) = F((x^i)_{i=1}^k) + F((y^i)_{i=1}^k).$$

Moreover, F is an increasing function of $\tau \subseteq T$, and let the following real positive increasing sequences,

$$(F^{\uparrow}(x_n^i)_{i=1}^k)_n, (F^{\uparrow}(y_n^i)_{i=1}^k)_n, (F^{\uparrow}((x_n^i)_{i=1}^k) \wedge (y_n^i)_{i=1}^k)_n, \quad (F^{\uparrow}((x_n^i)_{i=1}^k) \vee (y_n^i)_{i=1}^k))_n$$

by combining the following equality's,

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) \vee (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \vee (y_n^i)_{i=1}^k)) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \vee (y_n^i)_{i=1}^k)) \end{aligned}$$

and

$$\begin{aligned} F^{\uparrow}((x^i)_{i=1}^k) \wedge (y^i)_{i=1}^k) &= F^{\uparrow}(\vee_n^{\uparrow}((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \\ &= \sup_n{}^{\uparrow} F((x_n^i)_{i=1}^k \wedge (y_n^i)_{i=1}^k)) \end{aligned}$$

and Lemma 2.2, it's obviously verified that $F^{\uparrow}$ is a modular-form of $\tau^{\uparrow}$.

3. Let $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ with $((x_n^i)_{i=1}^k)_n$ is a sequence of $\tau^{\uparrow}$, for all integer n, we have $(x_n^i)_{i=1}^k = \vee_s^{\uparrow}(x_n^i)_{i=1}^k)^s$ with $((x_n^i)_{i=1}^k)^s \in \tau$.

Let the following sequence defined by, $(y^i_m)^k_{i=1} = \bigvee_{\substack{s\leq m\\ n\leq m}}((x^i_n)^k_{i=1})^s \in \tau$, for all integer m, if $((y^i_m)^k_{i=1})_m$ is an increasing sequence to $(x^i)^k_{i=1}$, we deducted that $\vee^{\uparrow}_m(y^i_m)^k_{i=1} \leq (x^i_n)^k_{i=1}$.

Conversely, for all integer n, s such that

$$(x^i_n)^k_{i=1} \leq (y^i_{\max(m,s)})^k_{i=1} \leq \vee^{\uparrow}_m(y^i_m)^k_{i=1}, \quad \text{for all integer } n$$

with $(y^i_m)^k_{i=1} \in \tau$, it follows that $(x^i)^k_{i=1} = \vee^{\uparrow}_m(y^i_m)^k_{i=1}$.

By Lemma 2.1, we have $F((x^i)^k_{i=1}) = \sup_m{}^{\uparrow} F((y^i_m)^k_{i=1})$.

On the other hand, the sequence $((x^i)^k_{i=1})^s)_s$ is increasing to $(x^i)^k_{i=1}$, for sufficiently large s, therefore we have for all integer n, such that $n \leq m$,

$$((x^i)^k_{i=1})^s \leq (x^i)^k_{i=1} \leq (x^i_m)^k_{i=1}, \text{ for all integer } m.$$

Consequently, we have $(y^i_m)^k_{i=1} \leq (x^i_m)^k_{i=1}$, for all integer m.

Using the increasing of F and the increasing of $F^{\uparrow}$, we obtains that

$$\sup_n{}^{\uparrow} F^{\uparrow}((x^i_n)^k_{i=1}) \leq F^{\uparrow}((x^i)^k_{i=1}) = \sup_n{}^{\uparrow} F((y^i_n)^k_{i=1}) \leq \sup_n{}^{\uparrow} F^{\uparrow}((x^i_n)^k_{i=1}).$$

On the other hand, similarly, we show the Proposition 2.2. For all $(x^i)^k_{i=1} \in T$, we consider the following sets,

$$\tau^{\uparrow}_{(x^i)^k_{i=1}} = \{(t^i)^k_{i=1} \in \tau^{\uparrow} / (t^i)^k_{i=1} \geq (x^i)^k_{i=1}\} \tag{2.3}$$

and

$$\tau^{\downarrow}_{(x^i)^k_{i=1}} = \{(t^i)^k_{i=1} \in \tau^{\downarrow} / (t^i)^k_{i=1} \leq (x^i)^k_{i=1}\}. \tag{2.4}$$

From stability properties in $\vee$-countable of $\tau^{\uparrow}$ and $\wedge$-countable of $\tau^{\downarrow}$, we can easily deduce the following Propositions.

Proposition 2.3. *Let $T \subseteq R^k$ be a 0-lattice; $(x^i)^k_{i=1}, (y^i)^k_{i=1}$ in T, $((x^i_n)^k_{i=1})_n$ is an increasing sequence of $T, \tau^{\uparrow}_{(x^i)^k_{i=1}}$ and $\tau^{\uparrow}_{(y^i)^k_{i=1}}$, are defined by* (2.3)*, we have*

1. *if $(x^i)^k_{i=1} \leq (y^i)^k_{i=1}$, then $\tau^{\uparrow}_{(y^i)^k_{i=1}} \subseteq \tau^{\uparrow}_{(x^i)^k_{i=1}}$;*
2. *$\tau^{\uparrow}_{(x^i)^k_{i=1}}$ is a stable lattice by $\vee$-countable;*
3. *$\tau^{\uparrow}_{\vee^{\uparrow}_n(x^i_n)^k_{i=1}} = \bigcap^{\downarrow}_n(\tau^{\uparrow}_{(x^i_n)^k_{i=1}})$.*

Proposition 2.4. *Let $T \subseteq R^k$ be a 0-lattice; $(x^i)^k_{i=1}, (y^i)^k_{i=1}$ in T, $((x^i_n)^k_{i=1})_n$ is an decreasing sequence of $T, \tau^{\downarrow}_{(x^i)^k_{i=1}}$ and $\tau^{\downarrow}_{(y^i)^k_{i=1}}$, are defined by* (2.4)*, we have*

1. *if $(x^i)^k_{i=1} \leq (y^i)^k_{i=1}$, then $\tau^{\downarrow}_{(x^i)^k_{i=1}} \subseteq \tau^{\downarrow}_{(y^i)^k_{i=1}}$;*

2. $\tau^{\downarrow}_{(x^i)_{i=1}^k}$ *is a stable lattice by* $\wedge$*-countable;*

3. $\tau^{\downarrow}_{\vee_n^{\downarrow}(x_n^i)_{i=1}^k} = \bigcup_n^{\uparrow}(\tau^{\downarrow}_{(x_n^i)_{i=1}^k})$.

Lemma 2.3. *Let* $T \subseteq \mathbb{R}^k$ *be a 0-lattice and* $\tau^{\uparrow}_{((x_n^i)_{i=1}^k)}$ *is defined by* (2.3) *for* $(x^i)_{i=1}^k$ *in* T *then, we have*

$$\mathcal{L}_{(x^i)_{i=1}^k} = \{(x^i)_{i=1}^k \in T / \tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset\} \ \text{ is a } \sigma-\text{lattice and } \ \tau \subseteq \mathcal{L}_{(x^i)_{i=1}^k}.$$

Proof of Lemma 2.3. If for all $n \in \mathbb{N}$, $(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}$, then, $\vee_n(t_n^i)_{i=1}^k \in \tau^{\uparrow}_{\vee_n(x_n^i)_{i=1}^k}$ so, we have $\tau^{\uparrow}_{\vee_n(x_n^i)_{i=1}^k} \subseteq \tau^{\uparrow}_{\wedge_n(x_n^i)_{i=1}^k}$.

Remark 2.3. In the following, consider the σ-lattice $\mathcal{L}_{(x^i)_{i=1}^k}$ in place of $T \subseteq \mathbb{R}^k$ and assume that for every $(x^i)_{i=1}^k \in T, \tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$.

If $(x^i)_{i=1}^k$ of $T \subseteq \mathbb{R}^k$, let F^* and F_* the extensions of F defined on T by,

$$F^*((x^i)_{i=1}^k) := \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \tag{2.5}$$

and

$$F_*((x^i)_{i=1}^k) := \sup_{(t^i)_{i=1}^k \in \tau^{\downarrow}_{(x^i)_{i=1}^k}} F^{\downarrow}((t^i)_{i=1}^k). \tag{2.6}$$

Note that the following inequality $F^*((x^i)_{i=1}^k) \geq F_*((x^i)_{i=1}^k)$ is always verified and the inequality's (2.5) and (2.6) are always true and we have the following propositions.

Proposition 2.5. *Let* F^* *defined as* (2.5) *then, we have*

1. F^* *is an increasing function on* T*;*

2. F^* *is a form under-modular on* T*;*

3. F^* *is order-s.c.i. on* T.

Proposition 2.6. *Let* F_* *defined as* (2.6) *and* $(x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k$ *such that* $(F_*((x_n^i)_{i=1}^k))_n$ *is finite for* n *so large, then, we have*

1. F_* *is an increasing function on* T*;*

2. F_* *is a form ever-modular on* T*;*

3. *If* $(x^i)_{i=1}^k = \wedge_n^{\downarrow}(x_n^i)_{i=1}^k$ *with* $F_*((x_n^i)_{i=1}^k)$ *is finite for all large integer* n, $F_*((x^i)_{i=1}^k) = \inf_n^{\downarrow} F_*((x_n^i)_{i=1}^k)$.

Proof of Proposition 2.5. 1. Let $(x^i)_{i=1}^k$ and $(y^i)_{i=1}^k$ be two elements of $T \subseteq \mathbb{R}^k$, such that $(x^i)_{i=1}^k \leq (y^i)_{i=1}^k$; since $\tau^{\uparrow}_{(y^i)_{i=1}^k}$ is included in $\tau^{\uparrow}_{(x^i)_{i=1}^k}$ then, we have

$$F^*((x^i)_{i=1}^k) := \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \leq \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) := F^*((y^i)_{i=1}^k)$$

so, F^* is increasing on T.

2. Let $(x^i)_{i=1}^k$ and $(y^i)_{i=1}^k$ are two elements of $T \subseteq \mathbb{R}^k$, we will show that,

$$F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) \geq F^*((x^i)_{i=1}^k) \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

If $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) = +\infty$, we have the inequality.

So, suppose that $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k)$ is a finite real number, then for any positive real number ε, there are $((t^i)_{i=1}^k), (s^i)_{i=1}^k) \in (\tau^{\uparrow}_{(x^i)_{i=1}^k})^2$ such that $F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) + \varepsilon \geq F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k)$.

On the other hand $F^{\uparrow}$ is a modular-form on $\tau^{\uparrow}$, then we have

$$F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k) = F^{\uparrow}((t^i)_{i=1}^k \vee (s^i)_{i=1}^k) + F^{\uparrow}((t^i)_{i=1}^k \wedge (s^i)_{i=1}^k)$$

for $(t^i)_{i=1}^k \vee (s^i)_{i=1}^k$ is an element of $\tau^{\uparrow}_{(x^i)_{i=1}^k \vee (y^i)_{i=1}^k}$ and $(t^i)_{i=1}^k \wedge (s^i)_{i=1}^k$ is an element of $\tau^{\uparrow}_{(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k}$ therefore, we have

$$\begin{aligned} F^{\uparrow}((t^i)_{i=1}^k) + F^{\uparrow}((s^i)_{i=1}^k) &\geq \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k \vee (y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \\ &+ \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k}} F^{\uparrow}((t^i)_{i=1}^k) \\ &\geq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k). \end{aligned}$$

On the other hand, we have

$$F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k) + \varepsilon \geq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

for ε is arbitrary. We deducted that F^* is an over-modular form on T.

3. Now, show that F^* is order-s.c.i. on $T \subseteq \mathbb{R}^k$.

Since $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and F^* is increasing, then we have the following inequality, $F^*(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) \geq \sup_n{}^{\uparrow} F^*((x_n^i)_{i=1}^k)$. For the other inequality, we can assume without restriction that $\sup_n^{\uparrow} F^*((x_n^i)_{i=1}^k)$ is finite. Given a strictly positive real ε and an integer n, then by definition $F^*((x_n^i)_{i=1}^k$, and (2.5) there exist a $(t_n^i)_{i=1}^k$ of $\tau^{\uparrow}(x_n^i)_{i=1}^k$ such that

$$F^{\uparrow}((t_n^i)_{i=1}^k) \leq F^*((x_n^i)_{i=1}^k) + \frac{\varepsilon}{2^{(n+1)}}.$$

Let $(e_n^i)_{i=1}^k = \vee_{s=0}^{s=n}(t_s^i)_{i=1}^k$, it is clear that $(e_n^i)_{i=1}^k$ is still an element of $\tau^{\uparrow}_{(x_n^i)_{i=1}^k}$ and since $(x^i)_{i=1}^k = \vee_n^{\uparrow}(x_n^i)_{i=1}^k \leq \vee_n^{\uparrow}(e_n^i)_{i=1}^k$, we have $\vee_n^{\uparrow}(e_n^i)_{i=1}^k$ is a $\tau^{\uparrow}(x^i)_{i=1}^k$ element which is stable $\vee$-countable.

Show by recurrent we have the following inequality

$$(I_n) \qquad F^{\uparrow}((e_n^i)_{i=1}^k) \leq F^*((x_n^i)_{i=1}^k) + \varepsilon\left(1 - \frac{1}{2^{(n+1)}}\right).$$

For $n = 0$, $e_0 = t_0$ then inequality (I_0) is satisfied.

Suppose (I_n) is satisfied for all integer n, first note that

$$(x_n^i)_{i=1}^k = (x_n^i)_{i=1}^k \wedge (x_{n+1}^i)_{i=1}^k \leq (e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k.$$

Since F^* is increasing, we have

$$F^*((x_n^i)_{i=1}^k) \leq F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k). \tag{2.7}$$

As $F^{\uparrow}$ is a modular-form, we have

$$F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k) + F^{\uparrow}((e_{n+1}^i)_{i=1}^k) = F^{\uparrow}((e_n^i)_{i=1}^k) + F^{\uparrow}((t_{n+1}^i)_{i=1}^k) \tag{2.8}$$

or if we define $(t_{n+1}^i)_{i=1}^k$ by,

$$F^{\uparrow}((t_{n+1}^i)_{i=1}^k) \leq F^*((x_{n+1}^i)_{i=1}^k) + \frac{\varepsilon}{2^{(n+2)}}.$$

Therefore, according to the inequality (2.3) and (2.4) and the recurrent's hypothesis, we obtain,

$$\begin{aligned} F^{\uparrow}((e_{n+1}^i)_{i=1}^k) &= F^{\uparrow}((e_n^i)_{i=1}^k + F^{\uparrow}((t_{n+1}^i)_{i=1}^k - F^{\uparrow}((e_n^i)_{i=1}^k \wedge (t_{n+1}^i)_{i=1}^k) \\ &\leq F^*((x_n^i)_{i=1}^k) + F^*((x_{n+1}^i)_{i=1}^k - F^*((x_n^i)_{i=1}^k) \\ &+ \varepsilon\left(1 - \frac{1}{2^{(n+1)}}\right) + \frac{\varepsilon}{2^{(n+2)}}. \end{aligned}$$

Consequently, we have

$$F^{\uparrow}((e_{n+1}^i)_{i=1}^k) \leq F^*((x_{n+1}^i)_{i=1}^k) + \varepsilon\left(1 - \frac{1}{2^{(n+2)}}\right).$$

So, $(I_{(n+1)})$ is still satisfied.

It follows the following inequalities

$$F^*((x_n^i)_{i=1}^k) \leq \overset{\uparrow}{\sup_n} F^*((e_n^i)_{i=1}^k) \leq \overset{\uparrow}{\sup_n} F^*((x_n^i)_{i=1}^k) + \varepsilon, \qquad \text{for all integer } n,$$

when ε is arbitrary, we concluded that F^* is order-s.c.i.

On the other hand, the Proposition 6 is shown similarly.

2.3 Extension of the set $F^* = F_*$

Now, it is assumed that any element $(x^i)_{i=1}^k$ of $T \subseteq \mathbb{R}^k$, we have $\tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$ (see Lemma 3) let the following sets

$$\tau^F = \{(x^i)_{i=1}^k \in T / F^*((x^i)_{i=1}^k) = F_*((x^i)_{i=1}^k)\} \tag{2.9}$$

and

$$\tau^F_{\mathbb{R}} = \{(x^i)_{i=1}^k \in T / F^*((x^i)_{i=1}^k) = F_*((x^i)_{i=1}^k) < +\infty\}. \tag{2.10}$$

On the set (2.9), we have $F^* = F_*$. So, we denote by $\overline{F}$ the common value on τ^F.

Proposition 2.7. *Let τ is a sub-lattice and 0-lattice $F : \tau \dashrightarrow \mathbb{R}^+$ is an increasing-order continues modular-form, τ^F as* (2.9) *and $\tau^F_{\mathbb{R}}$ as* (2.10) *then, we have*

1. *τ^F is a stable set in $\vee$-countable such that, for all $((x^i)_{i=1}^k,(y^i)_{i=1}^k) \in (\tau^F)^2$,*

$$\begin{aligned} \overline{F}((x^i)_{i=1}^k) + \overline{F}((y^i)_{i=1}^k) &= \overline{F}((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + \overline{F}((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &= \overline{F}((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k). \end{aligned}$$

2. *$\tau^F_{\mathbb{R}}$ is a stable trellis $\wedge$-countable containing τ, and $\overline{F}$ is an increasing-order continuous modular-form.*

Proof of Proposition 2.7 1. By Proposition 2.5 and Proposition 2.6, we have for all $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in T^2$,

$$\begin{aligned} F_*((x^i)_{i=1}^k) + F_*((y^i)_{i=1}^k) &\leq F_*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) + F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \\ &\leq F^*((x^i)_{i=1}^k) + F^*((y^i)_{i=1}^k), \end{aligned}$$

when $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F)^2$ so, we have

$$F^*((x^i)_{i=1}^k \vee (y^i)_{i=1}^k) \geq F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \geq \sup(F_*((x^i)_{i=1}^k, F_*((y^i)_{i=1}^k).$$

On the other hand if $F_*((x^i)_{i=1}^k)$ or $F_*((y^i)_{i=1}^k)$ is infinite, then $(x^i)_{i=1}^k \vee (y^i)_{i=1}^k \in \tau^F$.

If $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F_{\mathbb{R}})^2$, then $x \vee y$ and $(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau^F_{\mathbb{R}}$. So, we have the following Lemma.

Lemma 2.4. *Let $(a^i)_{i=1}^k, (b^i)_{i=1}^k, (c^i)_{i=1}^k$ and $(d^i)_{i=1}^k$ the real numbers of $\mathbb{R}^k$ such that $(a^i)_{i=1}^k \geq (b^i)_{i=1}^k$ and $(c^i)_{i=1}^k \geq (d^i)_{i=1}^k$ when $(a^i)_{i=1}^k + (c^i)_{i=1}^k = (b^i)_{i=1}^k + (d^i)_{i=1}^k$ then $(a^i)_{i=1}^k = (b^i)_{i=1}^k$ and $(c^i)_{i=1}^k = (d^i)_{i=1}^k$.*

2. b) According to Proposition 2.5 and Proposition 2.6, F^* is order-s.c.i. and F_* increasing, if $((x_n^i)_{i=1}^k)_n$ is an increasing sequence of elements of τ^F, it is obviously that

$$F^*(\vee_n^{\uparrow}(x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F^*((x_n^i)_{i=1}^k) = \sup_n{}^{\uparrow} F_*((x_n^i)_{i=1}^k) \leq F_*((x_n^i)_{i=1}^k),$$

and so it follows that $\vee_n^{\uparrow}(x_n^i)_{i=1}^k$ is still an element of τ^F.

Finally, F_* is order-s.c.s. in its domain, F^* increasing allows us the same way to verify that $\tau_{\mathbb{R}}^F$ was stable $\wedge$-countable, we have

$$F^*(\wedge_n^{\downarrow}(x_n^i)_{i=1}^k) \leq \overset{\downarrow}{\inf_n} F^*((x_n^i)_{i=1}^k) = \overset{\downarrow}{\inf_n} F_*((x_n^i)_{i=1}^k) = F_*(\wedge_n^{\downarrow}((x_n^i)_{i=1}^k).$$

Theorem 2.1. *Let F continuous modular form order on a trellis τ, then F is uniquely extension of the σ-lattice τ generated by an increasing modular form order-s.c.i. and order–s.c.s. in its domain.*

Proof of Theorem 2.1. Let $\tau_F = \{(x^i)_{i=1}^k \in T / \forall (y^i)_{i=1}^k \in \tau, (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau_{\mathbb{R}}^F\}$.

Step 1. We will show that τ_F is contained in τ^F, let x be an element of τ_F; since $\tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset$, there exists a increasing sequence $((x_n^i)_{i=1}^k)_n$ of τ elements for which we have $(x^i)_{i=1}^k = \vee_n^{\uparrow}((x^i)_{i=1}^k \wedge ((x_n^i)_{i=1}^k)$ as $(x^i)_{i=1}^k \in \tau_F$ and $(x_n^i)_{i=1}^k \in \tau, (x^i)_{i=1}^k \wedge (x_n^i)_{i=1}^k \in \tau_{\mathbb{R}}^F \subseteq \tau^F$.

On the other hand by Proposition 2.7 and because the latter is stable $\vee$-countable, we have x is in τ^F.

Step 2. It is observably verified that τ_F is a σ-lattice containing $\tau_{\mathbb{R}}^F$, on the other hand since $\tau_{\mathbb{R}}^F$ is $\wedge$-stable and contain τ, clearly τ_F containing $\tau_{\mathbb{R}}^F$.

Let $((x_n^i)_{i=1}^k$ a sequence of τ_F and $(y_n^i)_{i=1}^k \in \tau$, then

$$\wedge_n((x_n^i)_{i=1}^k) \wedge (y^i)_{i=1}^k = \wedge_n((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \in \tau_{\mathbb{R}}^F.$$

On the other hand, since, $\tau_{\mathbb{R}}^F$ is stable by $\wedge$-countable, and

$$\vee_n((x_n^i)_{i=1}^k) \wedge (y^i)_{i=1}^k = \vee_n((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \in \tau_{\mathbb{R}}^F$$

then, we have τ^F is stable by $\vee$-countable, and

$$F^*((x_n^i)_{i=1}^k \wedge (y^i)_{i=1}^k) \leq F^*((y^i)_{i=1}^k) < +\infty.$$

Step 3. Based on the above, for every $((x^i)_{i=1}^k, (y^i)_{i=1}^k) \in (\tau^F)^2$ we have

$$F^*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k) = F_*((x^i)_{i=1}^k \wedge (y^i)_{i=1}^k).$$

When $(x^i)_{i=1}^k \wedge (y^i)_{i=1}^k \in \tau_F$, then we deducted from Proposition 2.7, that F is a modular-form increasing order-s.c.i. and order - s.c.s. on τ_F in its domain.

As τ_F is an σ-lattice containing τ, it contains the σ-lattice generated by τ and the restriction of F to the latter answers the problem.

In the next, we will prove the uniqueness of the extension.

Let G be a regular extension of F, G order being continuous, it coincides with $F^{\uparrow}$ on $\tau^{\uparrow}$ and $F^{\downarrow}$, on $\tau^{\downarrow}$ therefore if $(x^i)_{i=1}^k \in T, G$ implies the increasing of the following inequalities, $\forall (t^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}$, $\forall (s^i)_{i=1}^k \in \tau^{\downarrow}_{(x^i)_{i=1}^k}$, we have

$$F^{\downarrow}((s^i)_{i=1}^k) \leq G((x^i)_{i=1}^k) \leq F^{\uparrow}((t^i)_{i=1}^k).$$

By definition of F^* and F_* , we have

$$F^*((x^i)_{i=1}^k) \leq G((x^i)_{i=1}^k) \leq F_*((x^i)_{i=1}^k).$$

The σ-lattice generated by τ being contained in the set of coincidence F^* and F_*, it is concluded that G and $\overline{F}$ are equal.

Remark 2.4. For $k = 1$, we have the Theorem of Carathéodory and Theorem of Daniell (see, [17]).

2.4 Multivariate case of the extension of Carathéodory and Daniell

In the following, we say that the σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$ is a monotone part if, for a increasing sequence $((\mathcal{A}_n^i)_{i=1}^k)_n$ of σ-lattice we have $\bigcup_n (\mathcal{A}_n^i)_{i=1}^k \in \sigma$-lattice; and for a decreasing sequence $((\mathcal{A}_n^i)_{i=1}^k)_n$ of σ-lattice we have $\bigcap_n (\mathcal{A}_n^i)_{i=1}^k$ $\in \sigma$-lattice.

From the constructions we have done before in this paper, we can deduce the multiple versions of the Theorem of Carathéodory and the Theorem of Daniell that we can call the multiple variate theorem of Carathéodory and the multiple variate theorem of Daniell.

2.4.1 Extension of Carathéodory

Corollary 2.1 (Caratheodory's theorem)**.** *Let $(\mathcal{A}^i)_{i=1}^k \subseteq \mathbb{R}^k$ a ring portions of Ω_k and $\mu_k : (\mathcal{A}^i)_{i=1}^k \dashrightarrow \mathbb{R}^+$ a measure, then there exit only one measure defined on the σ-ring generated by $(\mathcal{A}^i)_{i=1}^k$ and extending μ_k.*

Proof of Corollary 2.1. Denoted by $\mathcal{P}(\Omega_k)$ the all parts of Ω_k. Consider the order induced by the inclusion, let de following set,

$$T = \{(A^i)_{i=1}^k \in \mathcal{P}(\Omega)/(A^i)_{i=1}^k \subseteq \bigcup_n (A_n^i)_{i=1}^k \quad \text{with} \quad (A_n^i)_{i=1}^k \in (\mathcal{A}^i)_{i=1}^k\}.$$

T is a σ-lattice sequentially distributive with the smallest element is $\emptyset$, $(\mathcal{A}^i)_{i=1}^k$ is a under-lattice of T and μ is an increasing modular-form continuous order on $(\mathcal{A}^i)_{i=1}^k$.

By the Theorem 2.1, it has a only extension $\overline{\mu}_k$ on the σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$. $(\mathcal{A}^i)_{i=1}^k$ is an increasing modular form-order-s.c.i. and order-s.c.s. in its domain.

The σ-lattice generated by $(\mathcal{A}^i)_{i=1}^k$ is a monotone part (stable by increasing and decreasing) by intersection countable such that the ring $(\mathcal{A}^i)_{i=1}^k$ is included in the σ-lattice, then $((\mathcal{A}^i)_{i=1}^k)^\sigma$ is included in the σ-lattice.

We concluded that a moduler-form increasing order-s.c.i. on $((\mathcal{A}^i)_{i=1}^k)^\sigma$ is a positive measure.

2.4.2 Extension of Daniell

In this part, we suppose $T \subseteq \mathbb{R}^k$ is group (respectively, vectoriel space) of Riesz (See [8]) and τ is the set positive elements of under-group (respectively, under-vectoriel space) of Riesz of T.

Theorem 2.2. *Let $F : \tau \dashrightarrow \mathbb{R}^+$ is increasing, additive continuous order, then,*

1. *F has a only extension to σ-lattice of Riesz generated by τ, and this extension is increasing by additive, order-s.c.i. and order-s.c.s in its domain,*
2. *Under the hypothesis of* (1) *and F is positive and homogenous, then we have* (1) *and the extension is positive and homogenous in its domain.*

Remark 2.5. 1. The conditions $\lim_{n\to+\infty} F((x_n^i)_{i=1}^k) = 0$ when $\wedge_n (x_n^i)_{i=1}^k = 0$ (see [11]) maked in extension of Daniell is obviously equivalent to the order continuity.

2. It's easy to see that F is a modular-form, so if $(x, y) \in \tau^2$, then,

$$(x^i)_{i=1}^k \vee (y^i)_{i=1}^k + (x^i)_{i=1}^k \wedge (y^i)_{i=1}^k = (x^i)_{i=1}^k + (y^i)_{i=1}^k$$

and F is additive.

3. The set $\mathcal{L}(x^i)_{i=1}^k = \{(x^i)_{i=1}^k \in T/\tau^{\uparrow}_{(x^i)_{i=1}^k} \neq \emptyset\}$ is a under-group (respectively, under-space vectoriel) of Riesz T, since $\vee_n^{\uparrow}(x_n^i)_{i=1}^k + \vee_n^{\uparrow}(y_n^i)_{i=1}^k = \vee_n^{\uparrow}((x_n^i)_{i=1}^k + (y_n^i)_{i=1}^k)$ and $\alpha.\vee_n^{\uparrow}(x_n^i)_{i=1}^k = \vee_n^{\uparrow}\alpha.(x_n^i)_{i=1}^k$, for all $\alpha \geq 0$. We can assume that $\tau^{\uparrow}_{(x_n^i)_{i=1}^k} \neq \emptyset$ for all $(x^i)_{i=1}^k \in T$.

4. If the extension exists, it's a modular-form and by show in Theorem 2.1, this extension is exactly $\overline{F}$ on the σ-lattice generated by τ, the extension is stable by additive. In the case of the space-vectorial of the Riesz, it's obviously that the extensions $F^{\uparrow}, F^{\downarrow}, F^*$ and F_* are positively homogeneous, for this, we show the properties of F^* and F_*.

Lemma 2.5. *Let $F^{\uparrow}$, $F^{\downarrow}$, $\tau^{\uparrow}$ and $\tau^{\downarrow}$ as Lemma 2.1. and* (2.1), (2.2) *then, we have*

1. *$F^{\uparrow}$ is additif on $\tau^{\uparrow}$ and $F^{\downarrow}$ is additive on $\tau^{\downarrow}$;*
2. *For all $((x^i)_{i=1}, (y^i)_{i=1}^k) \in (\tau^F)^2$, we have*

$$\tau^{\uparrow}_{(x^i)_{i=1}^k} + \tau^{\uparrow}_{(y^i)_{i=1}^k} \subseteq \tau^{\uparrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}, \quad \tau^{\downarrow}_{(x^i)_{i=1}^k} + \tau^{\downarrow}_{(y^i)_{i=1}^k} \subseteq \tau^{\downarrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}.$$

3. F^* is under-additive and F_* is over-additive.

Corollary 2.2. *τ^F is stable by additive and $\overline{F}$ is additive on τ^F.*

Proof. By the Lemma 2.5, we have for all $((x^i)_{i=1},(y^i)_{i=1}^k)\in(\tau^F)^2$

$$\begin{aligned}F^*((x^i)_{i=1}^k+(y^i)_{i=1}^k) &\le F^*((x^i)_{i=1}^k)+F^*((y^i)_{i=1}^k)=F_*((x^i)_{i=1}^k)+F_*((y^i)_{i=1}^k)\\ &\le F_*((x^i)_{i=1}^k+(y^i)_{i=1}^k)\end{aligned}$$

so, τ^F is stable by additive.

If $((x^i)_{i=1}^k,(y^i)_{i=1}^k)\in(\tau^F)^2$ and $t\in\tau$, we have

$$((x^i)_{i=1}^k+(y^i)_{i=1}^k)\wedge(t^i)_{i=1}^k=((x^i)_{i=1}^k+(t^i)_{i=1}^k)\wedge((y^i)_{i=1}^k+(t^i)_{i=1}^k)\in\tau_{\mathbb{R}}^F.$$

τ_F is stable by additive and by the next $\overline{F}$ is additive on τ_F. □

Proof of Lemma 2.5. 1. Let $(x^i)_{i=1}^k=\vee_n^{\uparrow}(x_n^i)_{i=1}^k$ and $(y^i)_{i=1}^k=\vee_n^{\uparrow}(y_n^i)_{i=1}^k$ with $((x_n^i)_{i=1}^k,(y_n^i)_{i=1}^k)_n\in(\tau^2)^{\mathbb{N}}$; then

$$(x^i)_{i=1}^k+(y^i)_{i=1}^k=\vee_n^{\uparrow}((x_n^i)_{i=1}^k+(y_n^i)_{i=1}^k)$$

by Lemma 2.1 and Proposition 2.2, we have

$$\begin{aligned}F^{\uparrow}((x^i)_{i=1}^k+(y^i)_{i=1}^k) &= \sup_n{}^{\uparrow}F((x_n^i)_{i=1}^k+(y_n^i)_{i=1}^k)\\ &= \sup_n{}^{\uparrow}F((x_n^i)_{i=1}^k)+F((y_n^i)_{i=1}^k)\\ &= \sup_n{}^{\uparrow}F((x_n^i)_{i=1}^k)+\sup_n{}^{\uparrow}F((y_n^i)_{i=1}^k)\\ &= F^{\uparrow}((x^i)_{i=1}^k)+F^{\uparrow}((y^i)_{i=1}^k).\end{aligned}$$

The procedure is same for $F^{\downarrow}$.

2. If $(t^i)_{i=1}^k\in\tau^{\uparrow}_{(x^i)_{i=1}^k}$ and $(s^i)_{i=1}^k\in\tau^{\uparrow}_{(y^i)_{i=1}^k}$ then, $(t^i)_{i=1}^k+(s^i)_{i=1}^k)\in\tau^{\uparrow}$ and

$$((t^i)_{i=1}^k+(s^i)_{i=1}^k)\ge((x^i)_{i=1}^k+(y^i)_{i=1}^k)\quad\text{so}\quad((t^i)_{i=1}^k+(s^i)_{i=1}^k)\in\tau^{\uparrow}_{((x^i)_{i=1}^k+(y^i)_{i=1}^k)}.$$

The procedure is the same for the following statement,

$$\tau^{\downarrow}_{(x^i)_{i=1}^k}+\tau^{\downarrow}_{(y^i)_{i=1}^k}\subseteq\tau^{\uparrow}_{((x^i)_{i=1}^k+(y^i)_{i=1}^k)}.$$

3. By (2), we have

$$\begin{aligned}
F^*(((x^i)_{i=1}^k + (y^i)_{i=1}^k)) &= \inf_{(t^i)_{i=1}^k \in \tau^{\uparrow}_{((x^i)_{i=1}^k + (y^i)_{i=1}^k)}} F^{\uparrow}((t^i)_{i=1}^k) \\
&\leq \inf_{\substack{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}, \\ (v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})}} F^{\uparrow}((u^i)_{i=1}^k + (v^i)_{i=1}^k) \\
&\leq \inf_{\substack{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k}, \\ (v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})}} (F^{\uparrow}((u^i)_{i=1}^k) + F^{\uparrow}((v^i)_{i=1}^k)) \\
&\leq \inf_{((u^i)_{i=1}^k \in \tau^{\uparrow}_{(x^i)_{i=1}^k})} F^{\uparrow}((u^i)_{i=1}^k) + \inf_{((v^i)_{i=1}^k \in \tau^{\uparrow}_{(y^i)_{i=1}^k})} F^{\uparrow}((v^i)_{i=1}^k) \\
&= F^*((x^i)_{i=1}^k + F^*((y^i)_{i=1}^k).
\end{aligned}$$

By Similar proof, we have F_*.

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