Almost ω-continuous functions in bitopological spaces

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Abstract. In this paper, as a generalization of u-ω-continuous functions, we introduce the notion of almost ω-continuous functions in bitopological spaces and obtain several characterizations and some of its properties.

Keywords: bitopological spaces, u-ω-open sets, almost continuous function.

1. Introduction

The concept of bitopological spaces was first introduced by Kelly [12]. After the introduction of the definition of a bitopological space by Kelly, a large number of

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topologists have turned their attention to the generalization of different concepts of a single topological space in this space. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [10]. Several characterizations and properties of ω-closed sets were provided in [4, 2, 3, 10, 11, 15]. In this paper, we introduce and study upper (lower) (i,j)-almost ω-continuous functions on bitopological space and obtain several characterizations and some of its properties.

2. Preliminaries

A triple (X, τ_1, τ_2) consisting of a nonempty set X with two topologies τ_1 and τ_2 on X is called bitopological spaces. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$. For a A of (X, τ_1, τ_2), $i \operatorname{Int} A$ and $i \operatorname{cl} A$ means, respectively, the interior and closure of$A$ with respect to the topology τ_i. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X finer than τ. The intersection of all ω-closed sets containing A is called the ω-closure [10] of A and is denoted by $\omega \operatorname{cl}(A)$. The family of all ω-open sets of X is denoted by $\omega(\tau)$.

Definition 2.1. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

1. *(i,j)-sem-iopen [8] if $A \subset j \operatorname{cl}(i \operatorname{Int}(A))$;*
2. *(i,j)-preopen [14] if $A \subset i \operatorname{Int}(j \operatorname{cl}(A))$;*
3. *(i,j)-semi-preopen [14] if $A \subset j \operatorname{cl}(i \operatorname{Int}(j \operatorname{cl}(A)))$;*
4. *(i,j)-regular open [7] if $A = i \operatorname{Int}(j \operatorname{cl}(A))$,*

On each definition above, $i, j = 1, 2$ and $i \neq j$.

The complement of an (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-semi-preopen, (i,j)-regular open) set is called an (i,j)-semi-closed (resp. (i,j)-preclosed, (i,j)-semi-preclosed, (i,j)-regular closed) set. The family of all (i,j)-semi-open (resp. (i,j)-preopen, (i,j)-semi-preopen, (i,j)-regular open, (i,j)-semi-closed, (i,j)-preclosed, (i,j)-semi-preclosed, (i,j)-regular closed) sets of (X, τ_1, τ_2) is denote by (i,j)-$SO(X)$ (resp. (i,j)-$PO(X)$, (i,j)-$SPO(X)$, (i,j)-$RO(X)$, (i,j)-$SC(X)$, (i,j)-$PC(X)$, (i,j)-$SPC(X)$, (i,j)-$RC(X)$).

Definition 2.2. *A point x of X is said to be an (i,j)-δ-cluster point [13] of A if $A \cap U \neq \emptyset$ for every (i,j)-regular open set U containing x, the set of all (i,j)-δ-cluster points of A is called the (i,j)-δ-closure of A, a subset A of X is said to be (i,j)-δ-closed if the set of all (i,j)-δ-cluster points of A is a subset of A. The complement of an (i,j)-δ-closed set is an called (i,j)-δ-open set or a subset A of X is called (i,j)-δ-open if and only if there exist (i,j)-regular open sets A_k, $k \in I$ such that $A = \bigcup_{k\in I} A_k$.*

Remark 2.3. Let A be a subset of a bitopological space (X, τ_1, τ_2). Then (i,j)-$\mathrm{Int}_\delta(A)$=$\cup\{B : B \subset A,\ B$ an (i,j)-δ -open set$\}$ is called the (i,j)-δ-interior of A, and (i,j)-$\mathrm{cl}_\delta(A) = \cap\{B : B \supset A, B$ an (i,j)-δ-closed set$\}$ is called the (i,j)-δ-closure of A.

Definition 2.4 ([1])**.** *Let (X, τ_1, τ_2) be a bitopological space and let $A \subset X$. Then*

1. *A is said to be u-ω-open in (X, τ_1, τ_2) if $A \in \omega(\tau_1) \cup \omega(\tau_2)$,*
2. *A is said to be u-ω-closed in (X, τ_1, τ_2) if $X - A$ is u-ω-open in (X, τ_1, τ_2).*

The family of all u-ω-open sets in (X, τ_1, τ_2) is denoted by $\omega(\tau_1, \tau_2)$, and the family of all u-ω-closed sets in (X, τ_1, τ_2) is denoted by $\omega C(\tau_1, \tau_2)$.

Definition 2.5.
1. *The u-ω-closure of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega\,\mathrm{cl}(A)$ and defined as follows: (τ_1, τ_2)-$\omega\,\mathrm{cl}(A) = \omega\,\mathrm{cl}_{\tau_1}(A) \cap \omega\,\mathrm{cl}_{\tau_2}(A)$.*
2. *The u-ω-interior of A in (X, τ_1, τ_2) is denoted by (τ_1, τ_2)-$\omega\,\mathrm{Int}(A)$ and defined as follows: (τ_1, τ_2)-$\omega\,\mathrm{Int}(A) = \omega\,\mathrm{Int}_{\tau_1}(A) \cup \omega\,\mathrm{Int}_{\tau_2}(A)$.*

Lemma 2.6. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. Then*

1. *(τ_1, τ_2)-$\omega\,\mathrm{Int}(A)$ is u-ω-open;*
2. *(τ_1, τ_2)-$\omega\,\mathrm{cl}(A)$ is u-ω-closed;*
3. *A is u-ω-open if and only if $A = (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(A)$;*
4. *A is u-ω-closed if and only if $A = (\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$;*
5. *(τ_1, τ_2)-$\omega\,\mathrm{Int}(X \backslash A) = X \backslash (\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$;*
6. *(τ_1, τ_2)-$\omega\,\mathrm{cl}(X \backslash A) = X \backslash (\tau_1, \tau_2)$-$\omega\,\mathrm{Int}(A)$.*

Lemma 2.7. *Let (X, τ_1, τ_2) be a bitopological space and $A \subset X$. A point $x \in (\tau_1, \tau_2)$-$\omega\,\mathrm{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every u-ω-open set U of (X, τ_1, τ_2).*

Definition 2.8 ([1])**.** *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be u-ω-continuous if for each $x \in X$ and each $V \in \sigma_1 \cup \sigma_2$ containing $f(x)$, there exists an u-ω-open set U containing x such that $f(U) \subset V$.*

3. Properties of (i,j)-almost ω-continuous functions

Definition 3.1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be an (i,j)-almost ω-continuous at a point $x\in X$ if for each σ_i-open set V of Y containing $f(x)$, there exists an u-ω-open set U of X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. If f is (i,j)-almost ω-continuous at every point x of X, then it is called (i,j)-almost ω-continuous on X.*

It is obvious from the definition that u-ω-continuity implies (i,j)-almost ω-continuity. However, the converse is not true in general as it is shown in the following example.

Example 3.2. Take $X=\mathbb{R}$ and two topologies $\tau_1=\tau_2=\{\emptyset,\mathbb{R},\mathbb{R}\setminus\mathbb{Q}\}$ and $Y=\{a,b,c\}$ with topologies $\sigma_1=\sigma_2=\{\emptyset,Y,\{a\},\{a,b\}\}$. Define $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ as

$$f(x)=\begin{cases}a, & \text{if } x\in\mathbb{Q},\\ b, & \text{if } x\notin\mathbb{Q}.\end{cases}$$

It is easy to see that f is (i,j)-almost ω-continuous but does not is u-ω-continuous function.

Theorem 3.3. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *For each $x\in X$ and each (i,j)-regular open set V of Y containing $f(x)$, there exists an u-ω-open U in X containing x such that $f(U)\subset V$.*
3. *For each $x\in X$ and each (i,j)-δ-open set V of Y containing $f(x)$, there exists an u-ω-open U in X containing x such that $f(U)\subset V$.*

Proof. (1) ⇒ (2): Let $x\in X$ and let V be any (i,j)-regular open subset of Y containing $f(x)$. By (1), there exists an u-ω-open set U of X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. Since V is (i,j)-regular open, $i\operatorname{Int}(j\operatorname{cl}(V))=V$. Therefore, $f(U)\subset V$.

(2) ⇒ (3): Let $x\in X$ and let V be any (i,j)-δ-open set of Y containing $f(x)$. Then there exists a σ_i-open set G containing $f(x)$ such that $G\subset i\operatorname{Int}(j\operatorname{cl}(G))\subset V$. Since $i\operatorname{Int}(j\operatorname{cl}(G))$ is (i,j)-regular open set of Y containing $f(x)$. By (2), there exists an u-ω-open set U in X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(G))\subset V$.

(3) ⇒ (1): Let $x\in X$ and let V be any σ_i-open set of Y containing $f(x)$. Then $i\operatorname{Int}(j\operatorname{cl}(V)$ is (i,j)-δ-open set of Y containing $f(x)$. By (3), there exists an u-ω-open set U in X containing x such that $f(U)\subset i\operatorname{Int}(j\operatorname{cl}(V))$. Therefore, f is (i,j)-almost ω-continuous. □

Theorem 3.4. *For a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *$f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X for each σ_i-open set V in Y.*
3. *$f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is u-ω-closed set in X for each σ_i-closed set F in Y.*
4. *$f^{-1}(F)$ is u-ω-closed set in X for each (i,j)-regular closed set F of Y.*
5. *$f^{-1}(V)$ is u-ω-open set in X for each (i,j)-regular open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any σ_i-open set in Y. We have to show that $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X. Let $x \in f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$. Then $f(x) \in i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ and $i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ is an (i,j)-regular open set in Y. Since f is (i,j)-almost ω-continuous, by Theorem 3.3, there exists an u-ω-open set U of X containing x such that $f(U) \subset i\,\mathrm{Int}(j\,\mathrm{cl}(V))$. Which implies that $x \in U \subset f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$. Therefore, $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X.

(2) $\Rightarrow$ (3): Let F be any σ_i-closed set of Y. Then $Y\backslash F$ is a σ_i-open set of Y. By (2), $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(Y\backslash F)))$ is an u-ω-open set in X and $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(Y\backslash F))) = f^{-1}(i\,\mathrm{Int}(Y\backslash j\,\mathrm{Int}(F))) = f^{-1}(Y\backslash i\,\mathrm{cl}(j\,\mathrm{Int}(F))) = X\backslash f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-open set in X and hence $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-closed set in X.

(3) $\Rightarrow$ (4): Let F be any (i,j)-regular closed set of Y. Then F is a σ_i-closed set of Y. By (3), $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F)))$ is an u-ω-closed set in X, since F is (i,j)-regular closed set. Then $f^{-1}(i\,\mathrm{cl}(j\,\mathrm{Int}(F))) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is an u-ω-closed set in X.

(4) $\Rightarrow$ (5): Let V be any (i,j)-regular open set of Y. Then $Y\backslash V$ is an (i,j)-regular closed set of Y and by (4), we have $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$ is u-ω-closed set in X and hence $f^{-1}(V)$ is u-ω-open set in X.

(5) $\Rightarrow$ (1): Let $x \in X$ and let V be any (i,j)-regular open set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (5), we have $f^{-1}(V)$ is u-ω-open set in X. Therefore, we obtain $f(f^{-1}(V)) \subset V$. Hence by Theorem 3.3, f is (i,j)-almost ω-continuous. □

Theorem 3.5. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*

1. *f is (i,j)-almost ω-continuous.*
2. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$ for each (j,i)-semi-preopen set V of Y.*
3. *$f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$ for each (j,i)-semi-preclosed set F of Y.*
4. *$f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$ for each (j,i)-semi-closed set F of Y.*
5. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$ for each (j,i)-semi-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any (j,i)-semi-preopen set of Y. Since $i\,\mathrm{cl}(V)$ is an (i,j)-regular closed set in Y and f is (i,j)-almost ω-continuous. Then by Theorem 3.4, $f^{-1}(V)$ is u-ω-closed set in X. Therefore, we obtain (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(j\,\mathrm{cl}(V))$.

(2) $\Rightarrow$ (3): Let F be any (j,i)-semi-preclosed set of Y. Then $Y\backslash F$ is (j,i)-semi-preopen set of Y and by (2), we have

$$(\tau_1,\tau_2)-\omega\,\mathrm{cl}(f^{-1}(Y\backslash F)) \subset f^{-1}(i\,\mathrm{cl}(Y\backslash F)) \subset f^{-1}(Y\backslash i\,\mathrm{Int}(F)) \subset X\backslash f^{-1}(i\,\mathrm{Int}(F)).$$

Therefore, $f^{-1}(i\,\mathrm{Int}(F)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(F))$.

(3) $\Rightarrow$ (4): This is obvious since every (j,i)-semi-closed set is (j,i)-semi-preclosed set.

(4) $\Rightarrow$ (5): Let V be any (j,i)-semi open set of Y. Then $Y\backslash V$ is (j,i)-semi-closed set and by (4), we have $f^{-1}(i\,\mathrm{Int}(Y\backslash V)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(Y\backslash V)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(X\backslash f^{-1}(V)) \subset X\backslash(\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$.

Therefore, (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(i\,\mathrm{cl}(V))$.

(5) $\Rightarrow$ (1): Let F be any (i,j)-regular closed set of Y. Then F is (j,i)-semi-open set of Y. By (5), we have (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(F)) \subset f^{-1}(i\,\mathrm{cl}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is u-ω-closed set in X.

Therefore, by Theorem 3.4, f is (i,j)-almost ω-continuous. □

Theorem 3.6. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-almost ω-continuous if and only if $f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))$ for each σ_i-open set V of Y.*

Proof. Let V be any σ_i-open set of Y. Then $V \subset i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ and $i\,\mathrm{Int}(j\,\mathrm{cl}(V))$ is (i,j)-regular open set in Y. Since f is (i,j)-almost ω-continuous, by Theorem 3.4, $f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))$ is u-ω-open set in X and hence we obtain that $f^{-1}(V) \subset f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))$. Conversely, let V be any (i,j)-regular open set of Y. Then V is σ_i-open set of Y. By hypothesis, we have $f^{-1}(V) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V)))) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is u-ω-open set in X and hence by Theorem 3.4, f is (i,j)-almost ω-continuous. □

Corollary 3.7. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is (i,j)-almost ω-continuous if and only if (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(j\,\mathrm{cl}(i\,\mathrm{Int}(F)))) \subset f^{-1}(F)$ for each σ_i-closed set F of Y.*

Theorem 3.8. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an (i,j)-almost ω-continuous function and let V be any σ_i-open subset of Y.*

If $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(V)$, then $f(x) \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$.

Proof. Let $x \in X$ be such that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(V)$ and suppose $f(x) \notin (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$. Then there exists an u-ω-open set H containing $f(x)$ such that $H \cap V = \emptyset$. Then $j\,\mathrm{cl}(H) \cap V = \emptyset$ implies $i\,\mathrm{Int}(j\,\mathrm{cl}(H)) \cap V = \emptyset$ and

$i\,\mathrm{Int}(j\,\mathrm{cl}(H))$ is (i,j)-regular open set. Since f is (i,j)-almost ω-continuous, by Theorem 3.4, there exists an u-ω-open set U in X containing x such that $f(U) \subset i\,\mathrm{Int}(j\,\mathrm{cl}(H))$. Therefore, $f(U) \cap V = \emptyset$. However, since $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$, $U \cap f^{-1}(V) \neq \emptyset$ for every u-ω-open set U in X containing x, so that $f(U) \cap V \neq \emptyset$. We have a contradiction. Hence $f(x) \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(V)$. □

Theorem 3.9. *The set of all points x of X at which $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is not (i,j)-almost ω-continuous is identical with the union of the (τ_1,τ_2)-ω-frontiers of the inverse images of (i,j)-regular open subsets of Y containing $f(x)$.*

Proof. If f is not (i,j)-almost ω-continuous at $x \in X$, then there exists an (i,j)-regular open set V containing $f(x)$ such that for every u-ω-open set U of X containing x, $f(U) \subset (Y \backslash V) \neq \emptyset$. This means that for every u-ω-open set U of X containing x, we must have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$. Hence it follows that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(X \backslash f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(V))$. This means that x belongs to the (τ_1,τ_2)-ω-frontier of $f^{-1}(V)$. Conversely, suppose that x belongs to the (τ_1,τ_2)-ω-frontier of $f^{-1}(V_1)$ for some (i,j)-regular open subset V_1 of Y such that $f(x) \in V_1$. Suppose that f is (i,j)-almost ω-continuous at x. Then by Theorem 3.3, there exists an u-ω-open set U of X containing x such that $f(U) \subset V_1$. Then we have $U \subset f^{-1}(V_1)$. This shows that $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(V_1))$. Therefore, we have $x \in (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(X \backslash f^{-1}(V_1))$ and $x \notin (\tau_1,\tau_2)$-$\omega Fr(f^{-1}(V_1))$. But this is a contradiction. This means that f is not almost u-ω-continuous. □

Theorem 3.10. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$. Then the following statements are equivalent.*

1. *f is an (i,j)-almost ω-continuous mapping.*
2. *$f^{-1}(B)$ is an u-ω-open set of X for each (i,j)-δ-open set B of Y.*
3. *$f^{-1}(B)$ is an u-ω-closed set of X for each (i,j)-δ-closed set B of Y.*
4. *$f((\tau_1,\tau_2)$-$\omega\,\mathrm{cl}\,A) \subset (i,j)$-$\mathrm{cl}_\delta(f(A))$ for each set A of X.*
5. *(τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(B)) \subset f^{-1}((i,j)$-$\mathrm{cl}_\delta(B)$ for each set B of Y.*
6. *$f^{-1}((i,j)$-$\mathrm{Int}_\delta(B)) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B))$ for each set B of Y.*

Proof. (1)⇒ (2): Let B be any (i,j)-δ-open set of Y. Then $B = \bigcup_{\alpha\in I} B_\alpha$, where B_α is an (i,j)-regular open set of Y, for each $\alpha \in I$. From $f^{-1}(B) = f^{-1}(\bigcup_{\alpha\in I} B_\alpha) = \bigcup_{\alpha\in I} f^{-1}(B_\alpha)$ it follows that $f^{-1}(B)$ is an u-ω-open set as the union of u-ω-open sets.

(2)⇒(3): Can be proved by using the complement.

(3)⇒(4): Let A be any set of X. Then (i,j)-$\mathrm{cl}_\delta(f(A))$ is an (i,j)-δ closed set of Y. By assumption $f^{-1}((i,j)$-$\mathrm{cl}_\delta(f(A)))$ is an (i,j)-preclosed set of X. Hence

(τ_1,τ_2)-$\omega\,\mathrm{cl}(A) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(f(A))) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}((i,j)$-$\mathrm{cl}_\delta(f(A))) = f^{-1}((i,j)$-$\mathrm{cl}_\delta(f(A)))f((\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(A)) \subset (i,j)$-$\mathrm{cl}_\delta(f(A))$.

(4)⇒ (5): Let B be any set of Y. From the assumption it follows that (τ_1,τ_2)-$\omega\,\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(f(\tau_1,\tau_2)$-$\omega\,\mathrm{cl}(f^{-1}(B))) \subset f^{-1}((i,j)$-$\mathrm{cl}_\delta(B))$.

(5)⇒(6): Can be proved by using the complement.

(6)⇒(1): Let B be any (i,j)-regular open set of Y. Then $B = (i,j)$-$\mathrm{Int}_\delta(B)$. According to the assumption $f^{-1}(B) = f^{-1}((i,j)$-$\mathrm{Int}_\delta(B) \subset (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B)) \subset f^{-1}(B)$. Hence $f^{-1}(B) = (\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B))$, so $f^{-1}(B)$ is a u-ω-open set. Thus f is an (i,j)-almost ω-continuous function. □

Theorem 3.11. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a bijective function. The function f is (i,j)-almost ω-continuous if and only if (i,j)-$\mathrm{Int}_\delta(f(A)) \subset f((i,j)$-$\mathrm{Int}(A))$ for each subset A of X.*

Proof. We suppose that f is (i,j)-almost ω-continuous. Then

$$\begin{aligned} f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A)) &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A))) \\ &\subseteq (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}(f(A))) \\ &= (\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(A). \end{aligned}$$

Again, since f is surjective, we obtain

$$\begin{aligned} (i,j)\text{-}\mathrm{Int}_\delta(f(A)) &= f(f^{-1}((i,j)\text{-}\mathrm{Int}_\delta(f(A))) \\ &\subset f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(A)). \end{aligned}$$

Conversely, let B be any (i,j)-δ-open set of Y. Then (i,j)-$\mathrm{Int}_\delta(B) = B$. According to the assumption.

$$\begin{aligned} f((\tau_1,\tau_2)\text{-}\omega\,\mathrm{Int}(f^{-1}(B))) &\supset (i,j)\text{-}\mathrm{Int}_\delta(f(f^{-1}(B))) \\ &= (i,j)\text{-}\mathrm{Int}_\delta(B) \\ &= B. \end{aligned}$$

Thus, implies that $f^{-1}(f((\tau_1,\tau_2)$-$\omega\,\mathrm{Int}(f^{-1}(B)))) \supset f^{-1}(B)$. Since f is injective we obtain (τ_1,τ_2)-$\omega\,\mathrm{Int}(f^{-1}(B)) = f^{-1}(f((i,j)$-$\mathrm{Int}(f^{-1}(B)))) \supset f^{-1}(B)$. Hence, (τ_1,τ_2)-$\omega\,\mathrm{Int}(f^{-1}(B)) = f^{-1}(B)$, so $f^{-1}(B)$ is an u-ω-open set. Thus, f is (i,j)-almost ω-continuous function. □

For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, we define $D_{a\omega}(f)$ as follows: $D_{a\omega}(f) = \{x \in X : f$ is not (i,j)-almost ω-continuous at $x\}$.

Theorem 3.12. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties hold:*

$$\begin{aligned} D_{a\omega}(f) &= \bigcup_{V\in(j,i)-SPO(Y)} \{(\tau_1,\tau_2) - \omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(i\,\mathrm{cl}(V))\} \\ &= \bigcup_{F\in(j,i)-SPC(Y)} \{f^{-1}(i\,\mathrm{Int}(F))\backslash(\tau_1,\tau_2) - \omega\,\mathrm{Int}(f^{-1}(F))\} \end{aligned}$$

$$= \bigcup_{F\in(j,i)-SC(Y)} \{f^{-1}(i\,\mathrm{Int}(F))\backslash(\tau_1,\tau_2)-\omega\,\mathrm{Int}(f^{-1}(F))\}$$
$$= \bigcup_{V\in(j,i)-SO(Y)} \{(\tau_1,\tau_2)-\omega\,\mathrm{cl}(f^{-1}(V))\backslash f^{-1}(i\,\mathrm{cl}(V))\}$$
$$= \bigcup_{V\in\sigma_i} \{f^{-1}(V)\backslash(\tau_1,\tau_2)-\omega\,\mathrm{Int}(f^{-1}(i\,\mathrm{Int}(j\,\mathrm{cl}(V))))\}.$$

Proof. The proof follows from Theorem 3.5. □

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