A bi-endomorphism induces a new type of derivations on B-algebras

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Abstract. The goal of this paper is to introduce the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and to provide important properties. The study found that the composition of (l, r) and an (r, l)-τ-derivations is also an (l, r) and an (r, l)-τ-derivation on a 0-commutative B-algebra, respectively. In addition, the relationship among those derivations is also considered.

Keywords: B-algebra, bi-endomorphism, derivation.

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1. Introduction

The notion of B-algebras was introduced by Neggers and Kim [10] in 2002, it is a new algebraic structure, they took some properties from BCI and BCH-algebras (see, [5, 7]), called a B-algebra.

In 2010, Al-Shehrie [3] introduced the notion of a left-right and a right-left derivation of a B-algebra and investigated some related properties. Also, he studied the notion of derivation of 0-commutative B-algebra and investigated some of its properties. Ardekani and Davvaz [4] introduced a generalization of a derivation of a B-algebra, that is, the notion of an f-derivation and an (f, g)-derivation of a B-algebra and investigated some properties of an (f, g)-derivation of a commutative B-algebra in 2014. In a UP-algebra, Sawika et al. [11] introduced the concepts of an (l, r) and an (r, l)-derivation and a derivation, Iampan [6] introduced the concept of an f-derivation, and in the next year, Tippanya et al. [12] introduced the concepts of a left and a right-f-derivation of type I, and of a left and a right-f-derivation of type II. In 2021, Muangkarn et al. [9] studied some properties of an outside and an inside f_q-derivation of a B-algebra. In addition, they defined and studied some properties of a (right-left) and a (left-right) f_q-derivation on a B-algebra from the concept of Al-Kadi [2].

In this paper, we introduce the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and to provide important properties. In addition, the relationship among those derivations is also considered. Also, using the concept of a derivation in past investigate some of its properties.

2. Preliminaries

First, let's review some basic definitions and theorems that are required in our work.

Definition 2.1 ([10]). *By a B-algebra we mean an algebra* $B = (B, \rightsquigarrow, 0)$ *of type* $(2, 0)$ *satisfying the followings:*

(B1) $(\forall x \in B)(x \rightsquigarrow x = 0)$,

(B2) $(\forall x \in B)(x \rightsquigarrow 0 = x)$,

(B3) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = x \rightsquigarrow (z \rightsquigarrow (0 \rightsquigarrow y)))$.

A non-empty subset S *a* B*-algebra* B *is called a subalgebra of* B *if* $x \rightsquigarrow y \in S$*, for all* $x, y \in S$.

After that a B-algebra $(B, \rightsquigarrow, 0)$ is denoted by B.

Example 2.1 ([9])**.** Let $B = \{0, 1, 2, 3\}$ with the Cayley table as follows:

$\rightsquigarrow$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then, B is a B-algebra. Let $S = \{0, 3\}$. Then, S is a subalgebra of B.

Theorem 2.1 ([10])**.** *If B is a B-algebra, then*

(B4) $(\forall x, y \in B)((x \rightsquigarrow y) \rightsquigarrow (0 \rightsquigarrow y) = x)$,

(B5) $(\forall x, y, z \in B)(x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow (0 \rightsquigarrow z)) \rightsquigarrow y)$,

(B6) $(\forall x, y \in B)(x \rightsquigarrow y = 0 \Rightarrow x = y)$,

(B7) $(\forall x \in B)(0 \rightsquigarrow (0 \rightsquigarrow x) = x)$,

(B8) $(\forall x, y, z \in B)(x \rightsquigarrow z = y \rightsquigarrow z \Rightarrow x = y)$,

(B9) $(\forall x, y, z \in B)(z \rightsquigarrow x = z \rightsquigarrow y \Rightarrow x = y)$.

Theorem 2.2 ([10])**.** *An algebra B is a B-algebra if and only if it satisfies the followings:*

(B1) $(\forall x \in B)(x \rightsquigarrow x = 0)$,

(B7) $(\forall x \in B)(0 \rightsquigarrow (0 \rightsquigarrow x) = x)$,

(B10) $(\forall x, y, z \in B)((x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow z) = x \rightsquigarrow y)$,

(B11) $(\forall x, y \in B)(0 \rightsquigarrow (x \rightsquigarrow y) = y \rightsquigarrow x)$.

Definition 2.2 ([8])**.** *A B-algebra B is said to be* 0*-commutative if*

$$(\forall x, y \in B)(x \rightsquigarrow (0 \rightsquigarrow y) = y \rightsquigarrow (0 \rightsquigarrow x)).$$

Example 2.2. In Example 2.1, we have B is 0-commutative.

Theorem 2.3 ([8])**.** *If B is a 0-commutative B-algebra, then*

(B12) $(\forall x, y \in B)((0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y) = y \rightsquigarrow x)$,

(B13) $(\forall x, y, z \in B)((z \rightsquigarrow y) \rightsquigarrow (z \rightsquigarrow x) = x \rightsquigarrow y)$,

(B14) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = (x \rightsquigarrow z) \rightsquigarrow y)$,

(B15) $(\forall x, y \in B)((x \rightsquigarrow (x \rightsquigarrow y)) \rightsquigarrow y = 0)$,

(B16) $(\forall x, y, z, t \in B)((x \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow t) = (t \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x))$, (B17) $(\forall x, y, z \in B)((x \rightsquigarrow y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z))$,

(B18) $(\forall x, y \in B)(x \rightsquigarrow (x \rightsquigarrow y) = y)$.

For a B-algebra B, we denote $x \wedge y = y \rightsquigarrow (y \rightsquigarrow x)$, for all $x, y \in B$.

We can refer to (B7) and (B18) as follows:

(B7) $(\forall x, y \in B)(x \wedge 0 = x)$,

(B18) $(\forall x, y \in B)(x \wedge y = x)$ if B is a 0-commutative B-algebra.

Definition 2.3 ([3]). *A self-map d on B is called*

(1) *a (left-right)-derivation ((l, r)-derivation) of B if*

$$(\forall x, y \in B)(d(x \rightsquigarrow y) = (d(x) \rightsquigarrow y) \wedge (x \rightsquigarrow d(y))),$$

(2) *a (right-left)-derivation ((r, l)-derivation) of B if*

$$(\forall x, y \in B)(d(x \rightsquigarrow y) = (x \rightsquigarrow d(y)) \wedge (d(x) \rightsquigarrow y)),$$

(3) *a derivation of B if it is both an (l, r) and an (r, l)-derivation of B.*

Definition 2.4 ([3]). *A self-map d on B is said to be regular if $d(0) = 0$. Otherwise, d is said to be irregular.*

Example 2.3. In Example 2.1, we define a self-map d on B by

$$d(x) = \begin{cases} 3, & \text{if } x = 0 \\ 2, & \text{if } x = 1 \\ 1, & \text{if } x = 2 \\ 0, & \text{if } x = 3. \end{cases}$$

Then, d is a derivation of B, and we see that d is irregular.

Definition 2.5 ([1]). *A non-empty subset I of B is called an ideal of B if it satisfies the followings:*

(I1) $0 \in I$,

(I2) $(\forall x, y \in B)(x \rightsquigarrow y \in I, y \in I \Rightarrow x \in I)$.

We known that every subalgebra of B is an ideal.

Example 2.4. In Example 2.1, let $I = \{0, 2, 3\}$. Then, I is an ideal of B.

Proposition 2.1 ([1]). *Let I be an ideal of B. If $x \in B$ and $y \in I$ such that $x \rightsquigarrow y = 0$, then $x \in I$.*

Proposition 2.2 ([1]). *Let I be an ideal of B and $x, y \in I$. Then, $x \rightsquigarrow (0 \rightsquigarrow y) \in I$.*

Proposition 2.3 ([1]). *If I is an ideal of B such that $0 \rightsquigarrow x = x$, for all $x \in I$, then I is a subalgebra of B.*

3. Main results

In this section, we introduce bi-endomorphism on B-algebra and prove some results of some results of bi-endomorphisms of B-algebra B and its derivations as follows:

Definition 3.1. *A mapping $\tau : B \times B \to B$ is called*

(1) *a left bi-endomorphism on B if*

$$(\forall x, y, z \in B)(\tau(x \rightsquigarrow y, z) = \tau(x, z) \rightsquigarrow \tau(y, z)),$$

(2) *a right bi-endomorphism on B if*

$$(\forall x, y, z \in B)(\tau(x, y \rightsquigarrow z) = \tau(x, y) \rightsquigarrow \tau(x, z)),$$

(3) *a bi-endomorphism on B if it is a left and a right bi-endomorphism on B.*

Throughout this section, we assume that τ_l and τ_r are a left and a right bi-endomorphism of B, respectively.

Let $x \in B$. By (B2), we have

$$\tau_l(x, x) \rightsquigarrow 0 = \tau_l(x, x) = \tau_l(x \rightsquigarrow 0, x) = \tau_l(x, x) \rightsquigarrow \tau_l(0, x)$$

and

$$\tau_r(x, x) \rightsquigarrow 0 = \tau_r(x, x) = \tau_r(x, x \rightsquigarrow 0) = \tau_r(x, x) \rightsquigarrow \tau_r(x, 0).$$

By (B9), we have

(B19) $(\forall x \in B)(\tau_l(0, x) = 0)$,

(B20) $(\forall x \in B)(\tau_r(x, 0) = 0)$.

Example 3.1. In Example 2.1, we define a mapping $\tau_1 : B \times B \to B$ by

$$\tau_1(x, y) = \begin{cases} 3, & \text{if } (x, y) \in \{(1, 0), (2, 0)\} \\ 0, & \text{otherwise.} \end{cases}$$

Then, τ_1 is a left bi-endomorphism on B.

Definition 3.2. *A mapping $\tau : B \times B \to B$ is said to be symmetric if $\tau(x, y) = \tau(y, x)$, for all $x, y \in B$.*

Remark 3.1. For any B-algebra, define a mapping $0 : B \times B \to B$ by $0(x) = 0$, for all $x \in B$, which is a bi-endomorphism on B. Every symmetric left (right) bi-endomorphism on B is a bi-endomorphism.

Definition 3.3. *An ideal I of B is called a τ-ideal of B if $\tau(x, y) \in I$, for all $x, y \in I$.*

Example 3.2. From Example 2.4 and Example 3.1, we have $I = \{0, 2, 3\}$ is an ideal of B and $\tau_1(x, y) \in I$, for all $x, y \in I$. Thus I is a τ_1-ideal on B.

Next, we generalize derivations on B-algebras with a left (right) bi-endomorphism $\tau : B \times B \to B$ from the concept of the Definition 2.3.

Definition 3.4. *Let $\tau : B \times B \to B$ be a left (right) bi-endomorphism on B. A self-map d_τ on B is called*

(1) *an (l, r)-τ-derivation of B if*

$$(\forall x, y \in B)(d_\tau(x \rightsquigarrow y) = (d_\tau(x) \rightsquigarrow \tau(x, y)) \wedge (\tau(x, y) \rightsquigarrow d_\tau(y))),$$

(2) *an (r, l)-τ-derivation of B if*

$$(\forall x, y \in B)(d_\tau(x \rightsquigarrow y) = (\tau(x, y) \rightsquigarrow d_\tau(y)) \wedge (d_\tau(x) \rightsquigarrow \tau(x, y))),$$

(3) *a τ-derivation of B if it is both an (l, r)- and an (r, l)-τ-derivation of B.*

Example 3.3. In Example 2.1, we define $\tau = 0$ and $d_\tau : B \to B$ by $d_\tau(x) = 3$. It is obvious that τ is a bi-endomorphism on B. Then, d_τ is an (l, r)-τ-derivation of B.

Definition 3.5. *Let d be a self-map on B. An ideal I of B is said to be d-invariant if $d(I) \subseteq I$.*

Example 3.4. In Example 2.1 and Example 2.4, we have $I = \{0, 2, 3\}$ is an ideal of B. From Example 3.3, we have d_τ is an (l, r)-τ-derivation of B. It is obvious that $d_\tau(x) \in I$, for all $x \in I$, that is, $d_\tau(I) \subseteq I$. Hence, I is d_τ-invariant.

Theorem 3.1. *Let d_{τ_l} be an (l, r)-τ_l-derivation on B. Then, d_{τ_l} is regular if and only if $d_{\tau_l}(0 \rightsquigarrow x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
d_{\tau_l}(0 \rightsquigarrow x) &= (d_{\tau_l}(0) \rightsquigarrow \tau_l(0, x)) \wedge (\tau_l(0, x) \rightsquigarrow d_{\tau_l}(x)) \\
\text{(B19)} \quad &= (0 \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_l}(x)) \\
\text{(B1)} \quad &= 0 \wedge (0 \rightsquigarrow d_{\tau_l}(x)) \\
&= (0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow 0) \\
\text{(B2)} \quad &= (0 \rightsquigarrow d_{\tau_l}(x)) \rightsquigarrow (0 \rightsquigarrow d_{\tau_l}(x)) \\
\text{(B1)} \quad &= 0.
\end{aligned}$$

Conversely, suppose that $d_{\tau_l}(0 \rightsquigarrow x) = 0$, for all $x \in B$. Then

$$\begin{aligned}
0 &= d_{\tau_l}(0 \rightsquigarrow 0) \\
&= (d_{\tau_l}(0) \rightsquigarrow \tau_l(0, 0)) \wedge (\tau_l(0, 0) \rightsquigarrow d_{\tau_l}(0)) \\
\text{(B19)} \quad &= (d_{\tau_l}(0) \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_l}(0)) \\
\text{(B2)} \quad &= d_{\tau_l}(0) \wedge (0 \rightsquigarrow d_{\tau_l}(0)) \\
&= (0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow d_{\tau_l}(0)).
\end{aligned}$$

By (B6) and (B2), we get $(0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow 0 = 0 \rightsquigarrow d_{\tau_l}(0) = (0 \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow d_{\tau_l}(0)$. Using (B9), we have $d_{\tau_l}(0) = 0$. Hence, d_{τ_l} is regular. □

Theorem 3.2. *Let d_{τ_r} be an (l,r)-τ_r-derivation on B. Then, d_{τ_r} is regular and $\tau_r(0,x) = 0$, for all $x \in B$ if and only if $d_{\tau_r}(0 \rightsquigarrow x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_r} is regular and $\tau_r(0,x) = 0$, for all $x \in B$. Let $x \in B$. Then

$$
\begin{aligned}
d_{\tau_r}(0 \rightsquigarrow x) &= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\
&= (0 \rightsquigarrow 0) \wedge (0 \rightsquigarrow d_{\tau_r}(x)) \\
\text{(B1)} \quad &= 0 \wedge (0 \rightsquigarrow d_{\tau_r}(x)) \\
&= (0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow ((0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow 0) \\
\text{(B2)} \quad &= (0 \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow d_{\tau_r}(x)) \\
\text{(B1)} \quad &= 0.
\end{aligned}
$$

Conversely, suppose that $d_{\tau_r}(0 \rightsquigarrow x) = 0$, for all $x \in B$. By (B1), we have $d_{\tau_r}(0) = d_{\tau_r}(0 \rightsquigarrow 0) = 0$. Thus d_{τ_r} is regular. Let $x \in B$. Then

$$
\begin{aligned}
0 &= d_{\tau_r}(0 \rightsquigarrow x) \\
&= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\
&= (0 \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\
&= (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow ((\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow \tau_r(0,x))).
\end{aligned}
$$

By (B6) and (B2), we get $(\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow 0 = \tau_r(0,x) \rightsquigarrow d_{\tau_r}(x) = (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \rightsquigarrow (0 \rightsquigarrow \tau_r(0,x))$. Using (B9), we have $0 \rightsquigarrow \tau_r(0,x) = 0$. Using (B6) again, we obtain $\tau_r(0,x) = 0$. □

Next, we give a necessary and sufficient condition for (r,l)-τ_l-derivation and (r,l)-τ_r-derivation to be regular with some properties.

Theorem 3.3. *Let d_{τ_l} be an (r,l)-τ_l-derivation on B. Then, d_{τ_l} is regular and $\tau_l(x,0) = 0$, for all $x \in B$ if and only if $d_{\tau_l}(x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_l} is regular and $\tau_l(x,0) = 0$, for all $x \in B$. Let $x \in B$. Then

$$
\begin{aligned}
\text{(B2)} \quad d_{\tau_l}(x) &= d_{\tau_l}(x \rightsquigarrow 0) \\
&= (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \\
&= (0 \rightsquigarrow 0) \wedge (d_{\tau_l}(x) \rightsquigarrow 0) \\
\text{(B2)} \quad &= 0 \wedge d_{\tau_l}(x) \\
&= d_{\tau_l}(x) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow 0) \\
\text{(B2)} \quad &= d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(x) \\
\text{(B1)} \quad &= 0.
\end{aligned}
$$

Conversely, suppose that $d_{\tau_l}(x) = 0$, for all $x \in B$. Then, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
0 &= d_{\tau_l}(x) \\
&= d_{\tau_l}(x \rightsquigarrow 0) && \text{(B2)} \\
&= (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \\
&= (\tau_l(x,0) \rightsquigarrow 0) \wedge (0 \rightsquigarrow \tau_l(x,0)) \\
&= \tau_l(x,0) \wedge (0 \rightsquigarrow \tau_l(x,0)) && \text{(B2)} \\
&= (0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow ((0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0)).
\end{aligned}$$

By (B6) and (B2), we get $(0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow 0 = 0 \rightsquigarrow \tau_l(x,0) = (0 \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0)$. Using (B9), we have $\tau_l(x,0) = 0$. □

Theorem 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on B. Then, d_{τ_r} is regular if and only if $d_{\tau_r}(x) = 0$, for all $x \in B$.*

Proof. Suppose that d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned}
d_{\tau_r}(x) &= d_{\tau_r}(x \rightsquigarrow 0) && \text{(B2)} \\
&= (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0)) \\
&= (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0) && \text{(B20)} \\
&= 0 \wedge d_{\tau_r}(x) && \text{(B2)} \\
&= d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0) \\
&= d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x) && \text{(B2)} \\
&= 0. && \text{(B1)}
\end{aligned}$$

The converse is obvious that d_{τ_r} is regular. □

By (B19), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on B by $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, and obtain the following proposition.

Proposition 3.1. *Let d_{τ_l} be an (l,r)-τ_l-derivation on B. If $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, then d_{τ_l} is the zero self-map on B and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$. By (B19), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned}
d_{\tau_l}(x) &= d_{\tau_l}(x \rightsquigarrow 0) && \text{(B2)} \\
&= (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \wedge (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \\
&= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(x)) \wedge (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \\
&= 0 \wedge (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) && \text{(B1)} \\
&= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow ((d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow 0) \\
&= (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow d_{\tau_l}(0)) && \text{(B2)} \\
&= 0. && \text{(B1)}
\end{aligned}$$

□

Proposition 3.2. *Let d_{τ_l} be an (r,l)-τ_l-derivation on B. If $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$, then $d_{\tau_l}(0 \rightsquigarrow x) = 0 \rightsquigarrow d_{\tau_l}(x)$, for all $x \in B$ and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$, for all $x \in B$. By (B19), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_l}(0 \rightsquigarrow x) &= (\tau_l(0,x) \rightsquigarrow d_{\tau_l}(x)) \wedge (d_{\tau_l}(0) \rightsquigarrow \tau_l(0,x)) \\ \text{(B19)} \quad &= (0 \rightsquigarrow d_{\tau_l}(x)) \wedge (0 \rightsquigarrow 0) \\ \text{(B1)} \quad &= (0 \rightsquigarrow d_{\tau_l}(x)) \wedge 0 \\ \text{(B7)} \quad &= 0 \rightsquigarrow d_{\tau_l}(x). \end{aligned}$$

□

By (B20), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on B by $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, and obtain the following proposition.

Proposition 3.3. *Let d_{τ_r} be an (l,r)-τ_r-derivation on B. If $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, then $d_{\tau_r}(0 \rightsquigarrow x) = 0 \rightsquigarrow d_{\tau_r}(x)$, for all $x \in B$ and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$. By (B20), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_r}(0 \rightsquigarrow x) &= (d_{\tau_r}(0) \rightsquigarrow \tau_r(0,x)) \wedge (\tau_r(0,x) \rightsquigarrow d_{\tau_r}(x)) \\ &= (d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x)) \wedge (d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x)) \\ \text{(B1)} \quad &= (d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x)) \wedge 0 \\ \text{(B7)} \quad &= d_{\tau_r}(0) \rightsquigarrow d_{\tau_r}(x) \\ &= 0 \rightsquigarrow d_{\tau_r}(x). \end{aligned}$$

□

Proposition 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on B. If $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$, then d_{τ_r} is the zero self-map on B and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$, for all $x \in B$. By (B20), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned} d_{\tau_r}(x) &= d_{\tau_r}(x \rightsquigarrow 0) \\ &= (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0)) \\ \text{(B20)} \quad &= (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0) \\ \text{(B2)} \quad &= 0 \wedge d_{\tau_r}(x) \\ &= d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0) \\ \text{(B2)} \quad &= d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x) \\ \text{(B1)} \quad &= 0. \end{aligned}$$

□

Theorem 3.5. *Let d_{τ_l} be a regular (r,l)-τ_l-derivation on B. Every τ_l-ideal of B is d_{τ_l}-invariant.*

Proof. Let B be an element in a τ_l-ideal I of B. Then

$$
\begin{aligned}
&\text{(B2)} && d_{\tau_l}(x) \rightsquigarrow 0 = d_{\tau_l}(x)\\
&\text{(B2)} && = d_{\tau_l}(x \rightsquigarrow 0)\\
& && = (\tau_l(x,0) \rightsquigarrow d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
& && = (\tau_l(x,0) \rightsquigarrow 0) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
&\text{(B2)} && = \tau_l(x,0) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0))\\
& && = (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \rightsquigarrow ((d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)) \rightsquigarrow \tau_l(x,0))\\
&\text{(B10)} && = d_{\tau_l}(x) \rightsquigarrow (d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)).
\end{aligned}
$$

By (B9), we get $0 = d_{\tau_l}(x) \rightsquigarrow \tau_l(x,0)$. Since I is a τ_l-ideal of B and by (B6), we have $d_{\tau_l}(x) = \tau_l(x,0) \in I$. Hence, I is d_{τ_l}-invariant. □

Theorem 3.6. *Let d_{τ_r} be a regular (r,l)-τ_r-derivation on B. Every ideal of B is d_{τ_r}-invariant. In particular, every τ_r-ideal of B is d_{τ_r}-invariant.*

Proof. Let B be an element in an ideal I on B. Then

$$
\begin{aligned}
&\text{(B2)} && d_{\tau_r}(x) = d_{\tau_r}(x \rightsquigarrow 0)\\
& && = (\tau_r(x,0) \rightsquigarrow d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \rightsquigarrow \tau_r(x,0))\\
&\text{(B20)} && = (0 \rightsquigarrow 0) \wedge (d_{\tau_r}(x) \rightsquigarrow 0)\\
&\text{(B2)} && = 0 \wedge d_{\tau_r}(x)\\
& && = d_{\tau_r}(x) \rightsquigarrow (d_{\tau_r}(x) \rightsquigarrow 0)\\
&\text{(B2)} && = d_{\tau_r}(x) \rightsquigarrow d_{\tau_r}(x)\\
&\text{(B1)} && = 0 \in I.
\end{aligned}
$$

Hence, I is d_{τ_r}-invariant. □

Next, we focus on composition of τ_l and τ_r-derivations.

Theorem 3.7. *Let d_{τ_r} and D_{τ_r} be (l,r)-τ_r-derivations on a 0-commutative B-algebra B. If $\tau_r(x,y) = y$, for all $x, y \in B$, then $d_{\tau_r} \circ D_{\tau_r}$ is also an (l,r)-τ_r-derivation on B.*

Proof. Suppose that $\tau_r(x,y) = y$, for all $x, y \in B$. Then

$$
\begin{aligned}
&(d_{\tau_r} \circ D_{\tau_r})(x \rightsquigarrow y)\\
&= d_{\tau_r}(D_{\tau_r}(x \rightsquigarrow y))\\
&= d_{\tau_r}((D_{\tau_r}(x) \rightsquigarrow \tau_r(x,y)) \wedge (\tau_r(x,y) \rightsquigarrow D_{\tau_r}(y)))\\
&= d_{\tau_r}((D_{\tau_r}(x) \rightsquigarrow y) \wedge (y \rightsquigarrow D_{\tau_r}(y)))
\end{aligned}
$$

$$
\begin{aligned}
\text{(B18)} \quad &= d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow y) \\
&= (d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(D_{\tau_r}(x), y)) \wedge (\tau_r(D_{\tau_r}(x), y) \rightsquigarrow d_{\tau_r}(y)) \\
\text{(B18)} \quad &= d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(D_{\tau_r}(x), y) \\
&= d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow y \\
&= d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(x, y) \\
\text{(B18)} \quad &= (d_{\tau_r}(D_{\tau_r}(x)) \rightsquigarrow \tau_r(x, y)) \wedge (\tau_r(x, y) \rightsquigarrow d_{\tau_r}(D_{\tau_r}(y)) \\
&= ((d_{\tau_r} \circ D_{\tau_r})(x) \rightsquigarrow \tau_r(x, y)) \wedge (\tau_r(x, y) \rightsquigarrow (d_{\tau_r} \circ D_{\tau_r})(y)).
\end{aligned}
$$

Hence, $d_{\tau_r} \circ D_{\tau_r}$ is an (l, r)-τ_l-derivation on B. □

Theorem 3.8. *Let d_{τ_l} and D_{τ_l} be (r, l)-τ_l-derivations on a 0-commutative B-algebra B. If $\tau_l(x, y) = x$, for all $x, y \in B$, then $d_{\tau_l} \circ D_{\tau_l}$ is also an (r, l)-τ_l-derivation on B.*

Proof. Suppose that $\tau_r(x, y) = x$, for all $x, y \in B$. Then

$$
\begin{aligned}
&(d_{\tau_l} \circ D_{\tau_l})(x \rightsquigarrow y) \\
&= d_{\tau_l}(D_{\tau_l}(x \rightsquigarrow y)) \\
&= d_{\tau_l}((\tau_l(x, y) \rightsquigarrow D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \rightsquigarrow \tau_l(x, y))) \\
&= d_{\tau_l}((x \rightsquigarrow D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \rightsquigarrow x)) \\
\text{(B18)} \quad &= d_{\tau_l}(x \rightsquigarrow D_{\tau_l}(y)) \\
&= (\tau_l(x, D_{\tau_l}(y)) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(x) \rightsquigarrow \tau_l(x, D_{\tau_l}(y))) \\
\text{(B18)} \quad &= \tau_l(x, D_{\tau_l}(y)) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
&= x \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
&= \tau_l(x, y) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y)) \\
\text{(B18)} \quad &= (\tau_l(x, y) \rightsquigarrow d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(D_{\tau_l}(x)) \rightsquigarrow \tau_l(x, y)) \\
&= (\tau_l(x, y) \rightsquigarrow (d_{\tau_l} \circ D_{\tau_l})(y)) \wedge ((d_{\tau_l} \circ D_{\tau_l})(x) \rightsquigarrow \tau_l(x, y)).
\end{aligned}
$$

Hence, $d_{\tau_l} \circ D_{\tau_l}$ is an (r, l)-τ_l-derivation on B. □

Proposition 3.5. *Let d_{τ_r} be an (r, l)-τ_r-derivation and D_{τ_r} be an (l, r)-τ_r-derivation on a 0-commutative B-algebra B. Then*

(1) $(\forall x \in B)((d_{\tau_r} \circ D_{\tau_r})(x) = 0)$ *if d_{τ_r} is regular,*

(2) $(\forall x \in B)((D_{\tau_r} \circ d_{\tau_r})(x) = D_{\tau_r}(0) \rightsquigarrow \tau_r(0, d_{\tau_r}(0)))$,

(3) $(\forall x \in B)((d_{\tau_r} \circ D_{\tau_r})(x) = 0 = (D_{\tau_r} \circ d_{\tau_r})(x))$ *if d_{τ_r} and D_{τ_r} are regular.*

Proof. (1) Suppose that d_{τ_r} is regular. Let $x \in B$. Then

$$\begin{aligned}
&\text{(B2)} && (d_{\tau_r} \circ D_{\tau_r})(x) = (d_{\tau_r} \circ D_{\tau_r})(x \rightsquigarrow 0) \\
& && = d_{\tau_r}(D_{\tau_r}(x \rightsquigarrow 0)) \\
&\text{(B18)} && = d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow \tau_r(x, 0)) \\
&\text{(B20)} && = d_{\tau_r}(D_{\tau_r}(x) \rightsquigarrow 0) \\
&\text{(B18)} && = \tau_r(D_{\tau_r}(x), 0) \rightsquigarrow d_{\tau_r}(0) \\
&\text{(B20)} && = 0 \rightsquigarrow 0 \\
&\text{(B1)} && = 0.
\end{aligned}$$

(2) Let $x \in B$. Then

$$\begin{aligned}
&\text{(B2)} && (D_{\tau_r} \circ d_{\tau_r})(x) = (D_{\tau_r} \circ d_{\tau_r})(x \rightsquigarrow 0) \\
& && = D_{\tau_r}(d_{\tau_r}(x \rightsquigarrow 0)) \\
&\text{(B18)} && = D_{\tau_r}(\tau_r(x, 0) \rightsquigarrow d_{\tau_r}(0)) \\
&\text{(B20)} && = D_{\tau_r}(0 \rightsquigarrow d_{\tau_r}(0)) \\
&\text{(B18)} && = D_{\tau_r}(0) \rightsquigarrow \tau_r(0, d_{\tau_r}(0)).
\end{aligned}$$

(3) It is straightforward by (1) and (2). □

Let $\mathrm{Der}_\tau(B)$ be the set of all τ-derivations on a 0-commutative B-algebra B. For $d_\tau, D_\tau \in \mathrm{Der}_\tau(B)$, we define the binary operation $\curlywedge$ on $\mathrm{Der}_\tau(B)$ as follows:

$$(\forall x \in B)((d_\tau \curlywedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x)).$$

Indeed, let $x \in B$. Then

$$\begin{aligned}
& && (d_\tau \curlywedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x) \\
&\text{(B18)} && = d_\tau(x).
\end{aligned}$$

Hence, $d_\tau \curlywedge D_\tau = d_\tau$, for all $d_\tau, D_\tau \in \mathrm{Der}_\tau(B)$.

Therefore, we immediately get the result as the following proposition.

Proposition 3.6. *For a 0-commutative B-algebra B, $(\mathrm{Der}_\tau(B), \curlywedge)$ is the left zero semigroup.*

4. Conclusion and discussion

In this paper, we have introduced the concept of an (l, r) and an (r, l)-τ-derivation on a B-algebra which is induced by a left and a right bi-endomorphism and provided important properties. The study found that the composition of (l, r) and an (r, l)-τ-derivations is also an (l, r) and an (r, l)-τ-derivation on a 0-commutative B-algebra, respectively. In addition, we can show that the set of all τ-derivations on a 0-commutative B-algebra B is the left zero semigroup. Finally, the study of a bi-endomorphism on other algebras (d/BH/BF/BG-algebras) is an interesting open problem.

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