Some separation axioms on topological B-algebra and the quotient B-topological space

Katrina E. Belleza*
Department of Computer
Information Sciences and Mathematics
Faculty of Mathematics
University of San Carlos
Cebu City, Cebu
Philippines
kebelleza@usc.edu.ph

Jocelyn P. Vilela
Department of Mathematics and Statistics
Faculty of Mathematics
Mindanao State University-Iligan Institute of Technology
Iligan City, Lanao del Norte
Philippines
jocelyn.vilela@g.msuiit.edu.ph

Abstract. This paper provides some separation axioms on a topological B-algebra. In particular, a characterization of a discrete topological B-algebra is presented. Furthermore, this paper introduces the quotient B-topological space and presents some properties. Moreover, this paper gives some conditions which under a topological B-algebra (resp. quotient topological B-algebra), the quotient B-topological space (resp. B-topological space) have the topological properties T_0, T_1, and Hausdorff.

Keywords: B-topological space, topological B-algebra, quotient B-topological space, quotient topological B-algebra.

1. Introduction

In 1998, D.S. Lee and D.N. Ryu [7] defined a topological BCK-algebra and found some properties of this structure. On the following year, Y.B. Jun, X.L. Xin, and D.S. Lee [5] introduced the notion of topological BCI-algebras and gave a characterization in terms of neighborhoods. In 2002, J.Neggers and H.S. Kim [8] introduced and investigated B-algebras. In 2015, N. Kouhestani and S. Mehrshad [6] studied separation axioms and connected properties on topological quotient BCK-algebras. In 2019, an initial study on topological B-algebra was conducted by N.C. Gonzaga Jr. [4] where he investigated some properties of a topological B-algebra. Moreover, he characterized a topological B-algebra with respect to neighborhoods.

∗. Corresponding author

2. Preliminaries

Definition 2.1 ([8]). *A B-algebra is a triple $(X, *, 0)$ where X is a nonempty set with a constant 0 and a binary operation "$*$" satisfying the following axioms for all x, y, z in X:*

$$\text{(B1) } x * x = 0; \quad \text{(B2) } x * 0 = x;; \quad \text{(B3) } (x * y) * z = x * [z * (0 * y)].$$

Let us denote a B-algebra $(X, *, 0)$ simply as X.

Example 2.2 ([9]). Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then $(X, *, 0)$ is a B-algebra.

Definition 2.3 ([4]). *Let A and B be nonempty subsets of a B-algebra X. The product of A and B, denoted by $A * B$, is given by $A * B = \{a * b | a \in A, b \in B\}$.*

Lemma 2.4 ([8]). *Let $(X, *, 0)$ be a B-algebra. Then for any $x, y \in X$,*

(i) $x * y = 0$ *implies* $x = y$;

(ii) $0 * x = 0 * y$ *implies* $x = y$;

(iii) $0 * (0 * x) = x$.

Definition 2.5 ([10]). *Let X be a B-algebra. A nonempty subset N of X is called a subalgebra of X if $x * y \in N$ for any $x, y \in N$.*

The following remark follows directly from Definition 2.5 and axiom (B1) of a B-algebra.

Remark 2.6. Suppose X is a B-algebra and S a subalgebra of X. Then $0 \in S$.

Suppose X is a B-algebra and I a normal subalgebra of X. The relation "$\cong^I$" defined by $x \cong^I y$ if and only if $x * y, y * x \in I$ is a congruence relation on X for any $x, y \in X$. That is, $\cong^I$ is an equivalence relation and for each $a, b, x, y \in X$, if $x \cong^I y$ and $a \cong^I b$, then $a * x \cong^I b * y$. Let $I_x = \{y : y \cong^I x\}$ denote the equivalence class of x and $X/I = \{I_x : x \in X\}$. Then X/I is a B-algebra called the *quotient B-algebra* under the binary operation given by $I_x * I_y = I_{x*y}$ [3]. Define a mapping $\pi_I : X \to X/I$ by $\pi_I(x) = I_x$ for any $x \in X$. Then the mappping π_I is a B-epimorphism called the *canonical epimorphism* of B-algebras [9].

Definition 2.7 ([9])**.** Let X be a B-algebra. A nonempty subset S of X is said to be *normal* in X if for any $x * y, a * b \in S$, $(x * a) * (y * b) \in S$.

Theorem 2.8 ([10])**.** *Let N be a subalgebra of a B-algebra X. Then the following statements are equivalent:*

(*i*) *N is a normal subalgebra;*

(*ii*) *if $x \in X$ and $y \in N$, then $x * (x * y) \in N$.*

Definition 2.9 ([3])**.** Let X and Y be B-algebras. A mapping $\phi : X \to Y$ is called a *B-homomorphism* from X into Y if $\phi(x * y) = \phi(x) * \phi(y)$ for any $x, y \in X$. A B-homomorphism ϕ is called a *B-monomorphism, B-epimorphism,* or *B-isomorphism* if ϕ is one-to-one, onto, or a bijection, respectively.

Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) Each union of members of τ is also a member of τ;

(ii) Each finite intersection of members of τ is also a member of τ; and

(iii) $\emptyset$ and X are members of τ.

A couple (X, τ) consisting of a set X and a topology τ in X is called a *topological space*. We also say "τ is the topology of the space X". The members of τ are called *open sets* of (X, τ). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X) and f is called *open* if the image of each open set in X is open in Y (see, [1]).

Let (X, τ) be a topological space and $A \subset X$. By a *neighborhood of an element* x in X (denoted as $U(x)$) is meant any open set (that is, member of τ) containing x. The *interior* $Int(A)$ of A is the largest open set contained in A, that is, $\text{Int}(A) = \bigcup\{U | U \in \tau, U \subset A\}$. A point a is an interior point of A if $a \in \text{Int}(A)$, that is, there exists $U(a) \in \tau$ such that $U(a) \subset A$. A is open if and only if $\text{Int}\ (A) = A$. A set $Y \subset X$ is a *closed set* in X if its complement is open. A point $x \in X$ is *adherent* to Y if each neighborhood of x contains at least one point of Y. The set $\overline{Y} = \{x \in X | \forall U(x), U(x) \cap Y \neq \emptyset\}$ of all points in X adherent to Y is called the *closure* of Y (see, [1]).

Definition 2.10 ([1])**.** *Let $Y = (Y, \tau)$ be a space. Y is called a T_0-space if for each pair of distinct points, at least one has a neighborhood not containing the other. Y is called a T_1-space if for each pair of distinct points, each one has a neighborhood not containing the other. Y is Hausdorff (or T_2-space) if each two distinct points have nonintersecting neighborhoods, that is, if $p \neq q$, there exists $U(p), U(q)$ such that $U(p) \cap U(q) = \emptyset$.*

The next remark follows directly from Definition 2.10.

Remark 2.11. Every Hausdorff space is a T_1-space and every T_1-space is a T_0 space.

Lemma 2.12 ([1]). *Let Y be a Hausdorff space. Then $\bigcap_{i\in\mathcal{A}} U_i(p) = \{p\}$ for any $p \in Y$.*

Definition 2.13 ([1]). In the set X, Let $\mathcal{D} = \mathcal{P}(X)$. Then $\mathcal{D}$ is called the *discrete topology* where every set is an open set having basis of the form $\{\{x\}|x \in X\}$. Hence, (X, τ) is called a *discrete space.*

Remark 2.14. We note that a discrete space is Hausdorff while the converse does not necessarily hold. Consider the Euclidean 1-space (E^1, τ). An open set in E^1 is an open interval denoted as $B(x; r)$ where $B(x; r) = \{y : |y - x| < r\}$. (E^1, τ) is a Hausdorff space but is not a discrete space.

Definition 2.15 ([1]). *Let $\{Y_\alpha|\alpha \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology for Y_α. The cartersian product topology in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $< U_\beta >= \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.*

Definition 2.16 ([4]). *Let X be a B-algebra. A topology τ furnished on X is called a B-topology on X. A B-topological space (X, τ) is called a topological B-algebra if τ is a B-topology on X and the binary operation $* : X \times X \to X$ is continuous, where $X \times X$ is furnished by the Cartesian product topology.*

Theorem 2.17 ([4]). *Let $X = (X, *, 0)$ be a B-algebra and τ a B-topology on the set X. Then (X, τ) is a topological B-algebra if and only if for all x, y in X and for every neighborhood W of $x * y$, there are neighborhoods U and V of x and y, respectively, such that $U * V \subseteq W$.*

3. Topological B-algebra

Example 3.1. Let $X = \{0, a, b, c\}$ and $Y = \{0, 1, 2\}$ with binary operations "$*_1$ and "$*_2$, respectively, be defined on the Cayley tables provided.

$*_1$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$*_2$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

(i) $(X; *_1, 0)$ is a B-algebra (see [2]). Let $\tau = \big\{X, \emptyset, \{0, c\}, \{a, b\}\big\}$. Then τ is a B-topology on X and by routine calculations, (X, τ) is a topological B-algebra.

(ii) $(Y; *_2, 0)$ is a B-algebra (see [8]). Let $\tau = \big\{\{0, 2\}, \{1, 2\}, \{2\}, Y, \emptyset\big\}$. Then τ is a B-topology on Y. Note that for $0 * 1 = 2 \in \{0, 2\}$, $\{0, 2\} * \{1, 2\} = \{0, 2\} * Y = Y * \{1, 2\} = Y * Y = Y \nsubseteq \{0, 2\}$. Hence, (Y, τ) is not a topological B-algebra.

The following results involve some separation axioms on a topological B-algebra. Moreover, the succeeding theorem is a characterization of a discrete topological B-algebra.

Theorem 3.2. *Let X be a topological B-algebra. Then X is discrete if and only if $\{0\}$ is an open set.*

Proof. If X is discrete, $\{0\}$ is clearly an open set. Suppose $\{0\}$ is an open set in a topological B-algebra X. Note that by (B1), $x * x = 0 \in \{0\}$ for all $x \in X$. By Theorem 2.17, there exists $U(x), V(x) \in \tau$ such that $U(x) * V(x) \subset \{0\}$. Let $W = U(x) \cap V(x) \in \tau$. Then $W * W \subset U(x) * V(x) \subset \{0\}$. Since $x \in W$, it follows that $W * \{x\} = \{0\}$. By Lemma 2.4 (i), $W = \{x\}$. $\square$

The next theorem is a characterization of a topological B-algebra provided any B-topological space followed by a corollary under the condition that a B-topological space is discrete.

Theorem 3.3. *Suppose X is a B-topological space such that for any $x \in X$, $\{x\} \in \tau$. Then X is a topological B-algebra.*

Proof. Suppose X is a B-topological space such that for any $x \in X$, $\{x\} \in \tau$. Note that for any $x, y \in X$ and $U(x * y) \in \tau$, there exist $\{x\}, \{y\} \in \tau$ such that $\{x\} * \{y\} = \{x * y\} \subseteq U(x * y)$. Therefore, X is a topological B-algebra. $\square$

Corollary 3.4. *If a B-topological space X is a discrete space, then X is a topological B-algebra.*

In any topological space, every discrete space is Hausdorff while the converse is not always true as discussed in Remark 2.14. However in a topological B-algebra, the converse holds. This is stated in the next corollary which follows from Remark 2.14, Theorem 3.2, and Lemma 2.12.

Corollary 3.5. *Suppose X is a topological B-algebra such that $\bigcap_{i \in \mathcal{A}} U_i(0)$ is an open set. Then X is Hausdorff if and only if it is discrete.*

Theorem 3.6. *Let X be a topological B-algebra. Then the following are equivalent:*

(i) X is Hausdorff;

(ii) X is T_1;

(iii) X is T_0.

Proof. In view of Remark 2.11, it remains to show that (iii) $\Longrightarrow$ (i). Suppose X is a T_0-space and let $x \neq y$ for any $x, y \in X$. Then $x * y \neq 0$ or $y * x \neq 0$. Without loss of generality, consider $x * y \neq 0$. Since X is T_0, there exists $U(x * y)$ such that $0 \notin U(x * y)$. Since X is a topological B-algebra, there exists $V(x), V(y) \in \tau$ such that $V(x) * V(y) \subset U(x * y)$.

Claim: $V(x) \cap V(y) = \emptyset$. Assume on the contrary that $V(x) \cap V(y) \neq \emptyset$. Then there exists $z \in V(x) \cap V(y)$. By (B1), $0 = z * z \in V(x) \cap V(y) \subset U(x * y)$. This implies that $0 \in U(x * y)$ which is a contradiction. Hence, the claim holds. Therefore, X is Hausdorff. □

4. Quotient B-topological space

Henceforth, we shall assume that I is a normal subalgebra of a B-algebra X.

Definition 4.1. Let X be a B-algebra, I a subalgebra of X, and $\tilde{\tau}$ a topology on X/I. Then $\tilde{\tau}$ is called a *quotient B-topology* on X/I and the couple $(X/I, \tilde{\tau})$ is called a *quotient B-topological space*. If $(X/I, \tilde{\tau})$ is a topological B-algebra, we call the couple a *quotient topological B-algebra*.

The neighborhood of $I_x \in X/I$ shall be denoted by $\tilde{U}(I_x) \in \tilde{\tau}$ and a quotient B-topological space $(X/I, \tilde{\tau})$ shall be denoted as X/I unless otherwise specified.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then X is a B-algebra [2]. Let $I = \{0, a\}$. By routine calculations, I is a subalgebra of X. By Theorem 2.8, I is a normal subalgebra of X. Moreover, $X/I = \{I, I_b\}$ where $I = I_0 = I_a, I_b = I_c = \{c, b\}$. Let $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$. Then $\tilde{\tau}$ is a quotient B-topology on X/I and $(X/I, \tilde{\tau})$ is a quotient B-topological space. By Theorem 3.3, X/I is a quotient topological B-algebra.

Example 4.3. Consider the quotient B-topological space $X/I = \{I, I_b\}$ with $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$ from Example 4.2. Then, $\tilde{U}(I_0) = \tilde{U}(I_a) \in \{X/I, \{I\}\}$ and $\tilde{U}(I_b) = \tilde{U}(I_c) \in \{X/I, \{I_b\}\}$ are neighborhoods of I_0 and I_b, respectively.

Definition 4.4. Let X be a B-topological space and X/I a quotient B-topological space. We say that X satisfies *open condition* if for any subalgebra I, the canonical epimorphism π_I is an open map.

The next example shows that there exist a topological B-algebra that does not satisfy the open condition and π_I not continuous in which X/I is not a topological B-algebra.

Example 4.5. Consider the topological B-algebra $X = \{0, a, b, c\}$ with $\tau = \{X, \emptyset, \{0, c\}, \{a, b\}\}$ in Example 3.1. Let $I = I_0 = \{0\}$. Clearly, I is a normal subalgebra. Then $I_a = \{a\}, I_b = \{b\}, I_c = \{c\}$ so that $X/I = \{I, I_a, I_b, I_c\}$. Let $\tilde{\tau} = \{X/I, \emptyset, \{I, I_a, I_b\}, \{I_c\}\}$. Then $\tilde{\tau}$ is a quotient B-topology. Note that there exists $\{I_c\} \in \tilde{\tau}$ such that $\pi_I^{-1}(\{I_c\}) = \{c\} \notin \tau$ which implies that π_I is not continuous and there exists $\{0, c\} \in \tau$ such that $\pi_I(\{0, c\}) = \{I_0, I_c\} \notin \tilde{\tau}$ which implies that π is not open. Now, for $I_a, I_b \in X/I$ and for $I_a * I_b = I_{a*b} = I_c \in \{I_c\} = \tilde{U}(I_c)$, $\tilde{U}(I_a) * \tilde{U}(I_b) = X/I \nsubseteq \tilde{U}(I_c)$ for all $\tilde{U}(I_a), \tilde{U}(I_b) \in \tilde{U}$. This implies that X/I is not a topological B-algebra.

However, if the canonical epimorphism π_I is open and continuous on a topological B-algebra, then the quotient B-topological space is a topological B-algebra as shown in the next result.

Theorem 4.6. *If X is a topological B-algebra that satisfies open condition with π_I continuous, the quotient B-topological space X/I is a topological B-algebra. In this case, we say that X/I is a quotient topological B-algebra.*

Proof. Suppose X is a topological B-algebra that satisfies open condition with π_I continuous. Let $I_x, I_y \in X/I$, $z = x * y$, and $\tilde{W}(I_z) \in \tilde{\tau}$. Let $\tilde{W} = \tilde{W}(I_z)$. Since $I_z \in \tilde{W}$ and π_I is surjective, there exists $z \in X$ such that $\pi_I(z) = I_z \in \tilde{W}$. This implies that $x * y = z \in \pi_I^{-1}(\tilde{W})$. Since $\tilde{W} \in \tilde{\tau}$ and π_I^{-1} is continuous, $\pi_I^{-1}(\tilde{W}) \in \tau$. That is, $\pi_I^{-1}(\tilde{W})$ is an open set containing $x * y$. Since X is a topological B-algebra, there exist $U(x), U(y) \in \tau$ such that $U(x) * U(y) \subseteq \pi_I^{-1}(\tilde{W})$ by Theorem 2.17. Since $x \in U(x)$ and π_I is open, $\pi_I(x) \in \pi_I\big(U(x)\big) \in \tilde{\tau}$ or $I_x \in \pi_I\big(U(x)\big) \in \tilde{\tau}$. Similarly, $I_y \in \pi_I\big(U(y)\big) \in \tilde{\tau}$. That is, $\pi_I\big(U(x)\big)$ and $\pi_I\big(U(y)\big)$ are open sets containing I_x and I_y, respectively.
Claim: $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \pi_I\big(U(x) * U(y)\big)$.

Since π_I is a homomorphism, $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \{\pi_I(a) | a \in \pi_I\big(U(x)\big)\} * \{\pi_I(b) | b \in \pi_I\big(U(y)\big)\} = \{\pi_I(a) * \pi_I(b) | a \in U(x), b \in U(y)\} = \{\pi_I(a * b) | a \in U(x), b \in U(y)\} = \pi_I\big(U(x) * U(y)\big)$. This proves the claim.

By the claim and since $U(x) * U(y) \subseteq \pi_I^{-1}(\tilde{W})$, $\pi_I\big(U(x)\big) * \pi_I\big(U(y)\big) = \pi_I\big(U(x) * U(y)\big) \subseteq \pi_I\big(\pi_I^{-1}(\tilde{W})\big) = \tilde{W}$. Therefore, $(X/I, \tilde{\tau})$ is a topological B-algebra. □

The next example shows that there exists a quotient topological B-algebra X/I where X does not satisfy open condition and π_I not continuous in which X is not a topological B-algebra.

Example 4.7. Consider the quotient topological B-algebra $X/I = \{I, I_b\}$ with $\tilde{\tau} = \{X/I, \emptyset, \{I\}, \{I_b\}\}$ where $X = \{0, a, b, c\}$ from Example 4.2. Let $\tau = \{X, \emptyset, \{0, a, b\}, \{c\}\}$. Then τ is a B-topology on X. Note that there exists $\{0, a, b\} \in \tau$ such that $\pi_I(\{0, a, b\}) = \{I_a, I_b\} \notin \tilde{\tau}$ which implies that π_I is not open and there exists $\{I_b\} \in \tilde{\tau}$ such that $\pi_I^{-1}(\{I_b\}) = \{b\} \notin \tau$ which implies that π_I is not continuous. Now, for $b * c = a \in \{0, a, b\}$, $\{0, a, b\} * \{c\} = \{c, b, a\} \nsubseteq \{0, a, b\}$, $\{0, a, b, \} * X = X * \{c\} = X * X = X \nsubseteq \{0, a, b\}$. Hence, $U(b) * U(c) \nsubseteq \{0, a, b\}$ for all $U(b), U(c) \in \tau$. Therefore, X is not a topological B-algebra.

However, provided a B-topological space satisfying open condition with π_I continuous and X/I a topological B-algebra for any subalgebra I of X, then X is also a topological B-algebra which is presented in the next theorem.

Theorem 4.8. *Suppose X is a B-topological space satisfying open condition with π_I continuous. If X/I is a topological B-algebra, then X is also a topological B-algebra.*

Proof. Suppose X is a B-topological space satisfying open condition with π_I continuous and X/I a topological B-algebra. Let $x, y \in X$ and $U(x * y) \in \tau$. Since $x*y \in U(x*y)$ and π_I is open, $I_x * I_y = I_{x*y} = \pi_I(x*y) \in \pi_I\big(U(x*y)\big) \in \tilde{\tau}$. This imples that $\pi_I\big(U(x*y)\big)$ is an open set containing $I_x * I_y$. By Theorem 2.17, there exist $\tilde{U}(I_x), \tilde{U}(I_y) \in \tilde{\tau}$ such that $\tilde{U}(I_x) * \tilde{U}(I_y) \subseteq \pi_I\big(U(x*y)\big)$. Since $I_x \in \tilde{U}(I_x)$ and π_I continuous, $x = \pi_I^{-1}(I_x) \in \pi_I^{-1}\big(\tilde{U}(I_x)\big) \in \tau$. Hence, $\pi_I^{-1}\big(\tilde{U}(I_x)\big)$ is an open set containing x. Similarly, $\pi_I^{-1}\big(\tilde{U}(I_y)\big)$ is an open set containing y.
Claim: $\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big)$.

$\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \{a|\pi_I(a) \in \tilde{U}(I_x)\} * \{b|\pi_I(b) \in \tilde{U}(I_y)\} = \{a * b|\pi_I(a) \in \tilde{U}(I_x), \pi_I(b) \in \tilde{U}(I_y)\} = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big)$. This proves the claim. Hence, $\pi_I^{-1}\big(\tilde{U}(I_x)\big) * \pi_I^{-1}\big(\tilde{U}(I_y)\big) = \pi_I^{-1}\big(\tilde{U}(I_x) * \tilde{U}(I_y)\big) \subseteq \pi_I^{-1}(\pi_I\big(U(x*y)\big)) \subseteq U(x*y)$. Therefore, X is a topological B-algebra. □

Theorem 4.9. *Suppose X is a B-topological space and I a subalgebra of X. Define $\tilde{\tau}' = \{O \subseteq X/I | \pi^{-1}(O) \in \tau\}$ where π is the canonical epimorphism. Then $\tilde{\tau}'$ is a quotient B-topology on X/I.*

Proof. Suppose X is a B-topological space and I a subalgebra of X. Clearly, X/I and $\emptyset \in \tilde{\tau}'$. Let $O_1, O_2 \in \tilde{\tau}'$. Then $\pi^{-1}(O_1), \pi^{-1}(O_2) \in \tau$. Hence, $\pi^{-1}(O_1) \cap \pi^{-1}(O_2) \in \tau$.
Claim 1: $\pi^{-1}(O_1 \cap O_2) = \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$.

Suppose $x \in \pi^{-1}(O_1 \cap O_2)$. Then $\pi(x) \in O_1 \cap O_2$. That is, $\pi(x) \in O_1$ and $\pi(x) \in O_2$. Hence, $x \in \pi^{-1}(O_1)$ and $x \in \pi^{-1}(O_2)$. That is, $x \in \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$ implying that $\pi^{-1}(O_1 \cap O_2) \subseteq \pi^{-1}(O_1) \cap \pi^{-1}(O_2)$. The converse follows accordingly. This proves Claim 1.

By Claim 1, $\pi^{-1}(O_1 \cap O_2) \in \tau$. Hence, $O_1 \cap O_2 \in \tilde{\tau}'$. Next, let $\{O_i\}_{i \in \mathcal{A}}$ be an arbitrary family of sets in $\tilde{\tau}'$. Then for all $i \in \mathcal{A}$, $\pi^{-1}(O_i) \in \tau$. Hence, $\bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i) \in \tau$.
Claim 2: $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) = \bigcap_{i \in \mathcal{A}} \pi^{-1}(O_i)$.

Suppose $x \in \pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i)$. Then $\pi(x) \in \bigcup_{i \in \mathcal{A}} O_i$. Hence, $\pi(x) \in O_j$ for some $j \in \mathcal{A}$. This implies that $x \in \pi^{-1}(O_j)$ for some $j \in \mathcal{A}$. That is, $x \in \bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i)$ implying that $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) \subseteq \bigcup_{i \in \mathcal{A}} \pi^{-1}(O_i)$. The converse follows accordingly. This proves Claim 2.

By Claim 2, $\pi^{-1}(\bigcup_{i \in \mathcal{A}} O_i) \in \tau$. That is, $\bigcup_{i \in \mathcal{A}} O_i \in \tilde{\tau}'$. Therefore, $\tilde{\tau}'$ is a quotient B-topology on X/I. □

Theorem 4.10. *Suppose X/I is a quotient B-topological space. Define $\tau' = \{O \subseteq X | \pi(O) \in \tilde{\tau}\}$ where π is the canonical epimorphism. Then τ' is a B-topology on X.*

Proof. Suppose X/I is a quotient B-topological space. Clearly, X and $\emptyset \in \tau'$. Let $O_1, O_2 \in \tau'$. Then $\pi(O_1), \pi(O_2) \in \tilde{\tau}$.
Claim: $\pi(O_1 \cap O_2) = \pi(O_1) \cap \pi(O_2)$.

$\pi(O_1 \cap O_2) = \{\pi(x) | x \in O_1 \cap O_2\} = \{\pi(x) | x \in O_1 \text{ and } O_2\} = \{\pi(x) | x \in O_1\} \cap \{\pi(x) | x \in O_2\} = \pi(O_1) \cap \pi(O_2)$. This proves the claim. By the claim, $\pi(O_1 \cap O_2) = \pi(O_1) \cap \pi(O_2) \in \tilde{\tau}$. This implies that $O_1 \cap O_2 \in \tau'$. Now, suppose $O_i \in \tau'$ for all $i \in \mathcal{A}$. Then $\pi(O_i) \in \tilde{\tau}$ for each $i \in \mathcal{A}$.
Claim: $\pi(\bigcup_{i \in \mathcal{A}} O_i) = \bigcup_{i \in \mathcal{A}} \pi(O_i)$.

Suppose $z \in \pi(\bigcup_{i \in \mathcal{A}} O_i)$. Then $z = \pi(x)$ for some $x \in \bigcup_{i \in \mathcal{A}} O_i$. Then $x \in O_j$ for some $j \in \mathcal{A}$. Consequently, $z \in \pi(O_j)$ which implies that $z \in \bigcup_{i \in \mathcal{A}} \pi(O_i)$. The converse follows accordingly. This proves the claim. By the claim, $\pi(\bigcup_{i \in \mathcal{A}} O_i) = \bigcup_{i \in \mathcal{A}} \pi(O_i) \in \tilde{\tau}$. Hence, $\bigcup_{i \in \mathcal{A}} O_i \in \tau'$. Therefore, τ' is a B-topology on X. □

Note that in Examples 4.5 and 4.7, the canonical epimorphism π_I is not continuous and X does not satisfy open condition as shown in the examples. However, the next corollaries state that π_I is continuous with respect to $\tilde{\tau}'$ and X satisfies open condition with respect to τ'. These corollaries follow directly from Theorems 4.9 and 4.10.

Corollary 4.11. *Suppose X is a B-topological space and X/I a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. Then the canonical epimorphism π_I is a continuous map.*

Corollary 4.12. *Suppose X is a B-topological space with τ' as the B-topology and X/I is a quotient B-topological space. Then, X satisfies open condition.*

Proposition 4.13. *Let X be a B-algebra. Then for any $S \subseteq X$ and I a subalgebra of X,* $\pi_I^{-1}(\pi_I(S)) = \bigcup_{x \in S} I_x$ where π_I is the canonical epimorphism.

Proof. Suppose $y \in \pi_I^{-1}(\pi_I(S))$. Then $I_y = \pi_I(y) \in \pi_I(S)$. Since π_I is an epimorphism, there exists an $x \in S$ such that $\pi_I(x) = I_y$. Hence, $I_x = I_y$ or $x \cong^I y$. This implies that $y \in I_x$ for some $x \in S$. It follows that $y \in \bigcup_{x \in S} I_x$ so that $\pi_I^{-1}(\pi_I(S)) \subseteq \bigcup_{x \in S} I_x$. Conversely, suppose $y \in \bigcup_{x \in S} I_x$. Then $y \in I_x$ for some $x \in S$. This implies that $y \cong^I x$ or $I_y = I_x$. Hence, $\pi_I(y) = \pi_I(x)$. Since $\pi_I(x) \in \pi_I(S)$, it follows that $\pi_I(y) \in \pi_I(S)$ implying that $y \in \pi_I^{-1}(\pi_I(S))$. That is, $\bigcup_{x \in S} I_x \subseteq \pi_I^{-1}(\pi_I(S))$. Therefore, $\pi_I^{-1}(\pi_I(S)) = \bigcup_{x \in S} I_x$. □

Note that in Example 4.5, the canonical epimorphism π_I is not open with respect to $\tilde{\tau}$ and is not continuous with respect to τ as indicated. However, the next propositions show that π_I is open with respect to $\tilde{\tau}'$ and π_I is continuous with respect to τ'.

Proposition 4.14. *Suppose X is a B-topological space and X/I is a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. If $I_x \in \tilde{\tau}'$ for any $x \in X$, then X satisfies open condition.*

Proof. Suppose X is a B-topological space and X/I is a quotient B-topological space with $\tilde{\tau}'$ as the quotient B-topology. Note that for any $O \in \tau$ and by Proposition 4.13, $\pi_I^{-1}\big(\pi_I(O)\big) = \bigcup_{x\in O} I_x \in \tau$. By definition of $\tilde{\tau}'$, it follows that $\pi_I(O) \in \tilde{\tau}'$. □

Proposition 4.15. *Suppose X is a B-topological space with τ' as the B-topology and let X/I be a quotient B-topological space. Then π_I is continuous.*

Proof. Suppose X is a B-topological space with τ' as the B-topology and let X/I be a quotient B-topological space. Let $O \in \tilde{\tau}$. Since π_I is surjective, $\pi_I\big(\pi_I^{-1}(O)\big) = O \in \tilde{\tau}$. This implies that $\pi_I^{-1}(O) \in \tau'$. Hence, π_I is continuous. □

In view of Corollaries 4.11 and 4.12, Proposition 4.14 and 4.15, the next corollaries follow from Theorems 4.6, 4.8 and 3.6, and the succeeding corollary with respect to $\tilde{\tau}'$ and τ'. Specifically, Corollary 4.16 illustrates that the relationship between a topological B-algebra and a quotient topological B-algebra is that they are equivalent carrying with them the topologies (B-topology and quotient B-topology) τ' and $\tilde{\tau}'$, respectively. Also, Corollaries 4.17 and 4.18 states that every topological B-algebra and quotient topological B-algebra is a Hausdorff (equivalently, T_1, and T_0) space.

Corollary 4.16. *Suppose (X, τ') is a B-topological space and $(X/I, \tilde{\tau}')$ is a quotient B-topological space. Then X is a topological B-algebra if and only if X/I is a topological B-algebra.*

Corollary 4.17. *Suppose (X, τ') is a topological B-algebra. Then $(X/I, \tilde{\tau}')$ is a Hausdorff (equivalently, T_1, and T_0) space.*

Corollary 4.18. *Suppose $(X/I, \tilde{\tau}')$ is a topological B-algebra. Then (X, τ') is a Hausdorff (equivalently, T_1, and T_0) space.*

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