# $(p, q)$-order and $(p, q)$- lower order of entire functions of several complex variables on the basis of central index

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**Abstract.** In this paper we discuss about the growth rates of the central indices of composition of entire functions of several complex variables with their corresponding left or right factor.
**Keywords:** entire function, central index, $(p, q)$-order and $(p, q)$-lower order.

## 1. Introduction, definitions and notations

We denote complex $n$-space by $\mathbb{C}^n$ and indicate its elements (points):

$$(z_1, z_2, \ldots, z_n), (|z_1|, |z_2|, \ldots, |z_n|), (r_1, r_2, \ldots, r_n), (k_1, k_2, \ldots, k_n) \text{ etc.}$$

by their corresponding symbols $z, |z|, r, k$ etc. Throughout $\Omega = \Omega_n$ stands for a nonempty open complete $n$-circular region in $\mathbb{C}^n$(see §3, 3 of [1]) with centre at $(0, 0, \ldots, 0)$, the zero element of $\mathbb{C}^n$.

We write

$$|\Omega| = \{r : r = |z| \text{ for some } z \in \Omega\}$$

and

$$\Omega^+ = \{r : r \in |\Omega|, \text{ no } r_j = 0\}$$

and regard these as subsets of the $n$-dimensional Euclidean space $\mathbb{C}^n$. For any $r, s \in |\Omega|$ we say that

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$(i)$ $r \le s$ or $s \ge r$, if and only if $r_j \le s_j$ for $1 \le j \le n$

$(ii)$ $r < s$ or $s > r$, if and only if $r \le s$ but $r$ is not equal to $s$ and

$(iii)$ $r << s$ or $s >> r$, if and only if $r_j << s_j$ for $1 \le j \le n$.

A function $f(z)$, $z \in \mathbb{C}^n$ is said to be analytic at a point $\xi \in \mathbb{C}^n$ if it can be extended in some neighborhood of $\xi$ as an absolutely convergent power series. If we assume $\xi = (0, 0, \ldots, 0)$, then $f(z)$ has representation (see [3] and [5])

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1,k_2,\ldots\ldots,k_n} \, z_1^{k_1} \, z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k \, z^k,$$

where $k = (k_1, k_2, \ldots\ldots, k_n)$ belongs to $\mathcal{N} = \{k : k \in C^n$, each $k_j$ is rational integer$\}$ and $|k| = k_1 + k_2 + \ldots\ldots + k_n$.

Now, corresponding to an entire function $f(z)$, we define the functions (mappings): the maximum term

$$\mu(r) = \mu(r, f),$$

the maximum modulus

$$M(r) = M(r, f)$$

and the central index

$$\nu(r) = \nu(r, f) = (\nu_1(r, f), \nu_2(r, f), \ldots, \nu_n(r, f))$$

on $|\Omega|$ by (see, [3] and [4])

$$\mu(r) = \mu(r, f) = \max_{k \in \mathcal{N}} \{|a_k| \, r^k\},$$

$$M(r) = M(r, f) = \max_{|z|=r} \{|f(z)|\}$$

and

$$\nu_j(r) = \nu_j(r, f) = \begin{cases} \max[k_j : |a_k| r^k = \mu(r), & \text{if } \mu(r) > 0] \\ 0, & \text{if } \mu(r) = 0, \text{ for } 1 \le j \le n. \end{cases}$$

Also, the central index $\nu(r, f)$ for which maximum term is achieved

$$|\nu(r, f)| = \nu_1(r, f) + \nu_2(r, f) + \ldots + \nu_n(r, f).$$

We say that a mapping $f$ with domain $D$ in a euclidean space and also with range in a euclidean space is increasing (in $D$), if and only if $f(r) \le f(s)$ for any $r, s \in D$ such that $r \le s$. Then, in view of Hartogs theorem and maximum principal ([1], $p.21, p.51$), $M(r, f) = M(r_1, r_2, \ldots, r_n, f)$ is an increasing function of $r_1, r_2, \ldots, r_n$.

Krishna, J.G. ([3], Theorem 2.6) proved that $\mu(r)$ is increasing and continuous in $|\Omega|$. In the same paper Krishna, J.G. ([3], Corollary 2.9) has also proved that $\nu_j(r)$ is increasing and right continuous in $j$-th variable for $1 \le j \le n$. The following definitions are well known:

**Definition 1** ([1], p.339). *The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f(z) = f(z_1, z_2, ....., z_n)$ are defined by*

$$\rho_f = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

*and*

$$\lambda_f = \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)},$$

*where*

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, ... \text{and } \log^{[0]} x = x.$$

However, an entire function of $n$-complex variables, for which order and lower order are the same is said to be of regular growth. The function $\exp(z_1 z_2 \ldots z_n)$ is an example of regular growth of entire function of $n$-complex variables. Further, the functions which are not of regular growth is said to be of irregular growth.

Similarly, one can define the hyper order (hyper lower order), generalised order (generalised lower order) of entire function of $n$-complex variables as follows.

**Definition 2.** *The hyper order $\overline{\rho}_f$ and the hyper lower order $\overline{\lambda}_f$ of an entire function $f$ of $n$-complex variables are defined by*

$$\overline{\rho}_f = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[3]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

*and*

$$\overline{\lambda}_f = \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[3]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}.$$

**Definition 3** ([8]). *Let $l$ be an integer $\geq 1$. The generalised order $\rho_f^{[l]}$ and the generalised lower order $\lambda_f^{[l]}$ of an entire function $f$ of $n$-complex variables are defined as follows*

$$\rho_f^{[l]} = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[l+1]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}$$

*and*

$$\lambda_f^{[l]} = \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[l+1]} M(r_1, r_2, \ldots, r_n, f)}{\log(r_1 r_2 \ldots r_n)}.$$

Note that, for $l = 1$, we have the usual order (lower order) and for $l = 2$ , the hyper order (hyper lower order).

Now, in the line of Juneja, Kapoor and Bajpai [2], $(p,q)$-order and $(p,q)$-lower order of an entire function $f$ of $n$-complex variables are respectively defined as follows:

$$\rho_f(p,q) = \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M(r_1,r_2,\ldots,r_n,f)}{\log^{[q]}(r_1 r_2 \ldots r_n)} \tag{1}$$

and

$$\lambda_f(p,q) = \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p+1]} M(r_1,r_2,\ldots,r_n,f)}{\log^{[q]}(r_1 r_2 \ldots r_n)}, \tag{2}$$

where $p,q$ are any two positive integers with $p \geq q$. In paticular, if we consider $q = 1$, then the above definition is reduced to Definition 3. Further, an entire function $f$ of $n$-complex variables is said to be of regular $(p,q)$ growth if its $(p,q)$-order concides with its $(p,q)$-lower order, otherwise $f$ is said to be of irregular $(p,q)$ growth.

Recently, many authors (see [5], [6] and [7]) investigated the growth of entire functions of several complex variables in terms of their central index.

In this paper we would like to establish $(p,q)$-order and $(p,q)$-lower order of an entire function of several complex variables in term of its central index. The main purpose of this paper is to investigate the growth estimates of the central indices of composite entire functions of several complex variables as compared to their left or right factor on the basis of $(p,q)$-order ($(p,q)$-lower order), where $p,q$ are any two positive integers with $p \geq q$.

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** ([3])**.** *Let $p,r \in |\Omega|$ and let $\mu(p)$ and $\mu(r)$ be both positive. Then, the line integral,*

$$I = \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j$$

*taken over any connected polygon in $|\Omega|$ with sides parallel to the axes and from $p$ to $r$,*

$(i)$ *exists,*

$(ii)$ *is independent of the polygon and*

$(iii)$ *is such that $\log \mu(r) = \log \mu(p) + I$.*

**Lemma 2** ([3])**.** *Let $r \in |\Omega|$. Let $p \in |\mathbb{C}^n|$ and be such that $p >> (1,1,\ldots,1)$, while $pr = (p_1r_1, p_2r_2, \ldots, p_nr_n)$ still $\in |\Omega|$.*

Let

$$N_j = \max_{r \leq t \leq pr} \nu_j(t), \quad \text{for } 1 \leq j \leq n.$$

Then

(i) $\mu(r) \leq M(r) \leq \mu(r) \prod_{j=1}^{n} \left[ N_j + \frac{p_j}{p_j - 1} \right]$;

(ii) $\mu(r) = M(r)$, if and only if the series $\sum_{|k|=0}^{\infty} a_k r^k$ has at most one non vanishing term;

(iii) the last relation in (i) is an equality if and only if $\mu(r) = 0$.

## 3. Theorems

In this section we present the main results of the paper.

**Theorem 1.** *Let $f(z)$ be an entire function of $n$-complex variables with $(p,q)$-order $\rho_f(p,q)$ and $(p,q)$-lower order $\lambda_f(p,q)$, where $p,q$ are positive integers with $p \geq q$. If $\nu(r,f) = \nu(r_1, r_2, \ldots, r_n, f)$ be the central index of $f$, then*

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log^{[q]} (r_1 r_2 \ldots r_n)} = \rho_f(p,q) \tag{3}$$

*and*

$$\liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, f)|}{\log^{[q]} (r_1 r_2 \ldots r_n)} = \lambda_f(p,q). \tag{4}$$

**Proof.** Setting

$$f(z) = \sum_{k=(0,0,\ldots,0)}^{\infty} a_{k_1,k_2,\ldots,k_n} z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n} = \sum_{|k|=0}^{\infty} a_k z^k. \qquad \square$$

By Lemma 1, we see that the maximum term $\mu(r)$ of $f$ satisfies

$$\log \mu(r) = \log \mu(p) + \int_p^r \sum_{j=1}^{n} \frac{\nu_j(x)}{x_j} dx_j. \tag{5}$$

Since Krishna, J.G. ([3], Corollary 2.9) proved that $\nu_j(r)$ is increasing and right continuous in $j$-th variable for $1 \leq j \leq n$. Therefore, for any $p, r \in |\Omega|$ such that $\mu(r) > 0$ and $p >> (1, 1, \ldots, 1)$, we get for $1 \leq j \leq n$,

$$\nu_j(r) \leq \frac{1}{\log p_j} \int_p^r \nu_j(r_1, \ldots, r_{j-1}, \ldots, r_n) \frac{dx_j}{x_j}. \tag{6}$$

From (5) and (6) we get

$$\log \mu(r) \geq \log \mu(p) + \sum_{j=1}^{n} \nu_j(r) \log p_j. \tag{7}$$

By Lemma 2, we have

$$\mu(r,f) \le M(r,f). \tag{8}$$

It follows from (7) and (8) that

$$\sum_{j=1}^{n} \nu_j(r)\log p_j \le \log M(r,f) + C_1. \tag{9}$$

As $p >> (1,1,\dots,1)$ i.e., $p = (p_1,p_2,\dots,p_n) >> (1,1,\dots,1)$, choosing $p_j = 2$ for $1 \le j \le n$, we get

$$\begin{aligned}
&\sum_{j=1}^{n} \nu_j(r)\log 2 \le \log M(r,f) + C_1\\
&\Rightarrow |\nu(r,f)|\log 2 \le \log M(r,f) + C_1\\
&\Rightarrow \log^{[p]}|\nu(r,f)| + \log^{[p+1]} 2 \le \log^{[p+1]} M(r,f) + C_2,
\end{aligned}$$

where $C_j(>0)$ $(j=1,2)$ are suitable constants. By this and (1), we have

$$\limsup_{r_1,r_2,\dots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\dots,r_n,f)|}{\log^{[q]}(r_1r_2\dots r_n)} \le \rho_f\,(p,q)\,. \tag{10}$$

On the other hand, by choosing $p_j = 2$ for $1 \le j \le n$ i.e., $p = (2,2,\dots,2)$ in (i) of Lemma 2, we have

$$M(r,f) \le \mu(r,f)\prod_{j=1}^{n}[N_j+2]\,,$$

$$\text{where } N_j = \max_{r\le t\le pr} \nu_j(t) \text{ for } 1 \le j \le n.$$

$$\Rightarrow M(r,f) \le \left|a_{\nu(r,f)}\right| r^{\nu(r,f)} \prod_{j=1}^{n}[N_j+2]\,. \tag{11}$$

Since $\{|a_k|\}$ is bounded, from (11) we get

$$\begin{aligned}
\log M(r,f) &\le \sum_{j=1}^{n}\nu_j(r)\log r_j + \sum_{j=1}^{n}\log N_j + C_3\\
&\le \sum_{j=1}^{n}|\nu(r,f)|\log r_j + \sum_{j=1}^{n}\log N_j + C_3\\
&\le |\nu(r,f)|\log(r_1r_2\dots r_n) + \log(N_1N_2\dots N_n) + C_3\\
&\Rightarrow \log^{[p+1]} M(r,f) \le \log^{[p]}|\nu(r,f)| + \log^{[p+1]}(r_1r_2\dots r_n)\\
&+ \log^{[p+1]}(N_1N_2\dots N_n) + C_4,
\end{aligned}$$

where $C_j(>0)(j=3,4)$ are suitable constants. By this and (1), we get

$$\rho_f(p,q) \leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)}. \tag{12}$$

By (10) and (12), it follows that

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)} = \rho_f(p,q).$$

Similarly, we can prove that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,f)|}{\log^{[q]}(r_1r_2\ldots r_n)} = \lambda_f(p,q).$$

**Theorem 2.** *If $f$ and $g$ are entire functions $n$-complex variables such that $0<\lambda_{fog}(p,q)\leq\rho_{fog}(p,q)<\infty$ and $0<\lambda_g(m,q)\leq\rho_g(m,q)<\infty$, then*

$$\begin{aligned}\frac{\lambda_{fog}(p,q)}{\rho_g(m,q)} &\leq \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}\\ &\leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)},\end{aligned}$$

*where $p,q,m$ are positive integers with $p\geq q\geq m$.*

**Proof.** Using Theorem 1 for the entire function $fog$ and $g$ respectively, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r_1,r_2,\ldots,r_n$ that

$$\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)| \geq (\lambda_{fog}(p,q)-\varepsilon)\log^{[q]}(r_1r_2\ldots r_n) \tag{13}$$

and

$$\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)| \leq (\rho_g(m,q)+\varepsilon)\log^{[q]}(r_1r_2\ldots r_n). \tag{14}$$ □

Now, from (13) and (14) it follows for all sufficiently large values of $r_1,r_2,\ldots,r_n$,

$$\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\lambda_{fog}(p,q)-\varepsilon}{\rho_g(m,q)+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\lambda_{fog}(p,q)}{\rho_g(m,q)}. \tag{15}$$

Again, for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity

$$\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)| \leq (\lambda_{fog}(p,q)+\varepsilon)\log^{[q]}(r_1r_2\ldots r_n) \tag{16}$$

and for all large values of $r_1, r_2, \ldots, r_n$

$$\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)| \geq (\lambda_g(m,q) - \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{17}$$

So, combining (16) and (17) we get for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\lambda_{fog}(p,q) + \varepsilon}{\lambda_g(m,q) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{18}$$

Also, for a sequnce of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, fog)| \leq (\lambda_g(m,q) + \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{19}$$

Now, from (13) and (19) we obtain for a sequence of values of each of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \geq \frac{\lambda_{fog}(p,q) - \varepsilon}{\lambda_g(m,q) + \varepsilon}. \tag{20}$$

Choosing $\varepsilon \to 0$ we get that

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \geq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{21}$$

Also, for all large values of $r_1, r_2, \ldots, r_n$,

$$\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)| \leq (\rho_{fog}(p,q) + \varepsilon) \log^{[q]}(r_1 r_2 \ldots r_n). \tag{22}$$

So, from (17) and (22) it follows for all large values of $r_1, r_2, \ldots, r_n$,

$$\frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\rho_{fog}(p,q) + \varepsilon}{\lambda_g(m,q) - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[p]} |\nu(r_1, r_2, \ldots, r_n, fog)|}{\log^{[m]} |\nu(r_1, r_2, \ldots, r_n, g)|} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)}. \tag{23}$$

Thus, the theorem follows from $(15), (18), (21)$ and $(23)$.

**Theorem 3.** *Let $f$ and $g$ are entire functions of $n$-complex variables such that $0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$. Then*

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)} \leq \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|},$$

*where $p,q,m$ are positive integers with $p \geq q \geq m$.*

**Proof.** Using Theorem 1 for the entire function $g$, we get for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity that

$$\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)| \geq (\rho_g(m,q) - \varepsilon)\log^{[q]}(r_1 r_2 \ldots r_n). \tag{24}$$ □

Now, from (22) and (24) it follows for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity,

$$\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)+\varepsilon}{\rho_g(m,q)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \leq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{25}$$

Again, for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity

$$\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)| \geq (\rho_{fog}(p,q) - \varepsilon)\log^{[q]}(r_1 r_2 \ldots r_n). \tag{26}$$

So, combining (14) and (26) we get for a sequence of values of each of $r_1,r_2,\ldots,r_n$ tending to infinity,

$$\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\rho_{fog}(p,q)-\varepsilon}{\rho_g(m,q)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|} \geq \frac{\rho_{fog}(p,q)}{\rho_g(m,q)}. \tag{27}$$

Thus, the theorem follows from (25) and (27) .

The following theorem is a natural consequence of Theorem 2 and Theorem 3.

**Theorem 4.** *If $f$ and $g$ are entire functions of $n$-complex variables such that $0<\lambda_{fog}(p,q)\leq\rho_{fog}(p,q)<\infty$ and $0<\lambda_{g}(m,q)\leq\rho_{g}(m,q)<\infty$, then*

$$\liminf_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|}\leq\min\left\{\frac{\lambda_{fog}(p,q)}{\lambda_{g}(m,q)},\frac{\rho_{fog}(p,q)}{\rho_{g}(m,q)}\right\}$$
$$\leq\max\left\{\frac{\lambda_{fog}(p,q)}{\lambda_{g}(m,q)},\frac{\rho_{fog}(p,q)}{\rho_{g}(m,q)}\right\}\leq\limsup_{r_1,r_2,\ldots,r_n\to\infty}\frac{\log^{[p]}|\nu(r_1,r_2,\ldots,r_n,fog)|}{\log^{[m]}|\nu(r_1,r_2,\ldots,r_n,g)|},$$

*where $p,q,m$ are positive integers with $p\geq q\geq m$.*

**Remark 1.** If we take $0<\lambda_{f}(m,q)\leq\rho_{f}(m,q)<\infty$ instead of $0<\lambda_{g}(m,q)\leq\rho_{g}(m,q)<\infty$ and the other conditions remain the same, then also Theorem 2, Theorem 3 and Theorem 4 hold with $g$ replaced by $f$ in the denominator.

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