Some results on finite soluble groups and p-supersoluble groups

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Abstract. In this paper, motivated by the conjecture of Heliel, we firstly investigated the solubility of a group by the properties of primes 3 and 5. Further, we also studied the p-supersolubility and p-nilpotency under the conditions on primary subgroups with given order.

Keywords: solubility, p-supersolubility, p-nilpotency.

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1. Introduction

All groups considered in this paper are finite. We shall use the standard notation, as [1]. In particular, let G denote a group, $|G|$ denote the order of G, $\pi(G)$ denote the set of all prime divisors of $|G|$. Let H_G be the core of H in G when $H \leq G$ and let S_n be the symmetric group of degree n. Let $M < \cdot G$ denote M is a maximal subgroup of G.

As we all know, the study of the generalized supplementation of some primary subgroups is one of the most important topic in Finite Group Theory ([2], [3], [4]). Meanwhile, the embedded properties of subgroups are a crucial approach to characterize the structure of a group. For example, in 1998, Ballester-Bolinches and Pedraza-Aguilera [5] demonstrated that if all maximal subgroup of all Sylow subgroups of G are S-quasinormally embedded in G, then G is supersoluble. Recently, Miao [6] introduced the concept of $\mathcal{M}_p$-embedded subgroups which is closely related to $\mathcal{M}_p$-supplementation and obtained some characterization of p-supersolubility and saturated formations containing all supersoluble groups.

On the other hand, many scholars concentrated on the relationship between the solubility of some primary subgroups and their generalized supplementation. For instance, in 1937, Hall [7] proved that G is soluble if and only if each Sylow subgroup of G is complemented in G. In 1982, Arad and Ward [8] showed that G is soluble if and only if every Sylow 2-subgroup and Sylow 3-subgroup of G are complemented in G. In 2000, Ballester-Bolinches, Wang and Guo [9] asserted that G is soluble if and only if each Sylow subgroup of G is c-supplemented in G. In 2008, Asaad and Ramadan [10] proved that G is soluble if and only if every subgroup of prime order of G is c-supplemented in G. In 2009, Miao and Qian [11] stated that G is soluble if every Sylow subgroup of G is $\mathcal{M}$-supplemented in G. In 2014, Heliel [12] showed that G is soluble if and only if every Sylow subgroup of odd order of G is c-supplemented in G. Moreover, Heliel stated a conjecture as follows.

Conjecture 1.1. *Let G be a finite group such that every noncyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < D \leq P$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Is G soluble?*

It is easy to check that $A_5 \times C_5$ is a counterexample. With the further consideration, Our goal is to investigate the influence of the embedded property of some primary subgroups on the construction of a group by considering only two prime divisors $\{3, 5\}$ and obtain some new characterization about solubility of finite groups.

Definition 1.1 ([13]). *Let π be a set of primes. A subgroup H of a group G is called $\mathcal{M}_\pi$-supplemented in G, if there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H with $\pi(|H : H_1|) \subseteq \pi$. In particular, if π=$\{p\}$, then H is called $\mathcal{M}_p$-supplemented in G.*

Definition 1.2 ([6], Definition 1.2)**.** *A subgroup H of G is called $\mathcal{M}_p$-embedded in G, if there exists a p-nilpotent subgroup B of G such that $H_p \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G.*

The Example 1.3 in [6] indicates that an $\mathcal{M}_p$-embedded subgroup need not to be $\mathcal{M}_p$-supplemented in G.

2. Preliminaries

For the sake of convenience, we firstly list some known results here which will be useful in the sequel.

Lemma 2.1 ([6], Lemma 2.1)**.** *Let G be a group. Then*

(1) *Let $N \trianglelefteq G$ and $N \leq H$. If H is $\mathcal{M}_p$-embedded in G, then H/N is $\mathcal{M}_p$-embedded in G/N.*

(2) *Let π be a set of primes. Let N be a normal π'-subgroup and H be a π-subgroup of G. If H is $\mathcal{M}_p$-embedded in G, then HN/N is $\mathcal{M}_p$-embedded in G/N.*

Lemma 2.2 ([14], Lemma 4)**.** *If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.3 ([13], Lemma 4)**.** *Let H be a $\mathcal{M}_\pi$-supplemented subgroup in a group G and B be a $\mathcal{M}_\pi$-supplement to H. If H_1 is a maximal subgroup in H and $\pi(H : H_1) \subseteq \pi$, then $|G : H_1B| = |H : H_1|$.*

Lemma 2.4 ([13], Theorem 1)**.** *Let p be the smallest prime divisor of $|G|$ and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-nilpotent.*

Lemma 2.5 ([13], Theorem 2)**.** *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 = p < \ldots < p_n$, and H be a p-nilpotent subgroup containing a Sylow p-subgroup of G. If H is $\mathcal{M}_p$-supplemented in G, then G is p-supersoluble.*

Lemma 2.6 ([15], Lemma 2.8)**.** *Let G be a p-supersoluble group. If $O_{p'}(G) = 1$, then G is supersoluble.*

Lemma 2.7 ([16], Theorem 5.4)**.** *If $(15, |G|) = 1$, then G is soluble.*

Lemma 2.8. *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-soluble.*

Proof. It is easy to find that $O_{p'}(G) = 1$ by Lemma 2.1. Let H be a subgroup of P with $|H| = d$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. By Lemma 2.3, $|G : B_iK| = p$. Then $G/(B_iK)_G$ is isomorphic to a subgroup of the symmetric group S_p and so $(B_iK)_G \neq 1$. We may pick a minimal normal subgroup L of G such that $L \leq (B_iK)_G$. If $|L_p| < d$, then we may choose a subgroup F with order d such that $L_p < F$. By hypothesis, F is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B^* of G such that $F \in Syl_p(B^*)$ and B^* is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K_1 of G such that $G = B^*K_1 = FB^*_{p'}K_1 = LB^*_iK_1$, where $B^*_{p'}$ is the normal p-complement of B^*, $B^*_i = F_iB^*_{p'}$ and $B^*_iK_1 = F_iB^*_{p'}K_1 < G$ for every maximal subgroup F_i of F. Next, we consider subgroup B^*L. By Lemma 2.5, B^*L is p-supersoluble and L is p-supersoluble. By Lemma 2.6, L is supersoluble and L is a p-group. Then G/L satisfies the hypothesis of the theorem and so G is p-soluble by induction.

If $|L_p| \geq d$, then we may take a subgroup E with order d such that $L_p \geq E$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup T of G such that $E \in Syl_p(T)$ and T is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K_2 of G such that $G = TK_2 = ET_{p'}K_2 = LT_iK_2$, where $T_{p'}$ is the normal p-complement of T, $T_i = E_iT_{p'}$ and $T_iK_2 = E_iT_{p'}K_2 < G$ for every maximal subgroup E_i of E. Clearly, $L \nleq (T_iK_2)_G$ and so $L \cap (T_iK_2)_G = 1$. Then $L \cong L(T_iK_2)_G/(T_iK_2)_G$ is a subgroup of $G/(T_iK_2)_G$. Since $|G : (T_iK_2)_G| = 2^\alpha p$, L is soluble and so $|L| = p$. Hence $|L| = d = p$ and all subgroups of P with order p are $\mathcal{M}$-supplemented in G. By [[17], Corollary 3.6], G is p-soluble. □

Lemma 2.9. *Let G be a group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-soluble.*

Proof. By [[18], Lemma 2.6], Lemma 2.4 and the proof of Lemma 2.8, it is easy to be verified. □

Lemma 2.10. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. If every prime order subgroup H of P is complemented in G and $N_G(H)/C_G(H)$ is a p-group, then G is p-nilpotent.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $|P| > p$.

Assume that $|P| = p$. Since $N_G(P)/C_G(P)$ is a p-group and $N_G(P)/C_G(P) \hookrightarrow Aut(P)$, we get $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem, a contradiction.

(2) $G = P \rtimes Q$, where Q is a Sylow q-subgroup of G.

Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup A of the form $A = A_p \rtimes Q$, where Q is a Sylow q-subgroup of A. Assume that $A < G$. If $|A_p| = p$, then $N_A(A_p)/C_A(A_p) \hookrightarrow Aut(A_p)$. Further, $N_A(A_p)/C_A(A_p) \cong (N_G(A_p) \cap A)C_G(A_p)/C_G(A_p) \leq N_G(A_p)/C_G(A_p)$ is a p-group. Hence $N_A(A_p) = C_A(A_p)$ and A is p-nilpotent, a contradiction. If $|A_p| > p$, then A satisfies the hypothesis of the theorem and so A is p-nilpotent by the choice of G, a contradiction. Hence $G = A = P \rtimes Q$.

(3) $P = R_1 \times R_2 \times \cdots \times R_t$, where R_i is the minimal normal subgroup of G of order p, $i = 1, 2, \ldots, t$.

We assert that P is the product of the minimal normal subgroups of G with order p. If $P \cap \Phi(G) \neq 1$, then we pick a subgroup H of $P \cap \Phi(G)$ with order p. By hypothesis, there exists a subgroup B of G such that $G = HB$ and $H \cap B = 1$. Then $G = HB = B < G$, a contradiction. Hence $P \cap \Phi(G) = 1$. By [[11], Theorem 1.8.17], $P = R_1 \times R_2 \times \cdots \times R_t$, where R_i is the minimal normal subgroup of G, $i = 1, 2, \ldots, t$. For any $L \in \{R_1, R_2, \ldots, R_t\}$, we may choose a subgroup H of L with order p such that H is complemented in G. There exists a subgroup B of G such that $G = HB = LB$ and $H \cap B = 1$. It is easy to find that either $L \cap B = L$ or $L \cap B = 1$. If $L \cap B = L$, then $L \leq B$ and $G = LB = B < G$, a contradiction. Hence $L \cap B = 1$ and $L = |G : B| = p$.

(4) Final contradiction.

By hypothesis, $N_G(L)/C_G(L) = G/C_G(L)$ is a p-group. Since $N_G(L)/C_G(L) \hookrightarrow Aut(L)$, $G = C_G(L)$, $L \leq Z(G)$. Hence $P \leq Z(G)$ and G is p-nilpotent by [[21], Satz 5.5], a contradiction.

The final contradiction completes the proof. □

Lemma 2.11 ([19], Theorem 1.8.17). *Let N be a non-trivial soluble normal subgroup of G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.12 ([19], Theorem 1.8.19). *If G is a p-soluble group, where p is a prime divisor of $|G|$, then $C_G(F_p(G)) \leq F_p(G)$.*

3. Main results

Theorem 3.1. *Let G be a group. Then G is soluble provided G satisfies the following two conditions:*

(1) If $3 \in \pi(G)$, then every subgroup H_1 of every Sylow 3-subgroup P of G such that $|H_1| = d_1$ and $1 < d_1 \leq |P|$ is $\mathcal{M}_3$-embedded in G.

(2) If $5 \in \pi(G)$, then every subgroup H_2 of every Sylow 5-subgroup Q of G such that $|H_2| = d_2$ and $1 < d_2 \leq |Q|$ is $\mathcal{M}_5$-embedded in G.

Proof of Theorem 3.1. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $15 \mid |G|$ and G is 3-soluble.

If $(15, |G|) = 1$, then G is soluble by Lemma 2.7, a contradiction. If $(15, |G|) = 5$, then G is 5-soluble by Lemma 2.9. Moreover, the composition factor of G is 5-group or $5'$-group. Obviously, $5'$-group is soluble and so G is soluble, a contradiction. If $(15, |G|) = 3$, with the similar discussion as above, then G is soluble, a contradiction. Hence $15 \mid |G|$ and G is 3-soluble.

(2) $O_{\pi'}(G) = 1$, $\pi = \{3, 5\}$.

If $N = O_{\pi'}(G) \neq 1$, then we consider the factor group G/N. By Lemma 2.1 (2), G/N satisfies the condition of the theorem and so G/N is soluble by the choice of G. Hence G is soluble, a contradiction.

(3) If $O_p(G) \neq 1$ for $p = 3$ or $p = 5$, then every minimal normal subgroup L of G contained in $O_p(G)$ is order p and $L \nleq \Phi(G)$.

Set $d = d_1$ or d_2. We choose a minimal normal subgroup L of G such that $L \leq O_p(G)$. If $|L| < d$, then G/L satisfies the hypothesis of the theorem and so G is soluble by the minimality of G, a contradiction. Hence $|L| \geq d$. We may fix a subgroup H with order d such that $H \leq L$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K = LB_iK$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Clearly, $L \nleq \Phi(G)$ and $L \cap B_iK = 1$ by [[20], Lemma 2.8]. Then $|L| = p$ by Lemma 2.3.

(4) Final contradiction.

Since G is not soluble, $G'' \neq 1$. We may choose a minimal normal subgroup L of G such that $L \leq G''$. Since G is 3-soluble, L is 3-group or $3'$-group. If L is 3-group, then $|L| = 3$ by (3). If L is $3'$-group, then $L_5 \neq 1$ by (2).

If $|L_5| < d_2$, then we may pick a subgroup F with order d_2 such that $L_5 < F$. By hypothesis, F is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup C of G such that $F \in Syl_5(C)$ and C is $\mathcal{M}_5$-supplemented in G. Next, we consider subgroup CL. Set $CL = M$ and $P^* \in Syl_5(M)$. Since C is $\mathcal{M}_5$-supplemented in M, there exists a subgroup T such that $M = CT$ and $|M : C_iT| = 5$, where $C_{5'}$ is the normal 5-complement of C, $C_i = P_i^*C_{5'}$ and $C_iT = P_i^*C_{5'}T < M$ for every maximal subgroup P_i^* of P^*. Then $M/(C_iT)_M \hookrightarrow S_5$ and so $M/\cap(C_iT)_M$ is 5-soluble. Since $P^* \cap (\cap(C_iT)_M) = P^* \cap (\cap(P_i^*C_{5'}T)_M) = \cap P_i^* = \Phi(P^*)$, $\cap(C_iT)_M$ is 5-nilpotent by Lemma 2.2. Hence M is 5-soluble and L is 5-soluble. Then L is soluble and so $|L| = 5$ by (3).

If $|L_5| \geq d_2$, then we may fix a subgroup E with order d_2 such that $L_5 \geq E$. By hypothesis, E is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup U of G such that $E \in Syl_5(U)$ and U is $\mathcal{M}_5$-supplemented in G. There exists a subgroup W such that $G = UW = EU_{5'}W = LU_iW$, where $U_{5'}$ is the normal 5-complement of U, $U_iW = E_iU_{5'}W < G$ for every maximal subgroup U_i of U. Then $L/L \cap (U_iW)_G \cong L(U_iW)_G/(U_iW)_G$ is a subgroup of $G/(U_iW)_G \hookrightarrow S_5$. If $L \cap (U_iW)_G = 1$, then L is soluble and so $|L| = 5$ by (3). If $L \cap (U_iW)_G = L$, then $G = UW = EU_{5'}W = LU_iW = U_iW < G$, a contradiction.

Hence $|L|$ is a prime and $G/C_G(L)$ is isomorphic to a subgroup of $Aut(L)$. Then $G' \leq C_G(L)$ and $L \leq Z(G')$. Moreover, $L \leq Z(G') \cap G'' \leq \Phi(G') \leq \Phi(G)$ by [[1], A.9.3(d)], which contradicts (3).

The final contradiction completes the proof.

In general, the $\mathcal{M}_p$-embedded property is not subgroup closed. Note that the generalized Fitting subgroup $F^*(G)$ play an important role in studying the global structure of a group. Here we give the following theorem to investigate the solubility of $F^*(G)$ by considering its primary subgroups which are $\mathcal{M}_p$-embedded in a group G.

Theorem 3.2. *Let G be a group. Then $F^*(G)$ is soluble provided $F^*(G)$ satisfies the following two conditions:*

(1) *If $3 \in \pi(F^*(G))$, then every subgroup H_1 of every Sylow 3-subgroup P of $F^*(G)$ such that $|H_1| = d_1$ and $1 < d_1 \leq |P|$ is $\mathcal{M}_3$-embedded in G.*

(2) *If $5 \in \pi(F^*(G))$, then every subgroup H_2 of every Sylow 5-subgroup Q of $F^*(G)$ such that $|H_2| = d_2$ and $1 < d_2 \leq |Q|$ is $\mathcal{M}_5$-embedded in G.*

Proof of Theorem 3.2. By [[3], Lemma 2.6(2)], $F^*(G)/F(G) = R_1/F(G) \times \cdots \times R_s/F(G)$, $R_i/F(G)$ is minimal normal subgroups of $C_G(F(G))F(G)/F(G)$, $i = 1, \ldots, s$. If $(2, |R_i/F(G)|) = 1, i = 1, \ldots, s$, then $F^*(G)$ is soluble. Hence there exists $R_i/F(G)$ such that $2 \in \pi(R_i/F(G))$. We may take a normal subgroup L of G such that $R_i \leq L \leq F^*(G)$. If $(15, |L|) = 1$, then $F^*(G)$ is soluble by Lemma 2.7. If $(15, |L|) \neq 1$, without loss of generality, then we assume that $(15, |L|) = 15$.

Case one. $|L_3| \geq d_1$. We may fix a subgroup X with order d_1 such that $L_3 \geq X$. By hypothesis, X is $\mathcal{M}_3$-embedded in G, there exists a 3-nilpotent subgroup I of G such that $X \in Syl_3(I)$ and B is $\mathcal{M}_3$-supplemented in G. There exists a subgroup K of G such that $G = IK = XI_{3'}K = LI_iK$, where $I_{3'}$ is the normal p-complement of I, $I_i = X_iI_{3'}$ and $I_iK = X_iI_{3'}K < G$ for every maximal subgroup X_i of X. Then $L/L \cap (I_iK)_G \cong L(I_iK)_G/(I_iK)_G$ is a subgroup of $G/(I_iK)_G \hookrightarrow S_3$. If $L \cap (I_iK)_G = 1$, then L is soluble and R_i is soluble. Hence $F^*(G)$ is soluble. If $L \cap (I_iK)_G \neq 1$, then we set $L \cap (I_iK)_G = N_1$ and $L = N_0$. With the similar discussion of L, we have a series $1 = N_{s+1} \trianglelefteq N_s \trianglelefteq \cdots \trianglelefteq N_1 \trianglelefteq N_0 = L$ such that N_i/N_{i+1} is soluble, $i = 0, 1, \ldots, s$. Then L is soluble and R_i is soluble. Hence $F^*(G)$ is soluble.

Case two. $|L_3| < d_1$. We may choose a subgroup H with order d_1 such that $L_3 < H$. By hypothesis, H is $\mathcal{M}_3$-embedded in G, there exists a 3-nilpotent subgroup B of G such that $H \in Syl_3(B)$ and B is $\mathcal{M}_3$-supplemented in G. Next, we consider subgroup BL. By Lemma 2.9, BL is 3-soluble and so L is 3-soluble.

If $|L_5| < d_2$, then we may pick a subgroup F with order d_2 such that $L_5 < F$. By hypothesis, F is $\mathcal{M}_5$-embedded in G, there exists a 5-nilpotent subgroup C of G such that $F \in Syl_5(C)$ and C is $\mathcal{M}_5$-supplemented in G. Next, we consider subgroup $CL = M$. Set $P^* \in Syl_5(M)$. Since C is $\mathcal{M}_5$-supplemented in M, there exists a subgroup T such that $M = CT$ and $|M : C_iT| = 5$, where $C_{5'}$ is the normal 5-complement of C, $C_i = P_i^*C_{5'}$ and $C_iT = P_i^*C_{5'}T < M$ for every maximal subgroup P_i^* of P^*. Then $M/(C_iT)_M \hookrightarrow S_5$ and so $M/\cap(C_iT)_M$ is 5-soluble. Since $P^* \cap (\cap(C_iT)_M) = P^* \cap (\cap(P_i^*C_{5'}T)_M) = \cap P_i^* = \Phi(P^*)$, $\cap(C_iT)_M$ is 5-nilpotent by Lemma 2.2. Hence M is 5-soluble and L is 5-soluble. It follows from L is soluble that $F^*(G)$ is soluble.

If $|L_5| \geq d_2$, then L is 5-soluble by the similar discussion in Case one. It follows from L is soluble that $F^*(G)$ is soluble.

Theorem 3.3. *Let G be a p-soluble group and P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Proof of Theorem 3.3. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $O_{p'}(G)$=1.

If $N = O_{p'}(G) \neq 1$, then we consider the factor group G/N. By Lemma 2.1(2), G/N satisfies the condition of the theorem and so G/N is p-supersoluble by the choice of G. Hence G is p-supersoluble, a contradiction.

(2) $O_p(G) \neq 1$.

Since G is p-soluble and $O_{p'}(G)$=1, the minimal normal subgroup of G is a abelian p-group and $O_p(G) \neq 1$.

(3) $O_p(G) \cap \Phi(G) = 1$.

If $O_p(G) \cap \Phi(G) \neq 1$, then we may choose a minimal normal subgroup L of G such that $L \leq O_p(G) \cap \Phi(G)$. If $|L| < d$, then G/L satisfies the hypothesis of the theorem and so G is p-supersoluble by the minimality of G, a contradiction. Hence $|L| \geq d$. We may pick a subgroup H with order d such that $H \leq L$. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G, $G = BK = HB_{p'}K = LB_iK$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Since $L \leq O_p(G) \cap \Phi(G)$, $G < B_iK$, a contradiction.

(4) $O_p(G) = R_1 \times R_2 \times \cdots \times R_t$, where $R_1, R_2, \ldots, R_t$ are the minimal normal subgroups of G with order p.

By [[19], Theorem 1.8.17] and (3), $O_p(G) = R_1 \times R_2 \times \cdots \times R_t$, where R_i is a minimal normal subgroup of G, $i = 1, 2, \ldots, t$. Let $L \in \{R_1, \ldots, R_t\}$. If $|L| \geq d$, then we may take a subgroup E with order d such that $E \leq L$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup U of G such that $E \in Syl_p(U)$ and U is $\mathcal{M}_p$-supplemented in G. There exists a subgroup W of G, $G = UW = EU_{p'}W = LU_iW$, where $U_{p'}$ is the normal p-complement of U, $U_i = E_iU_{p'}$ and $U_iK = E_iU_{p'}W < G$ for every maximal subgroup E_i of E. By [[20], Lemma 2.8], $L \cap U_iW \trianglelefteq G$. If $L \cap U_iW = L$, then $G < U_iW$, a contradiction. Hence $L \cap U_iW = 1$ and $|L| = p$. If $|L| < d$, then we may fix a subgroup F with order d such that $L < F$. By hypothesis, F is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup C of G such that $F \in Syl_p(C)$ and C is $\mathcal{M}_p$-supplemented in G. There exists a subgroup T of G, $G = CT = FC_{p'}T$, where $C_{p'}$ is the normal p-complement of C, $C_i = F_iC_{p'}$ and $C_iT = F_iC_{p'}T < G$ for every maximal subgroup F_i of F. By (3), $G = CT = FC_{p'}T = LSC_{p'}T$, where $S < \cdot F$ such that $LS = F$. By Lemma 2.3, $|L| = p$.

(5) Final contradiction.

By [[19], Theorem 1.8.19], (1) and (4), $C_G(O_p(G)) = O_p(G) = \bigcap_{i=1}^{t} C_G(R_i)$. Since $G/O_p(G) = G/C_G(O_p(G)) = G/\bigcap_{i=1}^{t} C_G(R_i) \hookrightarrow G/C_G(R_1) \times \cdots \times G/C_G(R_t)$ and $G/C_G(R_i)$ is cyclic, $G/O_p(G)$ is p-supersoluble and G is p-supersoluble.

The final contradiction completes the proof.

Theorem 3.4. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G and $N_G(H)/C_G(H)$ is a p-group.*

Proof of Theorem 3.4. The necessity part is obvious. We only need to prove the sufficiency party. Assume that the assertion is false and choose G to be a counterexample of minimal order. Further, we have

(1) $O_{p'}(G)$=1.

Similar to step (1) in the proof of Theorem 3.3.

(2) G is not a simple group.

Assume that G is a simple group and we take a subgroup H with order d. By hypothesis, H is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup B of G such that $H \in Syl_p(B)$ and B is $\mathcal{M}_p$-supplemented in G. There exists a subgroup K of G such that $G = BK = HB_{p'}K$, where $B_{p'}$ is the normal p-complement of B, $B_i = H_iB_{p'}$ and $B_iK = H_iB_{p'}K < G$ for every maximal subgroup H_i of H. Since $(B_iK)_G = 1$, G is isomorphic to a subgroup of the symmetric group S_p and so $|P| = p$. Hence $H = P$, $N_G(P) = C_G(P)$ and G is p-nilpotent by Burnside's Theorem, a contradiction.

(3) For every minimal normal subgroup L of G, $|L| = p$.

We may choose a minimal normal subgroup L of G. If $|L_p| \geq d$, then we may pick a subgroup X with order d such that $L_p \geq X$. By hypothesis, X is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup C of G such that $X \in Syl_p(C)$ and I is $\mathcal{M}_p$-supplemented in G. There exists a subgroup T of G, $G = CT = XC_{p'}T = LC_iT$, where $C_{p'}$ is the normal p-complement of C, $C_i = X_iC_{p'}$ and $C_iT = X_iC_{p'}T < G$ for every maximal subgroup X_i of X. We assert that $(C_iT)_G \neq 1$ for every C_i. If there exists a $(C_jT)_G$ such that $(C_jT)_G = 1$, then G is isomorphic to a subgroup of the symmetric group S_p and so $d = |P| = p$. Hence G is p-nilpotent by Burnside's Theorem, a contradiction. Clearly, $L \nleq (C_jT)_G$ and $L \cap (C_jT)_G = 1$. Then L is isomorphic to a subgroup of the symmetric group S_p and $|L_p| = d = p$. By Lemma 2.10, G is p-nilpotent, a contradiction.

If $|L_p| < d$, then we may take a subgroup E with order d such that $L_p < E$. By hypothesis, E is $\mathcal{M}_p$-embedded in G, there exists a p-nilpotent subgroup U of G such that $E \in Syl_p(U)$ and U is $\mathcal{M}_p$-supplemented in G. There exists a subgroup W of G, $G = UW = EU_{p'}W$, where $U_{p'}$ is the normal p-complement of U, $U_i = E_iU_{p'}$ and $U_iW = E_iU_{p'}W < G$ for every maximal subgroup E_i of E. Next, we consider subgroup UL. If there exists a minimal normal subgroup L of G such that $UL < G$, then $|E| < |P|$, $E \in Syl_p(UL)$ and E is $\mathcal{M}_p$-embedded in

UL. Since $N_{UL}(E)/C_{UL}(E) \cong (N_G(E) \cap UL)C_G(E)/C_G(E) \leq N_G(E)/C_G(E)$ is a p-group, UL is p-nilpotent by the choice of G. Then L is p-nilpotent. By (1), L is a p-group and so $O_p(G) \neq 1$. If $UL = G$ for every minimal normal subgroup of G, then G/L is p-nilpotent and $|E| = |P|$. Since the class of all p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G. We assert that $(U_iW)_G \neq 1$ for every U_i. If not, then there exists a $(U_jW)_G$ such that $(U_jW)_G = 1$. Hence G is isomorphic to a subgroup of the symmetric group S_p and $|P| = d = p$. Hence L is a p'-group, contradicts (1). Since $P \cap (\cap(U_iW)_G) = P \cap (\cap(P_iU_{p'}W)_G) = \cap P_i = \Phi(P)$, $\cap(U_iW)_G$ is p-nilpotent by Lemma 2.2. Hence L is p-nilpotent and L is a p-group by (1). Further, $|L| = p$ by Lemma 2.3.

(4) Final contradiction.

By (3) and hypothesis, every subgroup H/L of P/L with order $d/|L|$ is $\mathcal{M}_p$-embedded in G/L. Set $C_{G/L}(H/L) = A/L$. Then $|N_{G/L}(H/L)/C_{G/L}(H/L)| = |N_G(H)/L/A/L| = |N_G(H)/C_G(H)/A/C_G(H)|$ is a p-group, and so G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G and $L = O_p(G) \nleq \Phi(G)$ by [[19], Theorem 1.8.17].

So G is p-soluble and $C_G(L) = L$ by (1) and [[19], Lemma 1.8.19]. Since $|L| < d$ and $|L| = p$ by (3), $G/C_G(L) = G/L$ is isomorphic to a subgroup of $Aut(L)$, $L = P$, a contradiction.

The final contradiction completes the proof.

4. Applications

The results in section 3 have many applications. By Lemma 2.8, 2.9, Theorem 3.1, 3.3, 3.4, we obtained the following corollaries.

Corollary 4.1. *Let G be a group and P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G.*

Corollary 4.2. *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Corollary 4.3. *Let G be a p-soluble group and P be a Sylow p-subgroup of $F_p(G)$ where p is a prime divisor of $|G|$. Suppose that $1 < d \leq |P|$. If every subgroup H of P with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is p-supersoluble.*

Corollary 4.4. *Let G be a group. Suppose that $1 < d \leq |P|$. If every subgroup H of every Sylow subgroup P of G with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is supersoluble.*

Corollary 4.5. *Let G be a group and P be a Sylow p-subgroup of G where p is an odd prime divisor of $|G|$. G is p-nilpotent if and only if every subgroup*

H of P with $|H| = d$ and $1 < d \leq |P|$ is $\mathcal{M}_p$-embedded in G and $N_G(H)$ is p-nilpotent.

Corollary 4.6. *Let G be a group. Suppose that $1 < d \leq |P|$. If every subgroup H of every Sylow subgroup P of odd order of G with $|H| = d$ is $\mathcal{M}_p$-embedded in G, then G is soluble.*

Corollary 4.7 ([13], Lemma 2.12)**.** *Let G be a group and P be a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. G is p-nilpotent if and only if P has a subgroup D such that $1 < D \leq P$ and every subgroup H of P with $|H| = |D|$ is $\mathcal{M}$-supplemented in G.*

Corollary 4.8 ([19], Corollary 3.6)**.** *Let G be a group, $\pi(G)=\{p_1, p_2 = p, \ldots, p_n\}$, $p_1 < p_2 < \ldots < p_n$ and P be a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < D \leq P$. If every subgroup H of P with $|H| = |D|$ is $\mathcal{M}$-supplemented in G, then G is p-supersoluble.*

Corollary 4.9 ([15], Theorem 3.1)**.** *Let G be a p-soluble group and P be a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Suppose that P is $\mathcal{M}$-supplemented in G, then G is p-supersoluble.*

Corollary 4.10 ([15], Theorem 3.4)**.** *Let G be a group. Suppose that every Sylow subgroup P of G is $\mathcal{M}$-supplemented in G, then G is soluble.*

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