Hyper BN-algebras: hyperstructure theory applied to BN-algebras

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Abstract. In this paper, we introduce the notion of hyper BN-algebras. We prove that it is a generalization of BN-algebras. We also compare it to other existing hyper algebras. We give some routine properties of hyper BN-algebra. We also give a certain condition for when a hyper BN-algebra becomes a hyper B-algebra and a hypergroup. Finally, we define the notion of hyper subBN-algebra and obtain some preliminary properties.

Keywords: hyper BN-algebra, hyper subBN-algebra.

1. Introduction

BCK-algebras were initiated by Y. Imai and K. Iséki [7] in 1966. In the same year, K. Iséki [9] introduced BCI-algebras to generalize BCK-algebras. BCI-algebras are logical algebras which are the algebraic formulations of the set difference and the implicational functor in logical systems. The names of BCK- and BCI-algebras are originated from the combinators B, C, K, and I in combinatory logic. Another generalization of BCK-algebras was introduced by Y. Komori [12] called BCC-algebras and proved that the class of BCC-algebras were not a variety.

On the other hand, B-algebras were introduced by Neggers, J. and Kim, H.S. [14] in 2002. It is a class of algebra which coincides with the class of groups. Many papers were published regarding the theory of B-algebra. In

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2013, another class of algebra was introduced by Kim, C.B., and Kim H.S. [11] called BN-algebras.

In 1934, F. Marty [13] brought the concept of the algebraic hyperstructure theory to the world at the 8th Congress of Scandinavian Mathematicians. This is a generalization of the classical algebraic structure. Several applications of the hyperstructure theory were given by P. Corsini and V. Leoreanu [3]. Papers were also published on providing examples of hyperstructures in inheritance issues in genetics [4] and in the interaction of elementary particles in physics [6].

In 2006, Xin, X.L. [17] applied the concept of hyperstructure to BCI-algebras, thereby forming hyper BCI-algebras. Hyper BCI-algebras are the generalization of the hyperstructure counterpart of BCK-algebras called hyper BCK-algebras introduced by Jun, Y.B. et al [10] in 2000. The theory of hyper BCK-algebras is widely studied specifically on β-relations [15] and state operators [18]. The hyperstructure counterpart of the BCC-algebras was defined by R. Borzooei, W. Dudek, and N. Koohestani [1] in 2006. They have shown that the class of hyper BCC-algebras were a generalization of the class of hyper BCK-algebras. In 2016, Indangan, R. and Petalcorin Jr., G. [8] introduced the notion of hyper GR-algebra, which is based on hyper BCI-algebra by culling out one property and replacing it with another property requirement. A hyperstructure counterpart of B-algebras was formulated by Endam J. in his dissertation and was called hyper B-algebras. Some other hyper algebras include hyper quasi-MV algebras as introduced by Chen, W. and Davvaz, B. [2] in 2018.

Motivated by the introduction of various hyper algebras, we will introduce the notion of hyper BN-algebras and show that this hyper algebra is a generalization of BN-algebras. Furthermore, we will differentiate hyper BN-algebras to hyper BCI-algebras, hyper BCC-algebras, hyper B-algebras, and hyper GR-algebras. Moreover, we will investigate the structure of hyper BN-algebra with regard to its properties and related concepts.

2. Preliminaries

This section provide some preliminary concepts and results needed for this paper.

Definition 2.1 ([11])**.** A *BN-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(BN_1) $x * x = 0$,

(BN_2) $x * 0 = x$, and

(BN_3) $(x * y) * z = (0 * z) * (y * x)$,

for any $x, y, z \in X$.

Definition 2.2 ([5])**.** Define $\mathcal{P}(H)$ to be the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. A *hyperoperation* on a nonempty set H is a function $\circledast : H \times H \to \mathcal{P}^*(H)$. The value $(x, y) \in H \times H$ under $\circledast$ is defined by $x \circledast y$. If $x \in H$ and $\emptyset \neq A, B \subseteq H$, then

$$(i)\ \ A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b;$$

$$(ii)\ \ A \circledast x = A \circledast \{x\} \text{ and } x \circledast B = \{x\} \circledast B.$$

Definition 2.3 ([3])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation. Then $(H, \circledast)$ is called a *hypergroup* if it satisfies the following axioms: for all $x, y, z \in H$,

$$(H_1)\ \ x \circledast (y \circledast z) = (x \circledast y) \circledast z, \text{ and}$$

$$(H_2)\ \ x \circledast H = H = H \circledast x.$$

A hypergroup $(H, \circledast)$ is said to be *commutative* if $x \circledast y = y \circledast x$ for all $x, y \in H$.

Definition 2.4 ([8])**.** Let $(H, \circledast, 0)$ be a hyper algebra, $x, y \in H$, and $\emptyset \neq A, B \subseteq H$. Then $x \ll y$ if and only if $0 \in x \circledast y$ and $A \ll B$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \ll b$. $\ll$ is called a *hyper order* on H. Let $x \in H$ and $A \subseteq H$. $x \ll A$ implies $\{x\} \ll A$. Similarly, $A \ll x$ implies $A \ll \{x\}$.

Definition 2.5 ([17])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BCI-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

$$(HI_1)\ \ (x \circledast z) \circledast (y \circledast z) \ll x \circledast y,$$

$$(HI_2)\ \ (x \circledast y) \circledast z = (x \circledast z) \circledast y,$$

$$(HI_3)\ \ x \ll x,$$

$$(HI_4)\ \ x \ll y \text{ and } y \ll x \text{ imply } x = y, \text{ and}$$

$$(HI_5)\ \ 0 \circledast (0 \circledast x) \ll x,\ x \neq 0.$$

Example 2.1 ([17])**.** Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

then $(H, \circledast, 0)$ is a hyper BCI-algebra.

Definition 2.6 ([1])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BCC-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HC_1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$,

(HC_2) $0 \circledast x = \{0\}$,

(HC_3) $x \circledast 0 = \{x\}$, and

(HC_4) $x \ll y$ and $y \ll x$ imply $x = y$.

Example 2.2 ([1])**.** Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0, 1\}$	$\{2\}$
3	$\{3\}$	$\{1, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 3\}$

then $(H, \circledast, 0)$ is a hyper BCC-algebra.

Definition 2.7 ([8])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper GR-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HGR_1) $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$,

(HGR_2) $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

(HGR_3) $x \ll x$,

(HGR_4) $0 \circledast (0 \circledast x) \ll x$, $x \neq 0$, and

(HGR_5) $(x \circledast y) \circledast z \ll y \circledast z$.

Example 2.3 ([8])**.** Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0\}$	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1, 2\}$

then $(H, \circledast, 0)$ is a hyper GR-algebra.

Definition 2.8 ([16])**.** Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper B-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HB_1) $x \ll x$,

(HB_2) $x \circledast H = H = H \circledast x$, and

(HB_3) $(x \circledast y) \circledast z = x \circledast [z \circledast (0 \circledast y)]$.

Example 2.4 ([16])**.** Let $H = \{0, a, b, c\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
a	$\{0\}$	$\{0\}$	$\{0, a, b\}$	$\{0, a, c\}$
b	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	$\{b, c\}$
b	$\{0, a, c\}$	$\{0, a, c\}$	$\{b, c\}$	$\{0, a, c\}$

then $(H, \circledast, 0)$ is a hyper B-algebra.

3. Hyper BN-algebras

We will formally define in this section what we meant by a hyper BN-algebra. We also present some of its routine properties. We will also present a characterization of a hyper BN-algebra.

Definition 3.1. Let H be a nonempty set and $\circledast$ be a hyperoperation on H. Then $(H, \circledast, 0)$ is called a *hyper BN-algebra*, if $0 \in H$ and the following conditions hold: for all $x, y, z \in H$,

(HBN_1) $x \ll x$,

(HBN_2) $x \circledast 0 = \{x\}$, and

(HBN_3) $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$.

The following example will show that the class of BN-algebras is just a subclass of the class of hyper BN-algebras.

Example 3.1. Let $(X, *, 0)$ be a BN-algebra. Define a hyperoperation "$\circledast$" on X by $x \circledast y = \{x * y\}$ for all $x, y \in X$. Note that, by (BN_1), $0 = x * x$. Thus, $x \circledast x = \{0\}$ which implies that $x \ll x$ holding (HBN_1). (HBN_2) and (HBN_3) follow directly from (BN_2) and (BN_3), respectively. Therefore, $(X, \circledast, 0)$ is a hyper BN-algebra.

Example 3.2. Let $H = \{0, a, b\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra. However, it is not a hyper BCI-algebra nor a hyper GR-algebra because it violates the condition $(x \circledast y) \circledast z = (x \circledast z) \circledast y$ which is common to both hyper algebras. Particularly, the condition is violated when $x = b$, $y = a$, and $z = b$, that is, $(b \circledast a) \circledast b = \{0, b\}$ and $(b \circledast b) \circledast a = \{a, b\}$ which imply that $(b \circledast a) \circledast b \neq (b \circledast b) \circledast a$. Also, $(H, \circledast, 0)$ is not a hyper BCC-algebra because for $x = a$, $0 \circledast a = \{a\} \neq \{0\}$. Finally, H is not a hyper B-algebra because it violates (HB_2). Particularly, when $x = b$, we have $b \circledast H = \{0, b\} \neq H$.

Remark 3.1. In Example 2.1, the hyper BCI-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it violates condition (HBN_3). In particular, the violation is incurred when $x = 1$, $y = 0$, and $z = 0$, that is, $(1 \circledast 0) \circledast 0 = \{1\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0, 1\}$ implying that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Also, in Example 2.2, the hyper BCC-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it does not satisfy (HBN_3) specifically when $x = 1$, $y = 0$, and $z = 0$, that is, $(1 \circledast 0) \circledast 0 = \{1\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0\}$ which says that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Also, in Example 2.3, the hyper GR-algebra $(H, \circledast, 0)$ is not a hyper BN-algebra because it violates condition (HBN_3). Particularly, when $x = 1$, $y = 0$ and $z = 0$, we have $(1 \circledast 0) \circledast 0 = \{0, 1, 2\}$ and $(0 \circledast 0) \circledast (0 \circledast 1) = \{0\}$ implying that $(1 \circledast 0) \circledast 0 \neq (0 \circledast 0) \circledast (0 \circledast 1)$. Finally, the hyper B-algebra in Example 2.4 is not a hyper BN-algebra because $2 \circledast 0 = \{0, 1, 2\} \neq \{2\}$. Together with Example 3.2, we can conclude that hyper BN-algebra is different from hyper BCI-algebra, hyper BCC-algebra, hyper GR-algebra, and hyper B-algebra.

Example 3.3. Let $H = \{0, 1, 2\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0, 2\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 1\}$	$\{0, 1\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra. Also, it is a hyper B-algebra. However, it is not a hyper BCI-algebra because $0 \ll 1$ and $1 \ll 0$ but $0 \neq 1$. It is also not a hyper GR algebra because it violates (HGR_1): $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$. The violation is incurred when $x = y = z = 0$, that is $(0 \circledast 0) \circledast (0 \circledast 0) = \{0, 1, 2\}$ and $0 \circledast 0 = \{0, 1\}$ but $\{0, 1, 2\} \not\ll \{0, 1\}$ since $2 \not\ll 0$ and $2 \not\ll 1$.

Example 3.4. Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{3\}$	$\{2\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$
2	$\{2\}$	$\{0, 1, 3\}$	$\{0, 1, 2, 3\}$	$\{0, 2\}$
3	$\{3\}$	$\{0, 1, 2\}$	$\{0, 3\}$	$\{0, 1, 2, 3\}$

then $(H, \circledast, 0)$ is a hyper BN-algebra.

Example 3.5. Let $\mathbb{Z}$ be the set of integers. Define a hyperoperation "$\circledast$" on $\mathbb{Z}$ by:

$$x \circledast y = \begin{cases} \{x\}, & \text{if } y = 0 \\ \{y\}, & \text{if } x = 0 \\ \{x - y, y - x, x + y\}, & \text{otherwise.} \end{cases}$$

Then, we can show that $(\mathbb{Z}, \circledast, 0)$ is a hyper BN-algebra. Note that, the same holds when $\mathbb{Z}$ is replaced by $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

Example 3.6. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Define a hyperoperation "$\circledast$" on $\mathbb{N}$ by:

$$x \circledast y = \begin{cases} \{x\}, & \text{if } y = 0 \\ \{y\}, & \text{if } x = 0 \\ \{0, x\}, & \text{if } x = y \neq 0 \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

Then we can show that $(\mathbb{N}, \circledast, 0)$ is a hyper BN-algebra.

Observe that by (HBN_1) and (HBN_2), we have $0 \circledast 0 = \{0\}$. Thus, we have the following remark:

Remark 3.2. In any hyper BN-algebra, $0 \circledast 0 = \{0\}$.

Hereinafter, we may write the singleton set $\{x\}$ just by x as needed for the purpose of simplification.

Theorem 3.1. *In any hyper BN-algebra H, the following hold: for any $x, y, z \in H$ and $\emptyset \neq A, B, C \subseteq H$,*

(*i*) $x \circledast x = \{x\} \Leftrightarrow x = 0$,

(*ii*) $x \ll 0 \Rightarrow x = 0$,

(*iii*) $0 \circledast (0 \circledast x) = \{x\}$,

(*iv*) $0 \circledast (x \circledast y) = y \circledast x$,

(*v*) $x \circledast y = (0 \circledast y) \circledast (0 \circledast x)$,

(*vi*) $(0 \circledast x) \circledast y = (0 \circledast y) \circledast x$,

(*vii*) $x \ll y \Rightarrow y \ll x$,

(*viii*) $0 \circledast x = 0 \circledast y \Rightarrow x = y$,

(*ix*) $(x \circledast z) \circledast (y \circledast z) = (z \circledast y) \circledast (z \circledast x)$,

(*x*) $A \ll A$,

(*xi*) $A \subseteq B \Rightarrow A \ll B$,

(*xii*) $A \subseteq B$ *and* $B \ll C$ *imply* $A \ll C$,

(*xiii*) $A \ll \{0\} \Rightarrow A = \{0\}$,

(*xiv*) $A \circledast \{0\} = \{0\} \Rightarrow A = \{0\}$, *and*

(*xv*) $(A \circledast B) \circledast C = (0 \circledast C) \circledast (B \circledast A)$.

Proof. Let H be a hyper BN-algebra, $x, y, z \in H$ and $\emptyset \neq A, B, C \subseteq H$.

(*i*) Note that by (HBN_1), $0 \in x \circledast x = \{x\}$. Thus, $x = 0$. The converse follows from Remark 3.2.

(*ii*) $x \ll 0$ means $0 \in x \circledast 0$. But $x \circledast 0 = \{x\}$ by (HBN_2). Thus, $x = 0$.

(*iii*) Let $y = z = 0$ in (HBN_3). Then $(x \circledast 0) \circledast 0 = (0 \circledast 0) \circledast (0 \circledast x)$. By (HBN_2) and Remark 3.2, we have $0 \circledast (0 \circledast x) = \{x\}$.

(*iv*) Let $x = y$, $y = x$ and $z = 0$ in (HBN_3). Then $(y \circledast x) \circledast 0 = (0 \circledast 0) \circledast (x \circledast y)$. By (HBN_2) and Remark 3.2, we have $0 \circledast (x \circledast y) = y \circledast x$.

(*v*) By (HBN_2) and (HBN_3), $x \circledast y = (x \circledast 0) \circledast y = (0 \circledast y) \circledast (0 \circledast x)$,

(*vi*) By (HBN_3), $(0 \circledast x) \circledast y = (0 \circledast y) \circledast (x \circledast 0)$. Applying (HBN_2), we get the desired result.

(*vii*) Since $0 \in x \circledast y$, we have, by (*iv*) $\{0\} = 0 \circledast 0 \subseteq 0 \circledast (x \circledast y) = y \circledast x$. Thus, $0 \in y \circledast x$ and the result follows.

(*viii*) If $0 \circledast x = 0 \circledast y$, then by (*iii*), we have $\{x\} = 0 \circledast (0 \circledast x) = 0 \circledast (0 \circledast y) = \{y\}$. Thus, $x = y$.

(*ix*) By (HBN_3) and (*iv*), we have $(x \circledast z) \circledast (y \circledast z) = (0 \circledast (y \circledast z)) \circledast (z \circledast x) = (z \circledast y) \circledast (z \circledast x)$.

(*x*) From (HBN_1), $0 \in a \circledast a$ for all $a \in A$. Hence, $A \ll A$.

(*xi*) Since $A \ll A$, by (*x*), for all $a \in A$, $a \ll a$. Take $b = a$ since $A \subseteq B$. Then there exists $b \in B$ such that $a \ll b$. Thus, $A \ll B$.

(*xii*) Let $a \in A$. Then $a \in B$ since $A \subseteq B$ and there is an element $c \in C$ such that $a \ll c$ because $B \ll C$. Thus, $A \ll C$.

(*xiii*) Assume $A \ll \{0\}$ and let $a \in A$. Then $a \ll 0$ and so $a = 0$ by (*ii*). Hence, $A = \{0\}$.

(*xiv*) Suppose $A \circledast \{0\} = \{0\}$ and let $a \in A$. Then $a \circledast 0 = \{0\}$. By (HBN_2), $a \circledast 0 = \{a\}$. Thus, $a = 0$. Consequently, $A = \{0\}$.

(*xv*) Straightforward. □

The next result says that the concepts of hyper BN-algebras and hyper BCC-algebras are almost entirely different.

Theorem 3.2. *A hyper BN-algebra with at least two elements is not a hyper BCC-algebra.*

Proof. Let H be a hyper BN-algebra and let $x \in H$ such that $x \neq 0$. Suppose on the contrary that H is also a hyper BCC-algebra. By (HC_2), $0 \circledast x = \{0\}$. This means that $0 \ll x$. By Theorem 3.1(*vii*), $x \ll 0$. Thus, $x = 0$ by Theorem 3.1(*ii*). This is a contradiction to our assumption that $x \neq 0$. Therefore, H must not be a hyper BCC-algebra. □

Remark 3.3. The only intersection of the class of hyper BN-algebras and the class of hyper BCC-algebras is when $H = \{0\}$.

Definition 3.2. A hyper algebra $(H, \circledast, 0)$ is said to be 0-*commutative* if $x \circledast (0 \circledast y) = y \circledast (0 \circledast x)$ for all $x, y \in H$.

Theorem 3.3. *A hyper BN-algebra H is* 0*-commutative.*

Proof. Let $x, y \in H$. Then by Theorem 3.1(iii), $x \circledast (0 \circledast y) = [0 \circledast (0 \circledast x)] \circledast (0 \circledast y)$. Applying (HBN_3), we have $[0 \circledast (0 \circledast x)] \circledast (0 \circledast y) = [0 \circledast (0 \circledast y)] \circledast [(0 \circledast x) \circledast 0]$. Finally, by Theorem 3.1(iii) and (HBN_2), $[0 \circledast (0 \circledast y)] \circledast [(0 \circledast x) \circledast 0] = y \circledast (0 \circledast x)$. Therefore, $x \circledast (0 \circledast y) = y \circledast (0 \circledast x)$. □

The next result is a characterization of a hyper BN-algebra.

Theorem 3.4. $(H, \circledast, 0)$ *is a hyper BN-algebra if and only if it satisfies the following conditions: for all $x, y \in H$,*

(i) $x \ll x$,

(ii) $x \circledast 0 = \{x\}$,

(iii) *H is* 0*-commutative, and*

(iv) $0 \circledast (x \circledast y) = y \circledast x$.

Proof. $(\Rightarrow)$: (i) and (ii) are just (HBN_1) and (HBN_2), respectively. (iii) follows from Theorem 3.3. Finally, (iv) follows from Theorem 3.1(iv).
$(\Leftarrow)$: We only need to show (HBN_3). Let $x, y, z \in H$. Now, by (iv), $(0 \circledast z) \circledast (y \circledast x) = (0 \circledast z) \circledast [0 \circledast (x \circledast y)]$. By (iii), $(0 \circledast z) \circledast [0 \circledast (x \circledast y)] = (x \circledast y) \circledast [0 \circledast (0 \circledast z)]$. Applying (iv) again, we have $(x \circledast y) \circledast [0 \circledast (0 \circledast z)] = (x \circledast y) \circledast (z \circledast 0)$. Finally by (ii), we have $(x \circledast y) \circledast (z \circledast 0) = (x \circledast y) \circledast z$. Therefore, $(x \circledast y) \circledast z = (0 \circledast z) \circledast (y \circledast x)$ holding (HBN_3). □

Definition 3.3. A hyper BN-algebra H is said to be *commutative* if for all $x, y \in H$, $x \circledast y = y \circledast x$.

Example 3.7. The hyper BN-algebras in Example 3.2 and Example 3.3 are commutative while the hyper BN-algebra in Example 3.4 is not commutative because $2 \circledast 0 = \{2\} \neq \{3\} = 0 \circledast 2$.

Theorem 3.5. *Let H be a hyper BN-algebra. Then H is commutative if and only if $0 \circledast x = \{x\}$ for all $x \in H$.*

Proof. Suppose H is a hyper BN-algebra. If H is commutative, then by (HBN_2), $\{x\} = x \circledast 0 = 0 \circledast x$ for all $x \in H$. For the converse, let $0 \circledast x = \{x\}$ for all $x \in H$ and let $y, z \in H$. Then by Theorem 3.3 and the assumption, $y \circledast z = y \circledast (0 \circledast z) = z \circledast (0 \circledast y) = z \circledast y$. Hence, H is commutative. □

Theorem 3.6. *Let* $(H_1, \circledast_1, 0)$ *and* $(H_2, \circledast_2, 0)$ *be hyper algebras such that* $H_1 \cap H_2 = \{0\}$. *Suppose* $H = H_1 \cup H_2$. *Define a hyperoperation* $\circledast$ *on* H *by*

$$x \circledast y = \begin{cases} x \circledast_1 y, & \text{if } x, y \in H_1 \\ x \circledast_2 y, & \text{if } x, y \in H_2 \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

H_1 *and* H_2 *are commutative hyper* BN*-algebras if and only if* H *is a commutative hyper* BN*-algebra.*

Proof. Suppose H_1 and H_2 are commutative hyper BN-algebras. Let $x \in H$. If $x \in H_1$, then (HBN_1) follows because H_1 is a hyper BN-algebra. Similarly, if $x \in H_2$, then (HBN_1) holds. Thus, for all $x \in H$, (HBN_1) holds for H. Note that $0 \in H$. Further, $0 \in H_1$ and $0 \in H_2$. Let $x \in H$. If $x \in H_1$, (HBN_2) holds because H_1 is a hyper BN-algebra. Same argument when $x \in H_2$. Thus, (HBN_2) holds for H. Since either H_1 or H_2 is commutative, whether $z \in H_1$ or $z \in H_2$, we have $0 \circledast z = \{z\}$ by Theorem 3.5. And so, (HBN_3) reduces to $(x \circledast y) \circledast z = z \circledast (y \circledast x)$. Let $x, y, z \in H$. (HBN_3) holds clearly when $x, y, z \in H_1$ and $x, y, z \in H_2$ since H_1 and H_2 are hyper BN-algebras. So we are left with the following cases: $x \in H_1$ and $y, z \in H_2$, $y \in H_1$ and $x, z \in H_2$, $z \in H_1$ and $x, y \in H_2$, $x, y \in H_1$ and $z \in H_2$, $x, z \in H_1$ and $y \in H_2$, and $y, z \in H_1$ and $x \in H_2$.

Case 1. $x \in H_1$ and $y, z \in H_2$.

$$(x \circledast y) \circledast z = \{x, y\} \circledast z = (x \circledast z) \cup (y \circledast_2 z) = \{x, z\} \cup (y \circledast_2 z),$$

and

$$z \circledast (y \circledast x) = z \circledast \{x, y\} = (z \circledast x) \cup (z \circledast_2 y) = \{x, z\} \cup (z \circledast_2 y).$$

Note that $y \circledast_2 z = z \circledast_2 y$ since H_2 is commutative. Thus, (HBN_3) holds for this case.

Case 2. $y \in H_1$ and $x, z \in H_2$.

$$(x \circledast y) \circledast z = \{x, y\} \circledast z = (x \circledast_2 z) \cup (y \circledast z) = (x \circledast_2 y) \cup \{y, z\},$$

and

$$z \circledast (y \circledast x) = z \circledast \{x, y\} = (z \circledast_2 x) \cup (z \circledast y) = (z \circledast_2 x) \cup \{y, z\}.$$

Note that $x \circledast_2 z = z \circledast_2 x$ since H_2 is commutative. Thus, (HBN_3) holds for this case.

Case 3. $z \in H_1$ and $x, y \in H_2$.

$$(x \circledast y) \circledast z = (x \circledast_2 y) \circledast z = \bigcup_{a \in x \circledast_2 y} \{a, z\}.$$

Note that $x \circledast_2 y = y \circledast_2 x$ since H_2 is commutative. Thus, we have

$$z \circledast (y \circledast x) = z \circledast (y \circledast_2 x) = \bigcup_{a \in y \circledast_2 x} \{a, z\}.$$

Thus, (HBN_3) holds for this case.
The other three cases can be proven similarly. Thus, for all $x, y, z \in H$, (HBN_3) holds. Therefore, $(H, \circledast, 0)$ is a hyper BN-algebra. Commutativity follows directly from the definition of the hyperoperation of H. The converse of this theorem is clear. □

4. Hyper BN-algebra, Hyper B-algebra, and hypergroup

We have shown in Example 3.2 that the given hyper BN-algebra is not a hyper B-algebra. This means that in general, a hyper BN-algebra may not be a hyper B-algebra. In this section, we provide a certain condition for when a hyper BN-algebra becomes a hyper B-algebra and a hypergroup.

Definition 4.1. A hyper algebra $(H, \circledast, 0)$ is said to have a *condition* (D) if $(x \circledast y) \circledast z = x \circledast (z \circledast y)$ for all $x, y, z \in H$.

Example 4.1. The hyper BN-algebra in Example 3.3 satisfies condition (D). The hyper BN-algebra in Example 3.2 does not satisfy condition (D) because $(a \circledast b) \circledast b = \{0, b\} \neq \{a, b\} = a \circledast (b \circledast b)$.

Theorem 4.1. *If H is a hyper BN-algebra with the condition (D), then*

(i) $0 \circledast x = \{x\}$ for all $x \in H$, and

(ii) H is commutative.

Proof. Let H be a hyper BN-algebra that satisfies condition (D).

(i) Let $x = z = 0$ in (D). Then $(0 \circledast y) \circledast 0 = 0 \circledast (0 \circledast y)$. Applying (HBN_2) on the left hand side and Theorem 3.1(iv) on the right hand side of the equation, we obtain $0 \circledast y = \{y\}$.

(ii) Since (i) holds, we have by Theorem 3.5 that H is commutative.

□

The converse of Theorem 4.1 is not true as shown in the following example:

Example 4.2. Consider the hyper BN-algebra in Example 3.2. Notice that H is commutative and it satisfies $0 \circledast x = \{x\}$ for all $x \in H$. But it does not satisfy condition (D) as shown in Example 4.1.

Notice that the hyper BN-algebra in Example 3.2 does not satisfy condition (D) as shown in Example 4.1. Furthermore, it is not a hyper B-algebra. On the other hand, the hyper BN-algebra in Example 3.3 satisfies condition (D) and as shown it is a hyper B-algebra. Now, consider another example.

Example 4.3. Consider $H = \{0, 1, 2, 3\}$ and a hyperoperation $\circledast$ on H defined by the following Cayley table:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,3\}$
2	$\{2\}$	$\{0,1,2\}$	$\{0,2\}$	$\{0,2,3\}$
3	$\{3\}$	$\{0,1,3\}$	$\{0,2,3\}$	$\{0,3\}$

By routine calculations, H is a hyper BN-algebra satisfying condition (D). Furthermore, it is a hyper B-algebra.

The next result generalizes our previous observation.

Proposition 4.1. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. If H satisfies condition (D), then H is a hyper B-algebra.*

Proof. Suppose H is a hyper BN-algebra satisfying condition (D). (HB_1) is just the same as (HBN_1). We will show (HB_3) first. To do that, let $x, y, z \in H$. Then by Theorem 4.1 and condition (D), $x \circledast [z \circledast (0 \circledast y)] = x \circledast (z \circledast y) = (x \circledast y) \circledast z$ holding (HB_3). We will use (HB_3) in proving (HB_2). Note that $x \circledast H \subseteq H$ and $H \circledast x \subseteq H$. We are left to show the other inclusions. Now, let $x, h \in H$. By Theorem 3.1(iv), (HBN_1), and (HB_3) we proved earlier, we have $\{h\} = 0 \circledast (0 \circledast h) \subseteq (x \circledast x) \circledast (0 \circledast h) = x \circledast [(0 \circledast h) \circledast (0 \circledast x)] \subseteq x \circledast H$. Thus, $h \in x \circledast H$ implying that $H \subseteq x \circledast H$. Hence, $x \circledast H = H$. Also, by (HBN_2), (HBN_1), Theorem 3.1(iv), and (HB_3), we have $\{h\} = h \circledast 0 \subseteq h \circledast (x \circledast x) = h \circledast [x \circledast (0 \circledast (0 \circledast x))] = [h \circledast (0 \circledast x)] \circledast x \subseteq H \circledast x$. Thus, $h \in H \circledast x$ implying that $H \subseteq H \circledast x$. Hence, $H = H \circledast x$. Thus, (HB_2) holds. Therefore, $(H, \circledast, 0)$ is a hyper B-algebra. □

In general, the converse of Proposition 4.1 is not true as shown in the next example.

Example 4.4. Consider the set $H = \{0, a, b\}$ with hyperoperation defined by the Cayley table as follows:

$\circledast$	0	a	b
0	$\{0\}$	$\{b\}$	$\{a\}$
a	$\{a\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{0,a,b\}$	$\{0,a,b\}$

Then H is a hyper BN-algebra. Note that H is also a hyper B-algebra but it does not satisfy condition (D) because $(0 \circledast a) \circledast 0 = \{b\}$ and $0 \circledast (0 \circledast a) = \{a\}$ implying that $(0 \circledast a) \circledast 0 \neq 0 \circledast (0 \circledast a)$.

The next result shows how a hyper BN-algebra becomes a hypergroup.

Theorem 4.2. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. If H satisfies condition (D), then H is a commutative hypergroup.*

Proof. Let $(H, \circledast, 0)$ be a hyper BN-algebra satisfying condition (D). Then by Theorem 4.1(ii), H is commutative. We will now show (H_1) and (H_2) of Definition 2.3. The proof of (H_2) is similar to the proof in Proposition 4.1. By condition (D), for all $x, y, z \in H$, we have $(x \circledast y) \circledast z = x \circledast (z \circledast y) = x \circledast (y \circledast z)$ since H is commutative. Therefore, $(H, \circledast)$ is a commutative hypergroup. □

5. Hyper SubBN-algebras

We focus our attention in this section to hyper subBN-algebras. We will also introduce the concepts of reflexivity and normality leading to the concept of reflexive normal hyper subBN-algebras.

Definition 5.1. Let $(H, \circledast, 0)$ be a hyper BN-algebra and let S be a subset of H containing 0. If S is a hyper BN-algebra with respect to the hyperoperation "$\circledast$" on H, we say that S is a *hyper subBN-algebra* of H.

Example 5.1. Consider the hyper BN-algebra H in Example 3.2. Let $S = \{0, a\}$ and $T = \{0, b\}$. By routine calculations, both S and T are hyper subBN-algebra of H. If we consider the hyper BN-algebra H in Example 3.3, then the set $L = \{0, 1\}$ is not a hyper subBN-algebra of H because $1 \circledast 1 = \{0, 2\}$ but $2 \notin L$. Also, the set $M = \{0, 2\}$ is not a hyper subBN-algebra of H because $2 \circledast 2 = \{0, 1\}$ but $1 \notin M$.

Proposition 5.1. *Let S be a nonempty subset of a hyper BN-algebra H. If $x \circledast y \subseteq S$ for all $x, y \in S$, then $0 \in S$.*

Proof. Assume that $x \circledast y \subseteq S$ for all $x, y \in S \subseteq H$ and let $a \in S$. Then $a \in H$ and $a \ll a$ implying that $0 \in a \circledast a \subseteq S$. □

Theorem 5.1. *Let S be a nonempty subset of a hyper BN-algebra. Then S is a hyper subBN-algebra if and only if $x \circledast y \subseteq S$, for all $x, y \in S$.*

Proof. Let H be a hyper BN-algebra and S be a nonempty subset of H. Suppose that S is a hyper subBN-algebra. Then by Definition 5.1, S is a hyper BN-algebra. Thus, $x \circledast y \subseteq S$ for all $x, y \in S$. Conversely, suppose that $x \circledast y \subseteq S$ for all $x, y \in S$. Now, by Proposition 5.1, $0 \in S$. Also, if $x, y, z \in S$, then $x, y, z \in H$. And so, the properties (HBN_1) to (HBN_3) of a hyper BN-algebra are already satisfied since H is a hyper BN-algebra. □

The following corollary is just a consequence of Theorem 5.1 and Remark 3.2.

Corollary 5.1. *The set $\{0\}$ is a hyper subBN-algebra of any hyper BN-algebra.*

Theorem 5.2. *The intersection of family of hyper subBN-algebras of a hyper BN-algebra H is a hyper subBN-algebra of H.*

The following example will show that the union of two hyper subBN-algebras of a hyper BN-algebra is not necessary a hyper subBN-algebra.

Example 5.2. Let $H = \{0, 1, 2, 3\}$ be a set. If we define a hyperoperation "$\circledast$" on H as follows:

$\circledast$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{1,3\}$	$\{2,3\}$
2	$\{2\}$	$\{1,3\}$	$\{0,2\}$	$\{3\}$
3	$\{3\}$	$\{2,3\}$	$\{3\}$	$\{0,3\}$

then H is a hyper BN-algebra. Let $S_1 = \{0, 1\}$ and $S_2 = \{0, 2\}$. By Theorem 5.1, S_1 and S_2 are hyper subBN-algebras of H. However, $S_1 \cup \S_2 = \{0, 1, 2\}$ is not a hyper subBN-algebra of H because $1, 2 \in S_1 \cup S_2$ but $1 \circledast 2 = \{1, 3\} \not\subseteq S_1 \cup S_2$.

Definition 5.2. Let N be a nonempty subset of a hyper BN-algebra. Then N is called *normal* if $(x \circledast a) \circledast (y \circledast b) \subseteq N$ whenever $x \circledast y, a \circledast b \subseteq N$.

Remark 5.1. *If N a nonempty subset of a hyper BN-algebra is normal, then $0 \in N$.*

Proof. Choose $x, y \in H$ such that $x \circledast y \subseteq N$. By normality of N, we have $0 = 0 \circledast 0 \subseteq (x \circledast x) \circledast (y \circledast y) \subseteq N$. Thus, $0 \in N$. □

Example 5.3. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 3.2. Let $N_1 = \{0, a\}$ and $N_2 = \{0, b\}$. Then it can be shown that N_1 is normal. However, N_2 is not normal because $0 \circledast b = \{b\} \subseteq N_2$ and $a \circledast b = \{b\} \subseteq N_2$ but $(0 \circledast a) \circledast (b \circledast b) = \{a, b\} \not\subseteq N_2$.

Definition 5.3. A nonempty subset I of a hyper BN-algebra H is said to be *reflexive* if $x \circledast x \subseteq I$ for all $x \in H$.

Let H be a hyper BN-algebra and I be a reflexive subset of H. Note that for all $x \in H$, $0 \in x \circledast x \subseteq I$. Thus, we have the following remark:

Remark 5.2. *If I is a reflexive subset of a hyper BN-algebra, then $0 \in I$.*

Example 5.4. Let $H = \{0, 1, 2\}$ with hyperoperation $\circledast$ defined by the following Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{2\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H is a hyper BN-algebra by routine calculations. Let $I = \{0, 1\}$. Then it is reflexive because $x \circledast x \subseteq I$ for $x = 0, 1, 2$. Let $J = \{0, 2\}$. Then J is not reflexive because $1 \circledast 1 \not\subseteq J$.

The concepts of reflexivity and normality are different as shown in the following example.

Example 5.5. The normal subset N_1 as shown in Example 5.3 is not reflexive because $b \circledast b = \{0, b\} \nsubseteq N_1$. Let $H = \{0, 1, 2\}$ with hyperoperation $\circledast$ defined by the folowing Cayley table:

$\circledast$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{1\}$	$\{0,1\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 1\}$. Then I is reflexive because $x \circledast x \subseteq I$ for all $x \in H$ but I is not normal because $1 \circledast 0 = \{1\} \subseteq I$ and $1 \circledast 2 = \{1\} \subseteq I$ but $(1 \circledast 1) \circledast (0 \circledast 2) = \{1, 2\} \nsubseteq I$.

Theorem 5.3. *Every normal subset N of a hyper BN-algebra H is a hyper subBN-algebra of H.*

Proof. Let H be a hyper BN-algebra and N a normal subset of H. Note that $0 \in N$ by Remark 5.1. Now, let $x, y \in N$. Then $x \circledast 0 = \{x\} \subseteq N$ and $y \circledast 0 = \{y\} \subseteq N$. By normality of N, $x \circledast y = (x \circledast y) \circledast 0 = (x \circledast y) \circledast (0 \circledast 0) \subseteq N$. By Theorem 5.1, N is a hyper subBN-algebra of H. □

The converse of Theorem 5.3 is not true in general. The following example will show that.

Example 5.6. Consider the hyper BN-algebra $H = \{0, a, b\}$ in Example 3.2. Let $N_2 = \{0, b\}$. It has been shown that N_2 is a hyper subBN-algebra in Example 5.1 but it is not normal as shown in Example 5.3.

Definition 5.4. A hyper subBN-algebra S of a hyper BN-algebra H is called *reflexive* (resp. *normal*) *hyper subBN-algebra* if it is reflexive (resp. normal). S is called a *reflexive normal hyper subBN-algebra* if it is both reflexive and normal.

Example 5.7. Consider the set $H = \{0, 1, 2, 3, 4\}$. Define the hyperoperation "$\circledast$" by the following Cayley table:

$\circledast$	0	1	2	3	4
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$
1	$\{1\}$	$\{0,3\}$	$\{3\}$	$\{1\}$	$\{4\}$
2	$\{2\}$	$\{3\}$	$\{0,3\}$	$\{2\}$	$\{4\}$
3	$\{3\}$	$\{1\}$	$\{2\}$	$\{0,3\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0,3\}$

By routine calculations, H is a hyper BN-algebra. Let $I = \{0, 3\}$. Then I is a reflexive normal hyper subBN-algebra of H.

Theorem 5.4. *The intersection of family of reflexive normal hyper subBN-algebras of a hyper BN-algebra H is a reflexive normal hyper subBN-algebra of H.*

Theorem 5.5. *Let $(H, \circledast, 0)$ be a hyper BN-algebra. Then the set $S_N := \{x \in H | x \circledast x = \{0\}\}$ is a hyper subBN-algebra of H whenever $\{0\}$ is normal in H.*

Proof. Note that $0 \in S_N$ by Remark 3.2. Thus, $S_N \neq \emptyset$. Let $x, y \in S_N$. Then $x \circledast x = \{0\}$ and $y \circledast y = \{0\}$. Let $a \in x \circledast y$. By normality of $\{0\}$, we have, $\{0\} \subseteq a \circledast a \subseteq (x \circledast y) \circledast (x \circledast y) \subseteq \{0\}$. Hence, $a \circledast a = \{0\}$. Thus, $a \in S_N$. Consequently, $x \circledast y \subseteq S_N$. Therefore, Theorem 5.1 says that S_N is a hyper subBN-algebra of H. □

6. Conclusion

We have defined a new concept of hyper algebra called hyper BN-algebras. We also successfully obtained some routine properties and gave a characterization of hyper BN-algebras. We showed some hyper BN-algebras that exhibit a certain condition to be a hyper B-algebra and a hypergroup. Finally, we developed the concept of hyper subBN-algebra and obtained some results. It is recommended to study further hyper subBN-algebras. It is also recommended to introduce the notion of ideals of a hyper BN-algebra and construct its quotient structure.

Acknowledgements

The authors thank the Department of Science and Technology (DOST) of the Philippines through the Accelerated Science and Technology Human Resource Development Program (ASTHRDP) for the financial support. Also, the authors thank Prof. Vilela, Prof. Dagondon, and Prof. Rara for their valuable comments and suggestions for the betterment of the paper. Finally, the authors would like to extend their gratitude to the anonymous referees for giving some comments for the improvement of this paper.

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Accepted: April 4, 2021