Lower bound for the index of convergence of the singular integral in a multidimensional analogue of Tarry's problem

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Abstract. In this paper, a general lower bound is obtained for the index of convergence of the singular integral in a multidimensional analogue of Tarry's problem. In all cases when the exact estimate is known, our estimate coincides with it.
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1. Introduction

Let $r \geq 1$ be a natural number, and

$$(1) \qquad A = \{(t_1, ..., t_r) \mid t_1 \geq 0, ..., t_r \geq 0\};\ (0, ..., 0) \notin A$$

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be a finite set of integer vectors $(t_1, ..., t_r)$. Consider the integral

$$\theta_k = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left| \int_0^1 ... \int_0^1 exp\{2\pi i F(x_1, ..., x_r)\} dx_1 ... dx_r \right|^{2k} d\alpha, \tag{2}$$

where $F(x_1, ..., x_r) = \sum_{(t_1,...,t_r)\in A} \alpha(t_1, ..., t_r)\, x_1^{t_1} ... x_r^{t_r}$. The integral (2) with r = 1 is present as a factor in the formula for the number of solutions of the system of diophantine equations obtained in 1938 by Hua Loo-Keng using the Vinogradov method of trigonometric sums [1]:

$$\begin{aligned} x_1 + ... + x_k &= y_1 + ... + y_k, \\ x_1^2 + ... + x_k^2 &= y_1^2 + ... + y_k^2, \\ &\cdots\cdots\cdots\cdots \\ x_1^n + ... + x_k^n &= y_1^n + ... + y_k^n, \end{aligned} \tag{3}$$

where $x_1, ..., x_k, y_1, ..., y_k$ take integer values from 1 to some sufficiently large value P. For $r > 1$, a similar formula was obtained in the works of G.I.Arkhipov, V.N.Chubarikov and A.A. Karatsuba [2].

The number γ is called the index of convergence of the integral (2) if, for any positive number ϵ, the integral (2) converges for $2k > \gamma + \epsilon$, and diverges for $2k < \gamma + \epsilon$. For $r = 1$, the Hua Loo-Keng problem of the index of convergence of the integral (2) was completely solved by G.I. Arkhipov, V.N. Chubarikov and A.A. Karatsuba [2, 3, 4, 5, 6]. In the multidimensional case, for the sets

(4) $A = A(n_1, ..., n_r) = \{(t_1, ..., t_r) \,|\, 0 \le t_1 \le n_1, ..., 0 \le t_r \le n_r\}, (0, ..., 0) \notin A,$

(5) $A = A(n) = \{(t_1, ..., t_r) \mid 1 \le t_1 + ... + t_r \le n\},$

the same authors obtained upper estimates for the convergence index of the singular integral (2), for the sets (4) and (5) they are equal, respectively, to $\gamma = \gamma_A \le n \cdot m$ and $\gamma = \gamma_A \le r\binom{n+r}{r+1} + r$. Here $n = \max(n_1, ..., n_r)$, $m = (n_1 + 1)...(n_r + 1) - 1$. I.A. Ikromov [7] obtained lower estimates

$$\gamma \ge \frac{1}{2} n \cdot m + 1; \quad \gamma \ge \binom{n+r}{r+1} + 1, \tag{6}$$

for the sets (4) and (5), respectively. Also in [7], in one particular case of the set A having the form

$$A = A(n) = \{(t_1, ..., t_r) \mid \left[\frac{n}{2}\right] + 1 \le t_1 \le n,\ 0 \le t_2 \le t_1, ..., 0 \le t_r \le t_1\},$$

the exact value of the convergence index was obtained:

$$\gamma = \gamma_A = \sum_{r=\left[\frac{n}{2}\right]+1}^{n} k(k+1)^{r-1}.$$

More exact lower bounds than (6) were obtained in [8], where exact bounds were also obtained in one particular case and the theorem was proved:

Theorem 1. *Let the integral* (2) *converge, and let the vector* $(0,...,0,n_s,0,...)$, *where* $n_s=\max\{t_s\,|\,(t_1,...,t_r)\,|\in A\}, s=1,...,r,$ *belong to the set* A. *Then*

$$\gamma \geq \sum_{(t_1,...,t_r)\in A} t_s.$$

The aim of the article is to remove the restriction that the vector $(0,...,0,n_s,0,...,0)$ belongs to the set A, which significantly expands the possibility of applying Theorem 1. Namely, we prove the theorem:

Theorem 2. *Let* $r\geq 2$ *be a natural number, let* $A=\{(t_1,...,t_r)\,|\,t_1\geq 0,...,t_r\geq 0\}\}, (0,...,0)\notin A$ *be some finite set of integer vectors. If the integral (2) converges, then*

$$\gamma \geq \max_{1\leq s\leq r} \sum_{(t_1,...,t_r)\in A} t_s.$$

2. Proof of Theorem 2

To prove Theorem 2, we need a lemma from [8, 9]:

Lemma 1. *Let* $f(x)=\alpha_n x^n+...+\alpha_1 x$ *be a polynomial of degree* n *with real coefficients. Then for* $k=1$ *the following representation holds*

$$I=\int_0^1 exp\{2\pi i\,(\alpha_n x^n+...+\alpha_1 x)\}dx$$
$$=\frac{c_n}{|\alpha_n|^{\frac{1}{n}}}+O\left(\frac{|\alpha_1|}{|\alpha_n|^{\frac{2}{n}}}+...+\frac{|\alpha_{n-2}|}{|\alpha_n|^{\frac{n-1}{n}}}+\frac{|\alpha_{n-1}|}{|\alpha_n|}\ln|\alpha_n|\right),$$
$$c_n=\int_0^{+\infty} exp\{2\,\pi\, i\, x^n sign(\alpha_n)\}dx\neq 0.$$

For definiteness, let us assume that $\max_{1\leq s\leq r} n_s=n_r=n$ and that the vector $(t_1^0,t_2^0,...,n)\in A$. We represent the integral $I=\int_0^1...\int_0^1\exp(2\,\pi\,i\,F(x_1,...,x_r))dx$ as a sum J_1+J_2, where

$$J_1=\int_0^{\varepsilon}\int_0^1...\int_0^1 \exp(2\pi i\,F(x_1,...,x_r))dx,$$
$$J_2=\int_{\varepsilon}^1\int_0^1...\int_0^1 \exp(2\pi i\,F(x_1,...,x_r))dx.$$

Then, $\theta_k(J_2)=\theta_k(I-J_1)\leq 2^{2k}\,\theta_k(I)+2^{2k}\,\theta_k(J_1)$. In the second integral we make the replacement $x_1=\varepsilon\, y_1$ and then put $\beta(t_1,...,t_r)=\varepsilon^{t_1}\,\alpha(t_1,...,t_r)$; $(t_1,...,t_r)\in A$. Then, we get $\theta_k(J_1)=\varepsilon^{2k-M}\theta_k(I)$, where $M=\sum_{(t_1,...,t_r)\in A} t_1$.

This implies $\theta_k(J_2) \leq 2^{2k}\,\varepsilon^{2k-M}\,\theta_k(I) + 2^{2k}\,\theta_k(I) \leq 2^{2k}\,(1+\varepsilon^{2k-M})\,\theta_k(I)$. That is, for convergence of $\theta_k(I)$, the convergence of the integral $\theta_k(J_2)$ is necessary. Similarly, variables $x_2, ..., x_{r-1}$ can be separated from zero. Let $J = \int_0^1 ... \int_0^1 \int_0^\varepsilon \exp(2\,\pi\,i\,F(x_1, ..., x_r))dx$. Then, for convergence of $\theta_k(I)$, the convergence of the integral $\theta_k(J)$ is necessary. We write the polynomial F in a different form, explicitly highlighting the summation over one variable: $F(x_1, ..., x_r) = \sum_{k=0}^n \beta_k x^k$, where $x = x_r$; $\beta_k = \sum_{(t_1,...,t_{r-1},k)\in A} \alpha(t_1, ..., t_{r-1}, k) x_1^{t_1} ... x_{r-1}^{t_{r-1}}$. Then

$$\theta_k(J) = \int_{R^N} \left| \int_\varepsilon^1 ... \int_\varepsilon^1 \exp\{2\,\pi i \beta_0\} dx_1 ... dx_{r-1} \int_0^1 \exp\{2\,\pi i f(x)\} dx \right|^{2k} d\alpha,$$

where $f(x) = \beta_n x^n + ... + \beta_1 x$. Define the set $\Omega \subset R^N$, where $N = \sum_{(t_1,...,t_r)\in A} 1$ is the number of elements of the set A, as follows:

$$\Omega = \{\alpha_n(t_1, ..., t_r), \quad (t_1, ..., t_r) \in A \,|\, \alpha_n = \alpha(t_1^0, ..., t_{r-1}^0, n) \geq \frac{2}{\varepsilon^{t_1^0 + ... + t_{r-1}^0}},$$

$$|\alpha(t_1, ..., t_{r-1}, n)| \leq \frac{\varepsilon}{N}\alpha_n; \quad (t_1, ..., t_{r-1}, n) \neq (t_1^0, ..., t_{r-1}^0, n),$$

$$|\alpha(t_1, ..., t_{r-1}, n-1)| \leq \frac{\varepsilon \cdot \alpha_n^{(n-1)/n}}{N \, \ln \alpha_n},$$

$$|\alpha(t_1, ..., t_{r-1}, s)| \leq \frac{\varepsilon}{N}\alpha_n^{s/n}, \quad s = 1, ..., n-2,$$

$$|\alpha(t_1, ..., t_{r-1}, 0)| \leq \frac{\delta}{N}\}. \tag{7}$$

Here, ε and δ are positive numbers that we define later. It's clear that $\theta_k(J) \geq \int_\Omega |J(\alpha)|^{2k} d\alpha$, where

$$J(\alpha) = \int_\varepsilon^1 ... \int_\varepsilon^1 \exp\{2\,\pi i \beta_0\}\, dx_1 ... dx_{r-1} \int_0^1 \exp\{2\,\pi i f(x)\}\, dx. \tag{8}$$

By the definition of the set Ω, we have:

$$|\beta_0| = \left| \sum_{(t_1,...,t_{r-1},0)\in A} \alpha(t_1, ..., t_{r-1}, 0)\, x_1^{t_1} ... x_{r-1}^{t_{r-1}} \right|$$

$$\leq \sum_{(t_1,...,t_{r-1},0)\in A} |\alpha(t_1, ..., t_{r-1}, 0)| \leq \sum_{(t_1,...,t_r)\in A} \frac{\delta}{N} = \delta,$$

$$|\beta_s| = \left| \sum_{(t_1,...,t_{r-1},s)\in A} \alpha(t_1, ..., t_{r-1}, s)\, x_1^{t_1} ... x_{r-1}^{t_r} \right|$$

$$\leq \sum_{(t_1,...,t_r)\in A} \frac{\varepsilon}{N}\, \alpha_n^{s/n} \leq \varepsilon \cdot \alpha_n^{s/n}, \quad s = 1, ..., n-2,$$

$$|\beta_{n-1}| = \left| \sum_{(t_1,...,t_{r-1},n-1)\in A} \alpha(t_1, ..., t_{r-1}, n-1)\, x_1^{t_1} ... x_{r-1}^{t_{r-1}} \right| \leq \frac{\varepsilon\, \alpha_n^{(n-1)/n}}{\ln \alpha_n},$$

$$|\beta_n - \alpha_n| = \left| \sum_{(n,t_2,...,t_r)\in A} \alpha(t_1,...,t_{r-1},n)\, x_1^{t_1} ... x_{r-1}^{t_{r-1}} - \alpha(t_1^0,...,t_{r-1}^0,n) \right|$$

(9) $$\leq \sum_{(t_1,...,t_r)\in A} \frac{\varepsilon}{N}\, \alpha_n = \varepsilon \cdot \alpha_n.$$

By Lemma 1, for the integral

$$I_1 = I(x_1,...,x_{r-1},\alpha) = \int_0^1 \exp\{2\pi i f(x)\}dx = \int_0^{\int} \exp\{2\pi i(\beta_n x^n + ... + \beta_1 x)\}dx,$$

we have:

$$I_1 = \frac{c_n}{\beta_n^{1/2}} + O\left(\frac{|\beta_1|}{\beta_n^{2/n}} + ... + \frac{|\beta_{n-2}|}{\beta_n^{(n-1)/n}} + \frac{|\beta_{n-1}|}{\beta_n} \cdot \ln\, \beta_n + \frac{1}{\beta_n} \right),$$

where $c_n = \int_0^{+\infty} \exp\{2\pi i x^n\}dx \neq 0$. Given the inequalities (9) and the fact that

$$\left| \frac{1}{\beta_n^{1/n}} - \frac{1}{\alpha_n^{1/n}} \right| \leq \frac{\varepsilon}{n\alpha_n^{1/n}},$$

we obtain

$$I_1 = \frac{c_n}{\alpha_n^{1/n}} + \varepsilon \cdot \left(\frac{1}{\alpha_n^{1/n}} \right).$$

Hence, for the integral (8) we have:

$$\begin{aligned} J_\alpha &= \int_\varepsilon^1 ... \int_\varepsilon^{\int} I_1 \cdot \exp\{2\pi i \beta_0\}\, dx_1 ... dx_{r-1} = \int_\varepsilon^{\int} ... \int_\varepsilon^1 I_1\, dx_1 ... dx_{r-1} \\ &+ \int_\varepsilon^1 ... \int_\varepsilon^1 (\exp\{2\pi i \beta_0\} - 1)\, I_1\, dx_1 ... dx_{r-1} \\ &= \frac{c_n}{\alpha_n^{1/n}} + \varepsilon \cdot O\left(\frac{1}{\alpha_n^{1/n}} \right) + \delta \cdot O\left(\frac{1}{\alpha_n^{1/n}} \right), \end{aligned}$$

because $|\exp\{2\pi i \beta_0\} - 1| \leq 2\pi\, |\beta_0| \leq 2\pi\delta$. Therefore, choosing ε and δ sufficiently small, we obtain:

$$|J(\alpha)| \geq \frac{|c_n|}{2\, \alpha_n^{1/n}},$$

for all $\alpha \in \Omega$. Given the definition of the set Ω, for the singular integral $\theta_k(J)$ we obtain the estimate

$$\theta_k(J) \geq \frac{|c_n|^{2k} \cdot \varepsilon^N \cdot \delta^N}{2^{2k} \cdot N^N} \int_{\frac{2}{\varepsilon^{t_1 + ... + t_{r-1}}}}^{+\infty} \frac{d\alpha_n}{\alpha_n^{\frac{2k}{n} - \frac{M-n}{n}} \cdot (\ln \alpha_n)^N},$$

where $M = \sum_{(t_1,...,t_{r-1},s)\in A} s$. Therefore, for the convergence of the integral $\theta_k(J)$, it is necessary to satisfy the inequality $2k \geq M$. The theorem is proved.

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References

[1] Hua Loo-Keng, *On the number of solutions of Tarry's problem*, Acta Sci. Sinica, 1 (1952), 1-76.

[2] G.I. Arkhipov, A. Karatsuba, V. Chubarikov, *Multiple trigonometric sums and their applications*, Mathematics of the USSR-Izvestiya, 17 (1981), 1-54.

[3] G.I. Arkhipov, A. Karatsuba, V. Chubarikov, *The index of convergence of the singular integral in Tarry's problem*, Sov. Math. Dokl., 20 (1979), 978–981.

[4] G.I. Arkhipov, A. Karatsuba, V. Chubarikov, *Theory of multiple trigonometric sums (Teoriya kratnykh trigonometricheskikh summ)*, Nauka, 1987. Russian.

[5] G.I. Arkhipov, A. Karatsuba, V. Chubarikov, *Trigonometric sums in number theory and analysis*, Volume 39 in the series De Gruyter Expositions in Mathematics, 2004.

[6] G.I. Arkhipov, A. Karatsuba, V. Chubarikov, *Trigonometric integrals*, Math. USSR Izv., 15 (1980), 211–239.

[7] I.A. Ikromov, *On the convergence exponent of trigonometric integrals*, Proceedings of the Steklov Institute of Mathematics, 218 (1997) 179-189.

[8] M.A. Chakhkiev, *On the convergence exponent of the singular integral in the multi-dimensional analogue of Tarry's problem*, Izvestiya: Mathematics, 67 (2003), 405-418.

[9] M.A. Chakhkiev, *Estimates for oscillatory integrals with convex phase*, Izvestiya: Mathematics, 70 (2006), 171–209.

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