Optimal investment of DC pension plan based on a weighted utility

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Abstract. We investigate the DC pension manager's portfolio problem when he bases decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. This setting leads to a non-concave objective utility and therefore a non-concave utility maximization problem. We apply the concavification technique to solve the non-concave optimization problem and obtain the closed-form representations of the optimal wealth process and the optimal strategies. Numerical results show that if the manager pays much more attention to the change of wealth relative to the reference point, then he will take a more conservative investment strategy due to risk aversion over gains.

Keywords: DC pension plan, weighted utility, non-concave utility, concavification.

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1. Introduction

Due to declining mortality and low interest rates, the literature on the asset allocation of pension plans has grown up significantly in recent years. There are mainly two kinds of pension fund: defined benefit (DB) pension plan and defined contribution (DC) pension plan. In a DB pension plan, the benefit is fixed in advance and therefore the plan sponsor bears the financial risks. In a DC plan, the member contributes part of the salary to the pension account and upon retirement, the member can receive regular income from the pension account. DC pension plans are becoming more popular than DB pension plans in many countries. The problem of optimal allocation during the accumulation phase for the DC pension plan has attracted a lot of attention, see Zhang et al. (2018), Wang et al. (2019), Dong and Zheng (2019, 2020).

Most earlier works on DC pension plans care about maximizing the expectation of a smooth utility of terminal wealth, see Dong et al. (2020) and Guan and Liang (2014). Under a smooth utility, people make decisions only based on absolute wealth. Prospect theory, proposed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992), states that people show risk aversion over gains and risk seeking over losses and base decisions on comparisons to a certain pre-defined reference point rather than evaluating absolute level of total wealth itself. These distinguished behaviors can be characterized by an asymmetric S-shaped utility function, convex in the domain of losses and concave in the domain of gains. Recently, more and more literatures care about maximizing the expectation of an S-shaped utility of terminal wealth, see Berkelaar et al. (2004), Guan and Liang (2016), Guo (2014) and Jin et al. (2008). However, none of the above-mentioned papers consider the optimization problem based on both absolute level of total wealth and prospect theory.

As the purpose of a pension plan is to provide an adequate retirement income for a DC member, risk management based on terminal portfolio insurance (PI) constraint has attracted much attention, see Deelstra et al. (2003) and He et al. (2019). The PI constraint can well protect the members' benefits by keeping the portfolio value above a minimum guarantee at retirement. In this paper, we consider a weighted utility of a smooth utility and an S-shaped utility under PI constraint, which can include the impact of both the absolute wealth and relative to the reference point on the investment decision in our modelling.

The main contribution of this paper is that we have solved an optimization problem of a DC plan under a weighted utility and PI constraint, such that the manager can base his decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. The rest of the paper is organized as follows. In Section 2 we formulate a DC investment problem under a weighted utility and PI constraint. In Section 3 we apply the concavification and the martingale method to solve the non-concave optimization problem and derive the explicit optimal solution. In Section 4 we numerically analysize the impact of the weight parameter on the distribution of the optimal terminal wealth. Section 5 concludes. The appendix contains the proofs of Lemma 3.2 and Proposition 3.4.

2. The model

In a continuous-time economy, we investigate the asset allocation problem of a DC plan among a risk-free asset and a risky asset. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space with the filtration $\mathbb{F} := \{\mathcal{F}_t | 0 \leq t \leq T\}$ being the natural filtration generated by a standard Brownian motion $\{W(t)\}_{0\leq t\leq T}$ and satisfying the usual conditions. The DC pension fund starts at time 0 and the retirement time is T.

The prices of the risk-free and risky assets evolve as follows:

$$dS_0(t) = rS_0(t)dt, \; S_0(0) = 1, \tag{2.1}$$

and

$$\begin{aligned} dS_1(t) &= S_1(t)(\mu dt + \sigma dW(t)) \\ &= S_1(t)(rdt + \sigma(dW(t) + \vartheta dt)), \end{aligned} \tag{2.2}$$

where $r > 0$ is a riskless interest rate, $\mu > r$ is the expected return of the risky asset, $\sigma > 0$ is the volatility of the risky asset and $\vartheta = \frac{\mu - r}{\sigma}$ is the market price of risk.

In a DC plan, the pension members contribute continuously to the pension plan before retirement time T. Let $c(t) > 0$ denote the aggregated amount of money contributed at time t of a cohort of fund participants. As explained in Dong and Zheng (2020), individual members' contribution rates may be random over the time, but the accumulated cash income of a large pension fund is in general deterministic and stable. We therefore for simplicity assume $c(t) \equiv c$ is a constant. Assume that there are no transaction costs or taxes in the financial market. The pension account is endowed with an initial endowment $x_0 \geq 0$.

Let $\pi(t)$ be the total amount allocated to the risky asset at time $t \in [0, T]$. Then, the wealth process $\{X^\pi(t)\}_{0\leq t\leq T}$ with the initial value $X^\pi(0) = x_0$ evolves as follows:

$$dX^\pi(t) = rX^\pi(t)dt + \pi(t)\sigma(dW(t) + \vartheta dt) + cdt. \tag{2.3}$$

Let $H(t)$ be the discounted value at time t of total pension contribution from t to T, where

$$H(t) = c\int_t^T e^{-r(s-t)}ds = \frac{c}{r}(1 - e^{-r(T-t)}). \tag{2.4}$$

To guarantee that a DC pension can meet the member's requirement of an elementary security supporting her living after retirement, a downside protection is necessary. Let $L > 0$ be a constant. We next define the set of admissible trading strategies.

Definition 2.1. *A portfolio strategy* $\{\pi = \pi(t) : t \in [0, T]\}$ *is said to be admissible if* $\pi(t)$ *is an* $\mathcal{F}$*-progressively measurable process, with* $\int_0^T \pi^2(t)dt < +\infty$*, a.s., for all* $t \in [0, T]$*, and* $X^\pi(t)$ *satisfies (2.3),* $X^\pi(T) \geq L$*. We denote the set of admissible portfolio strategies by* $\mathcal{A}$*.*

The goal of the pension fund manager is to find the optimal investment strategies within $[0, T]$. Most earlier existing works on DC pension investigate a concave utility maximization problem. Under a smooth utility, people make decisions based on absolute wealth. Kahneman and Tversky (1979) claim that people are risk averse over gains and risk seeking over losses and make decisions relative to some reference levels rather than absolute values directly. They adopt an S-shaped utility to capture these distinguished behaviors. In this paper, we consider a weighted utility of a smooth utility and an S-shaped utility, which can include the impact of both the absolute wealth and the change of wealth relative to some reference point on the investment decision in our modelling.

Firstly, we introduce an S-shaped utility, see Dong and Zheng (2020). Let $\theta > L$ be a reference point, which is chosen in advance. Assume that $U_1 : [L, \infty) \to R$ is an S-shaped utility (see, figure 1), which satisfies
1. U_1 is smooth and increasing on $[L, \infty)$, strictly convex on (L, θ) and strictly concave on $(\theta, +\infty)$.
2. $\lim_{x\to+\infty} U_1(x) = +\infty, \lim_{x\to+\infty} U_1'(x) = 0, \lim_{x\to\theta^-} U_1'(x) = \lim_{x\to\theta^+} U_1'(x) = +\infty$.
3. $\lim_{x\to\theta^-} U_1''(x) = +\infty, \lim_{x\to\theta^+} U_1''(x) = -\infty. \lim_{x\to+\infty} \frac{xU_1'(x)}{U_1(x)} < 1$.
4. $U_1''' > 0, x \in (L, \theta) \cup (\theta, \infty)$.

Note that U_1 is convex when x is below the reference point θ and concave when x is above θ, which demonstrates different behaviors of people in gain and loss situations relative to the reference point, that is, risk averse over gains and risk seeking over losses.

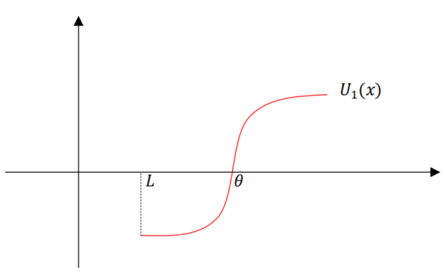

Figure 1: the utility U_1

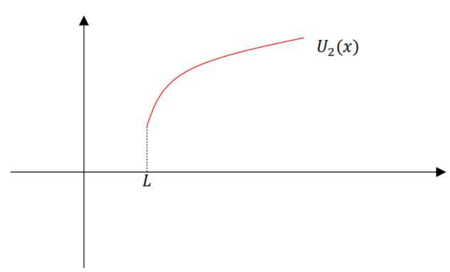

Figure 2: the utility U_2

The utility U_2 is strictly increasing, strictly concave, continuously differentiable, real-valued function defined on $[L, \infty)$ (see, figure 2) satisfying

$$\lim_{x\to+\infty} U_2(x) = +\infty, \lim_{x\to+\infty} U_2'(x) = 0, U_2'''(x) > 0,$$

and

$$\lim_{x\to+\infty}\frac{xU_2'(x)}{U_2(x)}<1.$$

We now consider a weighted utility defined by

$$U(x)=aU_1(x)+bU_2(x),x\geq L, \tag{2.5}$$

where $0\leq a,b\leq 1$ with $a+b=1$ reflect the aversion degrees of the absolute wealth and the change of wealth relative to the reference point, respectively. A larger a implies that the manager pays much more attention to the change of wealth relative to the reference point. The case $a=0$ degenerates to the optimization problem for a concave utility, while the case $a=1$ corresponds to the optimization problem for an S-shaped utility.

It is easy to conclude that the weighted utility U is strictly concave on $(\theta,+\infty)$ and the strictly decreasing function U' defined on (θ,∞) has a strictly decreasing inverse $I:(0,\infty)\to(\theta,\infty)$, that is,

$$U'(I(y))=y,\forall y>0,\ I(U'(x))=x,\forall x>\theta.$$

We consider the following optimization problem with minimum guarantee:

$$\begin{cases}\max_{\pi\in\mathcal{A}}E[U(X^\pi(T))],\\ s.t.\ X^\pi(t)\text{ satisfies (2.3) with }X^\pi(T)\geq L,a.s..\end{cases} \tag{2.6}$$

Since the market is complete, there exists the unique pricing kernel defined by

$$\xi(t)=e^{-(r+\frac{\vartheta^2}{2})t-\vartheta W(t)},\xi(0)=1. \tag{2.7}$$

Here $\xi(t)$ is the Arrow-Debreu value per probability unit of a security which pays out 1 at time t if the scenario ω happens, and 0 else. As this value is high in a recession and low in prosperous times, $\xi(t)$ directly reflects the overall state of the economy.

Following Cox and Huang (1989) and Guan and Liang (2016), we can find the optimal solution for (2.6) via the martingale method. The problem (2.6) can be rewritten as follows:

$$\begin{cases}\max_{X^\pi(T)}E[U(X^\pi(T))],\\ s.t.\ E[\xi(T)X^\pi(T)]\leq x_0+H(0),X^\pi(T)\geq L,a.s..\end{cases} \tag{2.8}$$

3. Optimal trading strategy under PI constraint

In this section, we will solve the optimization problem (2.6) and obtain the optimal wealth and optimal investment strategies.

Recall that $\lim_{x\to\theta^-}U''(x)=+\infty$ and U'' is continuous, increasing on (L,θ) and is strictly concave on (θ,∞). We can easily conclude that

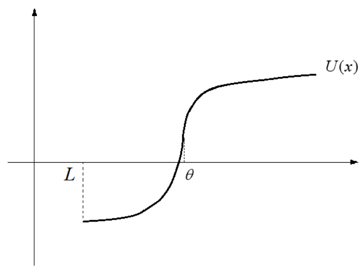

Figure 3: $U''(L+) \geq 0$

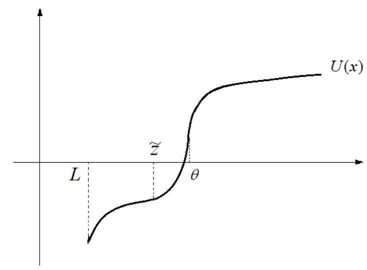

Figure 4: $U''(L+) < 0$

Case A. for $U''(L+) \geq 0$, U is strictly convex on (L, θ) and U is strictly concave on (θ, ∞) (see, figure 3).
Case B. for $U''(L+) < 0$, there exists $L < \tilde{z} < \theta$ (see, figure 4) such that

$$U''(\tilde{z}) = 0. \tag{3.1}$$

Note that, in Case B, U is strictly concave on $(L, \tilde{z}) \cup (\theta, \infty)$ and strictly convex on $(\tilde{z}, \theta)$. Furthermore, U' defined on $(L, \tilde{z})$ has a strictly decreasing inverse $\overline{I} : (U'(\tilde{z}), U'(L+)) \to (L, \tilde{z})$, that is,

$$U'(\overline{I}(y)) = y, \forall\ U'(\tilde{z}) < y < U'(L+),\ \overline{I}(U'(x)) = x, \forall\ L < x < \tilde{z}.$$

Lemma 3.1. *For $\theta > L$, there exists a unique tangent point $z > \theta$ of the straight line starting at $(L, U(L))$ to the curve $U(x), x > \theta$ satisfying the following equation*

$$U(x) - (x - L)U'(x) - U(L) = 0. \tag{3.2}$$

Proof. Denote by $f(x) = U(x) - (x - L)U'(x) - U(L)$. It is easy to check that $f(x)$ is a decreasing, continuous function in (θ, ∞) and $\lim_{x \to \theta} f(x) < 0$, $\lim_{x \to \infty} f(x) = \infty$, which yields the result. □

Lemma 3.2. *Let $\tilde{z}$ and z be given by (3.1) and (3.2), respectively. For $U'(L+) > U'(z)$, there exist $L < z_1 < \tilde{z}, z_2 > z$ satisfying*

$$\frac{U(z_2) - U(z_1)}{z_2 - z_1} = U'(z_1) = U'(z_2). \tag{3.3}$$

and

$$U(x) < U'(z_2)(x - z_2) + U(z_2) = U'(z_1)(x - z_1) + U(z_1),\ x \in [z_1, z_2]. \tag{3.4}$$

Proof. See Appendix. □

Following Carpenter(2000), we shall use the concavification technique to derive the optimal terminal wealth. We first investigate the following problem for any $y > 0$:

$$\max_{x \geq L} \{U(x) - xy\}. \tag{3.5}$$

Proposition 3.3. *Let $\tilde{z}$ and z be given by (3.1) and (3.2), respectively. For $y > 0$,*

Case A. If $U'(L+) \leq U'(z)$, then the concave envelope of U is given by

$$U^c(x) = \begin{cases} U(z) + U'(z)(x - z), & L \leq x < z, \\ U(x), & x \geq z. \end{cases} \tag{3.6}$$

Case B. If $U'(L+) > U'(z)$, then the concave envelope of U is given by

$$U^c(x) = \begin{cases} U(x), & L \leq x < z_1, \\ U(z_1) + U'(z_1)(x - z_1), & z_1 \leq x < z_2, \\ U(x), & x \geq z_2, \end{cases} \tag{3.7}$$

where $L < z_1 < \tilde{z}$ and $z_2 > z$ are determined by (3.3).

Proof. The concave envelope of U can be directly obtained from Lemma 3.2 and Lemma A.1 of Dong and Zheng (2020). □

Proposition 3.4. *Let z be given by (3.2). Assume $x_0 \geq Le^{-rT} - H(0)$. Then for the optimization problem (2.6), the optimal terminal wealth is as follows:*

For $U'(L+) \leq U'(z)$,

$$X^{\pi^*,\beta^*}(T) \quad = \quad \begin{cases} I(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U'(z)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U'(z)}{\beta^*}. \end{cases} \tag{3.8}$$

For $U'(L+) > U'(z)$,

$$X^{\pi^*,\beta^*}(T) \quad = \quad \begin{cases} I(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U'(z_1)}{\beta^*}, \\ \overline{I}(\beta^*\xi(T)), & \frac{U'(z_1)}{\beta^*} \leq \xi(T) < \frac{U'(L+)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U'(L+)}{\beta^*}. \end{cases} \tag{3.9}$$

Here the multiplier $\beta^* > 0$ is determined by

$$E[\xi(T)X^{\pi^*,\beta^*}(T)] = x_0 + H(0). \tag{3.10}$$

Proof. See Appendix. □

Remark 3.5. Note that if $a = 0$, then the optimal terminal wealth under a concave utility U_2 is given by

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I_2(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U_2'(L+)}{\beta^*}, \\ L, & \xi(T) \geq \frac{U_2'(L+)}{\beta^*}, \end{cases} \tag{3.11}$$

where I_2 is the inverse function of U_2.

If $a = 1$, then the optimal terminal wealth under an S-shaped utility U_1 is given by

$$X^{\pi^*,\beta^*}(T) = \begin{cases} I_1(\beta^*\xi(T)), & 0 < \xi(T) < \frac{U_1'(\hat{z})}{\beta^*}, \\ L, & \xi(T) \geq \frac{U_1'(\hat{z})}{\beta^*}, \end{cases} \tag{3.12}$$

where I_1 is the inverse function of U_1, $\hat{z}$ is the unique solution to the following equation

$$U_1(x) - (x - L)U'(x) - U(L) = 0.$$

The optimal terminal wealth $X^{\pi^*,\beta^*}(T)$ can be expressed as a function of the state price density at maturity $\xi(T)$ and the multiplier β^*. $X^{\pi^*,\beta^*}(T)$ may take a two- or three-region form according to the values of $U'(L+)$ and $U'(z)$. In all cases, the optimal terminal wealth ends up with L when the state price increases above the boundary point of the bad-states region. The PI constraint well protects the members's benefits by keeping the optimal terminal wealth always above the level L. Due to the budget constraint, we can conclude that the protection in bad economic states is at the expense of the optimal terminal wealth in good economic states.

After obtaining the optimal terminal wealth, we can derive the closed-form expression for the optimal investment strategy given in the following result.

Proposition 3.6. *Assume that $x_0 \geq Le^{-rT} - H(0)$. Then for the optimization problem (2.6), the optimal wealth process and the optimal strategy are given as follows:*

For $U'(L+) \leq U'(z)$, the optimal wealth process at time t is given by

$$X^{\pi^*,\beta^*}(t) = A_1(\xi(t), t) - H(t), \tag{3.13}$$

where

$$A_1(\xi(t), t) = F_1(t, \xi(t), d_{1,t}(U'(z))) + Le^{-r(T-t)}\Phi(d_{2,t}(U'(z))),$$

with

$$\begin{aligned} F_1(t, \xi(t), x) &= \int_x^\infty I(\beta^*\xi(t)\chi(t,u))\chi(t,u)\varphi(u)du, \\ d_{1,t}(x) &= \frac{-\ln(\frac{x}{\xi(t)\beta^*}) - (r + \frac{\vartheta^2}{2})(T-t)}{\vartheta\sqrt{T-t}}, \; d_{2,t}(x) = d_{1,t}(x) + \vartheta\sqrt{T-t}, \\ \chi(t,x) &= e^{-(r+\frac{\vartheta^2}{2})(T-t)-\vartheta\sqrt{T-t}x}, \end{aligned}$$

the multiplier $\beta^* > 0$ determined by (3.10), Φ and φ the standard normal cumulative distribution function and the standard normal density function, respectively.

The amount of wealth invested in the risky asset is given by:

$$\pi^{*,\beta^*}(t) = -\frac{\vartheta}{\sigma}(\overline{F}_1(t,\xi(t),d_{1,t}(U'(z))) + (L-z)\overline{\varphi}(U'(z))), \tag{3.14}$$

where

$$\overline{F}_1(t,\xi(t),x) = \int_x^\infty I'(\beta^*\xi(t)\chi(t,u))\beta^*\chi^2(t,u)\varphi(u)du,$$

and $\overline{\varphi}(u) = \frac{u\varphi(d_{1,t}(u))}{\vartheta\sqrt{T-t}\beta^*\xi(t)}$.

For $U'(L+) > U'(z)$, the optimal wealth process at time t is given by

$$X^{\pi^*,\beta^*}(t) = A_2(\xi(t),t) - H(t), \tag{3.15}$$

where

$$\begin{aligned} A_2(\xi(t),t) &= F_1(t,\xi(t),d_{1,t}(U'(z_1))) + F_2(t,\xi(t),d_{1,t}(U'(L+))) \\ &\quad -F_2(t,\xi(t),d_{1,t}(U'(z_1))) + Le^{-r(T-t)}\Phi(d_{2,t}(U'(L+))), \end{aligned}$$

with

$$F_2(t,\xi(t),x) = \int_x^\infty \overline{I}(\beta^*\xi(t)\chi(t,u))\chi(t,u)\varphi(u)du.$$

The amount of wealth invested in the risky asset is given by:

$$\begin{aligned} \pi^{*,\beta^*}(t) = -\frac{\vartheta}{\sigma}(&\overline{F}_1(t,\xi(t),d_{1,t}(U'(z_1))) + \overline{F}_2(t,\xi(t),d_{1,t}(U'(L+))) \\ &-\overline{F}_2(t,\xi(t),d_{1,t}(U'(z_1))) + (\overline{I}(U'(z_1)) - z_2)\overline{\varphi}(U'(z_1))), \end{aligned} \tag{3.16}$$

where

$$\overline{F}_2(t,\xi(t),x) = \int_x^\infty \overline{I}'(\beta^*\xi(t)\chi(t,u))\beta^*\chi^2(t,u)\varphi(u)du.$$

Proof. The price of $X^{\pi^*,\beta^*}(T)$ at t is calculated as follows:

$$X^{\pi^*,\beta^*}(t) = \frac{1}{\xi(t)}E[\xi(T)X^{\pi^*,\beta^*}(T)|\mathcal{F}_t] - H(t). \tag{3.17}$$

Note that $\log(\frac{\xi(T)}{\xi(t)})$ is a normal distribution with a mean $-(r+\frac{\vartheta^2}{2})(T-t)$ and a variance $\vartheta^2(T-t)$. By substituting the expressions for $X^{\pi^*,\beta^*}(T)$ given in Proposition 3.4 into (3.17), we can easily obtain the formulas for $X^{\pi^*,\beta^*}(t)$ by some straightforward calculations.

Let $X^{\pi^*,\beta^*}(t) = A_i(\xi(t), t)$. Then

$$dX^{\pi^*,\beta^*}(t) = \hbar_i(t, \xi(t))dt - \frac{\partial A_i}{\partial \xi}\xi(t)\vartheta dW(t),$$

for some $\hbar_i(t, \xi(t))$, and we are only interested in the diffusion part. Comparing it with (2.3), we have

$$\pi^{*,\beta^*}(t) = -\frac{\vartheta}{\sigma}\frac{\partial A_i}{\partial \xi}\xi(t).$$

Some simple calculations yield the results. □

4. Numerical analysis

In this section, we carry out some numerical calculations to investigate the impact of the weight parameter and some other model parameters on the optimal terminal wealth. Assume that $U_1(x)$ is given by:

$$U_1(x) = \begin{cases} -B(\theta - x)^{\gamma_1}, & L \le x < \theta, \\ (x - \theta)^{\gamma_2}, & x \ge \theta, \end{cases} \tag{5.1}$$

where $\theta > 0$ is a reference point, $B > 1$ is called loss aversion degree, $0 < \gamma_1 < 1, 0 < \gamma_2 < 1$ are constants, which characterizes the degree of loss aversion and risk aversion, respectively. The concave utility $U_2(x)$ is as follows:

$$U_2(x) = x^p, x \ge L, 0 < p < 1. \tag{5.2}$$

For all numerical computations, the benchmark data used are fixed: $r = 0.03, \mu = 0.07, \sigma = 0.3, c = 1, B = 2.25, \gamma_1 = 0.2, \gamma_2 = 0.3, a = 0.5, L = 60, \theta = 100, x_0 = 20, T = 40$.

Table 1 lists expectations, standard deviations and the probabilities for different a. We can see from it that the expectation, the standard deviation and the probability $P(X^{\pi^*,\beta^*}(T) = L)$ all decrease in a. Note that the reference point can be easily obtained by investing a large proportion of wealth in the cash bond. Due to loss aversion, the manager will take a more conservative allocation strategy when he pays much more attention to the change of wealth relative to the reference point.

Table 1: expectations, standard deviations, and probabilities

a	0	0.3	0.5	0.7	1
mean	409.85	342.37	299.91	266.64	225.92
std dev	999.95	670.60	515.35	385.54	228.25
$P(X^{\pi^*,\beta^*}(T) = L)$	0.2574	0.0852	0.0384	0.0179	0.0058

Table 2 lists expectations, standard deviations and the probabilities for different (θ, L). It is observed that a higher value of L leads to a lower mean, standard deviation and $P(X^{\pi^*,\beta^*}(T)=L)$, since the manager shall allocate less money in the risky asset to attain a higher protection level. We can also see that the mean, the standard deviation and $P(X^{\pi^*,\beta^*}(T)=L)$ also decreases in θ. As pointed out above, the reference point can be obtained by investing much money in the riskless asset for a relatively low value of θ. To attain a higher reference point, the manager will allocate more money in the riskless asset.

Table 2: expectations, standard deviations, and probabilities

(L,θ)	$(60,80)$	$(60,100)$	$(60,120)$	$(40,100)$	$(80,100)$
means	319.87	299.91	292.15	321.53	278.27
std dev	611.11	515.35	453.85	577.47	467.11
$P(X^{\pi^*,\beta^*}(T)=L)$	0.0071	0.0384	0.0860	0.0578	0.0125

Figure 5 plots the relationship between the expectation and the parameter p for $a=0$ and $a=0.5$. We can observe that the expectation increases with p for a fixed a. This is due to the fact that the effect of higher risk aversion, measured by a lower p,leads to less investment in a risky asset, which brings a lower expectation.

Figure 6 plots the relationship between the standard deviation and the parameter p for $a=0$ and $a=0.5$. It is seen that the standard deviation increases with p for a fixed a. As explained in Figure 5, a lower p leads to a less risky portfolio, which corresponds to a lower standard deviation.

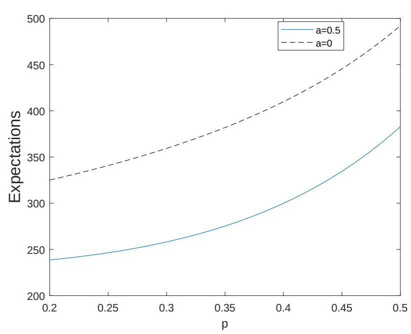

Figure 5: relationship between expectation and p

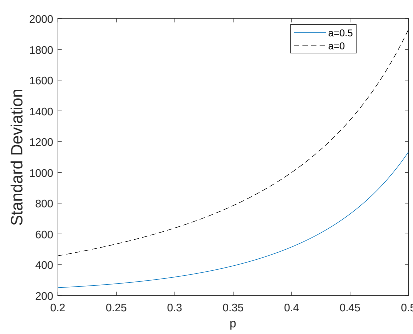

Figure 6: relationship between standard deviation and p

5. Conclusions

In this paper, we investigate the optimal portfolio selection problem of a DC pension manager when he bases decisions on both absolute level of total wealth and comparisons to a certain pre-defined reference point. By using a martingale method and a concavification technique, we derive the optimal wealth process and the optimal investment strategies. The form for the optimal terminal wealth can take a two- or three-region form. In all cases, the optimal terminal wealth ends up with the protection level from a certain value of the price density. Numerical results show that the impact of the weight parameters which reflect the aversion degrees of the absolute wealth and the change of wealth relative to the reference point is significant. If the manager pays much more attention to the change of wealth relative to the reference point, then he will take a more conservative allocation strategy due to risk aversion over gains.

The present work might be extended in at least two directions. Firstly, we can extend a PI-constrained optimization problem to a VaR- or mutiple-VaR constrained one. Secondly, we can consider a performance fee arrangement in DC pension plan management under a weighted utility. We will leave these questions for future research.

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6. Appendix

The proof of Lemma 3.2

Proof. First, we prove the existence of z_1, z_2. Note that U is strictly concave on $(L,\tilde{z})\cup(\theta,\infty)$ and is strictly convex on $(\tilde{z},\theta)$ for the case $U'(L+) > U'(z)$. It is obvious that $U'(z) > U'(\tilde{z})$. Then there must exist unique $L < \eta_1 \le \tilde{z}$ and $\eta_2 \ge \tilde{z}$, Such that

$$U'(\eta_1) = U'(z), \qquad U'(\eta_2) = U'(\tilde{z}). \tag{6.1}$$

For any $y \in [U'(\tilde{z}), U'(z)]$£¬ we define $L < c_1(y) \le \tilde{z}, c_2(y) \ge z$ by the relation

$$U'(c_1(y)) = U'(c_2(y)) = y. \tag{6.2}$$

It is obvious that

$$\lim_{y\to U'(z)} c_1(y) = \eta_1,\ \lim_{y\to U'(z)} c_2(y) = z,\ \lim_{y\to U'(\tilde{z})} c_1(y) = \tilde{z},\ \lim_{y\to U'(\tilde{z})} c_2(y) = \eta_2. \tag{6.3}$$

Furthermore, $c_1(y)$ and $c_2(y)$ are strictly decreasing and continuous on $[U'(\tilde{z}), U'(z)]$. Define $\kappa(y) = \frac{U(c_2(y))-U(c_1(y))}{c_2(y)-c_1(y)}$. It is easy to check that $\kappa(y)$ is continuous on $[U'(\tilde{z}), U'(z)]$. It follows from concave/convex properties that

$$U(\eta_2) - U(\theta) > U'(\eta_2)(\eta_2-\theta),\ U(\theta) - U(\tilde{z}) > U'(\tilde{z})(\theta-\tilde{z}).$$

Hence, we have

$$\kappa(U'(\tilde{z})) = \frac{U(\eta_2)-U(\tilde{z})}{\eta_2-\tilde{z}} > U'(\tilde{z}).$$

Furthermore,

$$U(z) - U(L) = U'(z)(z - L),\ U(\eta_1) - U(L) > U'(\eta_1)(\eta_1 - L).$$

Therefore, we can obtain that

$$\kappa(U'(z)) = \frac{U(z) - U(L)}{z - L} < U'(z).$$

Then, there must exist $U'(\tilde{z}) < y_0 < U'(z)$ such that $\kappa(y_0) = y_0$. Define $L < z_1 < \tilde{z}, z_2 > z$ by the relation $U'(z_1) = U'(z_2) = y_0$, which concludes the existence of z_1, z_2. Additionally, we can conclude that the linear interpolation between $U(z_1)$ and $U(z_2)$ has to be higher than $U(.)$ on (z_1, z_2), which yields (3.4).

Now, we turn to the uniqueness of z_1, z_2. Assume there exists another pair $(\widehat{z}_1, \widehat{z}_2)$ with $L < \widehat{z}_1 < \tilde{z}, \widehat{z}_2 > z$ satisfying

$$U(\widehat{z}_2) - U(\widehat{z}_1) = U'(\widehat{z}_1)(\widehat{z}_2 - \widehat{z}_1) = U'(\widehat{z}_2)(\widehat{z}_2 - \widehat{z}_1),$$

and

$$U(x) < U(\widehat{z}_1) + U'(\widehat{z}_1)(x - \widehat{z}_1), x \in (\widehat{z}_1, \widehat{z}_2).$$

Without loss of generality, we assume that $\widehat{z}_1 > z_1$. From (3.4), we have $U(\widehat{z}_1) < U(z_2) + U'(z_2)(\widehat{z}_1 - z_2)$. So, for $z_2 \in (\widehat{z}_1, \widehat{z}_2)$,

$$U(z_2) > U(\widehat{z}_1) + U'(z_2)(z_2 - \widehat{z}_1) > U(\widehat{z}_1) + U'(\widehat{z}_1)(z_2 - \widehat{z}_1),$$

which leads to a contradiction. □

The proof of Proposition 3.2

Proof. We use the Lagrange dual theory as well as the concavification technique to solve problem (2.8). Define the Lagrangian of problem (2.8) as follows:

$$\mathcal{L}(X(T), \beta) = E[U(X(T)) - \beta\xi(T)X(T)]. \tag{6.4}$$

So, the original problem (2.8) is equivalent to the following problem:

$$\begin{cases} \inf_{\beta>0} \max_{X(T)} \mathcal{L}(X(T), \beta), \\ s.t.\ X(T) \geq L, a.s.. \end{cases}$$

To solve the above problem, we first use the concavification technique to find the optimal terminal wealth $X^{\pi^*,\beta}(T)$ for a fixed multiplier $\beta > 0$, which solves the following problem

$$\begin{cases} \max_{X(T)} \mathcal{L}(X(T), \beta), \\ s.t.\ X(T) \geq L, a.s.. \end{cases} \tag{6.5}$$

To solve (6.5), for each $y > 0$, we shall find the maximizer $x^*(y)$ solving

$$\max_{x \geq L}\{U(x) - xy\}.$$

For $U'(L+) \leq U'(z)$, it is easy to obtain the superdifferential of $U^c(x)$ given by (3.6) as follows:

$$(U^c)'(x) = \begin{cases} [U'(z), \infty), & x = L, \\ \{U'(z)\}, & L < x < z, \\ \{U'(x)\}, & x > L. \end{cases} \tag{6.6}$$

Then we can find the point $x^*(y) \in \{x | U(x) = U^c(x)\}$ solving both $\max_{x \geq L}\{U(x) - xy\}$ and $\max_{x \geq L}\{U^c(x) - xy\}$ for which 0 is in the superdifferential of $U^c(x) - xy$ given as follows:

$$x^*(y) = \begin{cases} I(y), & 0 < y < U'(z), \\ L, & y \geq U'(z). \end{cases} \tag{6.7}$$

For $U'(L+) > U'(z)$, similar to deriving (6.7), we can obtain that the maximizer is given by

$$x^*(y) = \begin{cases} I(y), & 0 < y < U'(z_1), \\ \overline{I}(y), & U'(z_1) \leq y < U'(L+), \\ L, & y \geq U'(L+). \end{cases} \tag{6.8}$$

Therefore, for each β, an optimal solution to (6.5) is

$$X^{\pi^*, \beta}(T) = x^*(\beta \xi(T)),$$

with $x^*(y)$ given by (6.7) and (6.8).

It remains to find the multiplier $\beta^* > 0$ to the following problem

$$\inf_{\beta > 0} \mathcal{L}(X^{\pi^*, \beta}(T), \beta) \tag{6.9}$$

via the complementary slackness condition (3.10).

It is easy to see that $V(\beta) = E[\xi(T) X^{\pi^*, \beta}(T)]$ is continuous and strictly decreasing in β. Furthermore, $\lim_{\beta \to 0^+} V(\beta) = +\infty$ and $\lim_{\beta \to \infty} V(\beta) = Le^{-rT}$. Thus, there exists a unique β^* satisfying (3.10), which implies that $X^{\pi^*, \beta^*}(T)$ is feasible to (2.8). Following the same arguments as used in Dong et al. (2020), we can check $X^{\pi^*, \beta^*}(T)$ is indeed the optimal terminal wealth to the problem (2.8). □

References

[1] A. B. Berkelaar, R. Kouwenberg, T. Post, *Optimal portfolio choice under loss aversion*, Rev. Econ. Stat., 86 (2004), 973-987.

[2] J. N. Carpenter, *Does option compensation increase managerial risk appetite*? J. Financ., 55 (2000), 2311-2331.

[3] J. Cox, C. F. Huang, *Optimun consumption and portfolio policies when asset prices follow a diffusion process*, J. Econom. Theor., 49 (1989), 33-83.

[4] G. Deelstra, M. Grasselli, P.F. Koehl, *Optimal investment strategies in the presence of a minimum guarantee*, Insurance Mathematics and Economics, 33 (2003), 189-207.

[5] Y. H. Dong, W. X. Lv, S. Y. Wei, Y. Y. Gong, *Optimal investment of DC pension plan under incentive schemes and loss aversion*, Math. Probl. Eng., Article ID 5145848(2020), 14 pages.

[6] Y. H. Dong, H. Zheng, *Optimal investment of DC pension plan under short-selling constraints and portfolio insurance.* Insur. Math. Econom., 85 (2019), 47-59.

[7] Y. H. Dong, H. Zheng, *Optimal investment with S-shaped utility and trading and value at risk constraints: an application to defined contribution pension plan*, Eur. J. Oper. Res., 281 (2020), 341-356.

[8] G.H. Guan, Z. X. Liang, *Optimal management of dc pension plan in a stochastic interest rate and stochastic volatility framework*, Insur. Math. Econom., 57 (2014), 58-66.

[9] G. H. Guan, Z. X. Liang, *Optimal management of DC pension plan under loss aversion and value-at-risk constraints*, Insur. Math. Econom., 69 (2016), 224-237.

[10] W. J. Guo, *Optimal portfolio choice for an insurer with loss aversion*, Insur. Math. Econom., 58 (2014), 217-222.

[11] L. He, Z. X. Liang, Y. Liu, M. Ma, *Optimal control of DC pension plan management under two incentive schemes*, N. Am. Actuar. J., 23 (2019), 120-141.

[12] H. Jin, Z. Xu, X.Y. Zhou, *A convex stochastic optimization problem arising from portfolio selection*, Math. Financ., 18 (2008), 171-183.

[13] D. Kahneman, A. Tversky, *Prospect theory: an analysis of decision under risk*, Econometrica, 47 (1979), 263-292.

[14] A. Tversky, D. Kahneman, *Advances in prospect theory: cumulative representation of uncertainty*, Journal of Risk and Uncertainty, 5 (1992), 297-323.

[15] S. X. Wang, X. M. Rong, H. Zhao, *Optimal investment and benefit payment strategy under loss aversion for target benefit pension plans*, Applied Mathematics and Computation, 346 (2019), 205-218.

[16] L. Zhang, H. Zhang, H. X. Yao, *Optimal investment management for a defined contribution pension fund under imperfect information*, Insurance Mathematics and Economics, 79 (2018), 210-224.

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