The cycle-complete graph Ramsey numbers $R(C_7, K_9)$ and $R(C_8, K_9)$

A. Baniabedalruhman*
Department of Mathematics
Yarmouk University
Irbid
Jordan
ahmad_a@yu.edu.jo

Ahmad Alrifai
Department of Mathematics
Yarmouk University
Irbid
Jordan
ahmad12982@yahoo.com

Ahmad A. Al-Rhayyel
Department of Mathematics
Yarmouk University
Irbid
Jordan
Al-Rhayyel@yu.edu.jo

Abstract. The Ramsey number $R(G, H)$ is the smallest integer n such that any graph of order n contains the graph G or its complement contains the graph H. In this paper, we prove that $R(C_s, K_9) = 8s - 7$ for $s = 7, 8$, where C_s is a cycle of order s and K_r is the complete graph of order r.
Keywords: Ramsey number, cycle graph, complete graph.

1. Introduction

In this paper, all graphs are considered simple graphs. The Ramsey number $R(G, H)$ is the smallest integer n such that any graph of order n contains the graph G, or its complement contains the graph H. Note that if a complement of a graph Y, denoted by Y^c, contains K_r, then $\alpha(Y) \geqslant r$, where $\alpha(Y)$ is the number of independence of a graph Y. The vertex set of a graph Y is denoted by $V(Y)$ and the edge set by $E(Y)$. The order of a graph Y is the number of vertices in the graph, denoted by $\mid V(Y) \mid$ or simply by $\mid Y \mid$. A path of order s is denoted by P_s. The graph $K_1 + P_s$ is obtained by adding one vertex to the path P_s and connecting this vertex to each vertex of P_s. The neighborhood $N(u)$ of a vertex u is the set of vertices adjacent to u and $N[u] = N(u) \cup \{u\}$. The

*. Corresponding author

neighborhood of a vertex u in a graph Y is denoted by $N_Y(u)$. For $U \subseteq V(G)$, $N(U)$ is defined as $\bigcup_{u \in U} N(u)$ and $N[U] = N(U) \cup U$. The degree of a vertex u of a graph G is $\mid N(u) \mid$ and is denoted by $d(u)$. The minimum degree of a graph Y is denoted by $\delta(Y)$. The graph obtained by deleting the vertex u and all edges adjacent to u from the graph G is denoted by $G - \{u\}$ and $G - U = \cup_{i=1}^{n}(G - \{u_i\})$ where $n = |U|$ or simply by $G \setminus \{u\}$ and $G \setminus U$.

In 1976, it has been conjectured by Erdös et al. [10] that

$$R(C_m, K_n) = (n-1)(m-1) + 1; \quad \forall \ 3 \leq n \leq m,$$

except for $n = m = 3$, since $R(3, 3) = 6$. The conjecture was proved for $m \geq n^2 - 2$ by Bondy and Erdös [6]. Moreover, in the early works on Ramsey number, the conjecture was proved for $n = 3$ [11, 19]. Yang et al. [22] and Bollobás et al. [5] confirmed the conjecture for $n = 4$ and $n = 5$, respectively. Schiermeyer [20] showed that the conjecture is true for $n = 6$. Furthermore, Cheng et al. [8] confirmed the result in the conjecture for $n = 7$ and proved that $R(C_7, K_8) = 43$. Jaradat and Alzaleq [15] proved the conjecture for $n = m = 8$. Moreover, Bataineh, Jaradat and Al-Zaleq [4] proved the conjecture for $m = 9$ and $n = 8$. Greenwood and Gleason [12] and Clancy [9] showed that $R(C_3, K_4) = 9$ and $R(C_4, K_5) = 14$, respectively. Whereas, $R(C_4, K_8) = 26$ and $R(C_4, K_9) = 30$ were proved by Radziszowski and Kung-Kuen Tse[18]. Grinstead and Roberts [13] proved that $R(C_3, K_8) = 28$ and $R(C_3, K_9) = 36$. Recently, Grinstead and S. Roberts [16] showed that $R(C_5, K_8) = 29$, $R(C_6, K_9) = 41$ and $R(C_5, K_9) \leq 36$. Hendry [14] proved that $R(K_1 + P_4, K_5) = 19$. The graph $H = (n-1)K_{m-1}$ does not contain C_m and H^c doesn't contain K_n, which proves $R(C_m, K_n) \geq (n-1)(m-1) + 1$.

2. Main results

The purpose of this section is to find the Ramsey number for a cycle of order 7 and 8 versus the complete graph of order 9. Since $R(C_m, K_n) \geq (m-1)(n-1)+1$, then it is sufficient to prove that $R(C_s, K_9) \leq 8(s-1) + 1 = 8s - 7$, for $s = 7, 8$.

Theorem 2.1. *$R(C_s, K_9) = 8s - 7$, where $s = 7, 8$.*

To prove Theorem 2.1, we first prove a sequence of lemmas that are necessary to the proof.

Lemma 2.1. *Let G be a graph of order $n \geq 8s - 7$, $s = 7, 8$, that contains neither C_s nor a 9-element independent set, then $\delta(G) \geq s - 1$.*

Proof. Let u be a vertex of G such that $d(u) < s - 1$, then $|V(G - N[u])| \geq 7s - 6$. Since $R(C_s, K_8) = 7s - 6; s = 7, 8$, then $G - N[u]$ contains an 8-element independent set. This set together with the vertex u is a 9-element independent set, thus a contradiction. Therefore, $d(u) \geq s - 1$. The proof is done. □

Note that, with a similar arguments as in Lemma 2.1, $|N(\{v_1, \ldots, v_t\})| \geq t(s-2)+1$ for any idependent set $\{v_1, \ldots, v_t\}$, $2 \leq t \leq 8$ and $s = 7, 8$. Now, in what follows we let $s = 7, 8$, and we only consider graphs G of order $8s-7$ with the following conditions:

(1) $\delta(G) \geq s-1$.

(2) G does not contain C_s.

Lemma 2.2. *If G contains K_{s-1}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-1}\}$ be the vertex set of K_{s-1}, $R = G - U$ and $U_i = N(u_i) \cap V(R)$; $1 \leq i \leq s-1$.

By (1)

$$(*) \qquad U_i \neq \phi, \quad 1 \leq i \leq s-1.$$

Note that, between any two vertices of K_{s-1} there are paths of orders $s-1, s-2, \ldots, 2$. By $(*)$ and (2), for $1 \leq i < j \leq s-1$, we have the following observations:

1. $U_i \cap U_j = \phi$.
2. $N(U_i) \cap N(U_j) \cap R = \phi$.
3. if $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, then x and y are independent.

If $\alpha(N[U_i] \cap R) \geq 2$ for all $i = 1, \ldots, s-1$, then each $N[U_i] \cap R$ contains two independent vertices, hence we get at least $12 \geq 9$ independent vertices in G. If $\alpha(N[U_i] \cap R) = 1$ for some i, we may assume $\alpha(N[U_1] \cap R) = 1$, then $N[U_1] \cap R$ is complete. By condition (1) $|N[U_1] \cap R| \geq s-1$. Hence, by (2) if $x, y \in (N[U_1] \cap R)$, then $N(x) \cap N(y) \cap (R \setminus N[U_1]) = \phi$ and $uv \notin E(G)$ for any $u \in N(x) \cap (R \setminus N[U_1])$ and $v \in N(y) \cap (R \setminus N[U_1])$, and hence $\alpha(R \cap N(N[U_1])) > s-2$.

Also, by (2) $N(N[U_1]) \cap R \cap U_i = \phi$; $2 \leq i \leq s-1$ and if $w \in N(N[U_1]) \cap R$ and $v \in U_i$; $2 \leq i \leq s-1$, then v and w are independent. Therefore, $\alpha(G) \geq \alpha((N(N[U_1]) \cap R) \cup (\cup_{i=2}^{s-1} U_i)) \geq s-2+s-2 \geq 9$. The proof is done. □

Lemma 2.3. *If G contains K_{s-2} and doesn't contain K_{s-1}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-2}\}$ be the vertex set of K_{s-2}, $R = G - U$ and $U_i = N(u_i) \cap R$, $i = 1, \ldots, s-2$. By (1) $|U_i| \geq 2$; $i = 1, \ldots, s-2$. To complete the proof, we consider two cases:

Case 2.3.1. *$U_i \cap U_j \neq \phi$, for some $1 \leq i < j \leq s-2$.* □

Proof. Without loss of generality, assume that $U_1 \cap U_2 \neq \phi$ and let $u \in U_1 \cap U_2$.

By (1) $U_i - \{u\} \neq \phi; 1 \leq i \leq s-2$ and by (2) $xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $x \neq y$, $1 \leq i < j \leq s-2$. Moreover, by (2) $(U_i - \{u\}) \cap (U_j - \{u\}) = \phi$ and $N(U_i - \{u\}) \cap N(U_j - \{u\}) \cap R = \phi$; $1 \leq i < j \leq s-2$ and $\{i, j\} \neq \{1, 2\}$.

Now, let $v_i \in U_i - \{u\}$; $i = 3, \ldots, s-2$ and $V_i = N[v_i] \cap R$, then by (1) $|V_i| \geq s-1$. Since G doesn't contain K_{s-1}, then $\alpha(V_i) \geq 2$.

Also, by (2) if $x \in V_i$ and $y \in V_j$, then x and y are independent, $3 \leq i < j \leq s-2$. Thus, $\alpha(V_3 \cup \ldots \cup V_{s-2}) \geq 2(s-4)$.

Now, we consider three cases:

Case 2.3.1.1. $|U_1 \cap U_2| \geq 3$. □

Proof. Suppose that $|U_1 \cap U_2| \geq 3$. Then, by (2) $xy \notin E(G)$ for any $x \in U_1$ and $y \in U_2$ and hence $\{x, y, u\}$ is a 3-element independent set. Therefore, $\alpha(G) \geq 2(s-4) + 3 = 2s - 5 \geq 9$, because s=7,8.

Case 2.3.1.2. $|U_1 \cap U_2| = 2$. □

Proof. Suppose that $|U_1 \cap U_2| = 2$, say $\{u, a\} = U_1 \cap U_2$. By (2), if $x \in (N(u) \cap R)$ and $y \in (N(a) \cap R)$, then $\{y, x, u_1\}$ is a 3-element independent set. Therefore, $\alpha(G) \geq 9$.

Case 2.3.1.3. $|U_1 \cap U_2| = 1$. □

Proof. Suppose that $|U_1 \cap U_2| = 1$. Then by (2) u is isolated in U_1 and U_2, so $\alpha(U_1 \cup U_2) \geq 3$ and hence $\alpha(G) \geq 9$.

Case 2.3.2. *$U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$.* □

Proof. By (2) $N[U_i] \cap N[U_j] \cap R = \phi$; $1 \leq i < j \leq s-2$. Also, by (2), if $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, then x and y are independent.

Now, by (1) $|N[U_i] \cap R| \geq s-1$; $1 \leq i \leq s-2$. Since G doesn't contain K_{s-1}, then

$\alpha(N[U_i] \cap R) \geq 2$; $1 \leq i \leq s-2$. Since $s-2 \geq 5$, then $\alpha(G) \geq \alpha(\cup_{i=1}^{s-2} N[U_i] \cap R) \geq 2(s-2) \geq 10 \geq 9$. The proof is done. □

Lemma 2.4. *If G contains $K_1 + P_{s-2}$ and does not contain K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-2}\}$ be the vertices of P_{s-2} and v be the vertex of K_1. Suppose that $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$; $i = 1, \ldots s-2$. To complete the proof, we consider two cases:

Case 2.4.1. *$U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$.* □

Proof. By (2) if $x \in U_i$ and $y \in U_j$, then x and y are independent. Also, $N[U_i] \cap N[U_j] \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $i = 1, \ldots s-2$. Now, since G does not contain K_{s-2} and $|N[U_i] \cap R| \geq s-1$, then $\alpha(N[U_i] \cap R) \geq 2$. Therefore, $\alpha(G) \geq 2(s-2) = 2s - 4 \geq 9$.

Case 2.4.2. *$U_i \cap U_j \neq \phi$ for some $1 \leq i < j \leq s-2$. By (2) $\{i,j\}$ can be $\{2,4\}$ when s=7, and any pair of $\{2,4\},\{2,5\},\{3,5\}$ when s=8.* □

Proof. Note that, by (2) only one pair of the $U_i's$ is allowed to intersect, so we may assume this pair to be U_2 and U_4, $U_2 \cap U_4 \neq \phi$. Let $u \in U_2 \cap U_4$. By(2), $U_i \cap U_j = \phi$; $1 \leq i < j \leq s-2$ and $\{i,j\} \neq \{2,4\}$. If $x \in U_i$ and $y \in U_j$ and $x \neq y$, then x and y are independent, $1 \leq i < j \leq s-2$. Also, by (2) $N[U_i] \cap N[U_j] \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $1 \leq i < j \leq s-2$ and $\{i,j\} \neq \{2,4\}$. By (1) $|N[U_i] \cap R| \geq s-2$, hence $\alpha(N[U_i] \cap R) \geq 2$; $i \in \{1,\ldots,s-2\} \setminus \{2,4\}$. If s=8, then $\alpha(G) \geq 4(2) + |\{u\}| = 9$. For s=7, by (2) $xv \notin E(G)$ for any $x \in N[U_i] \cap R$, $i = 1,3,5$, and $xv \notin E(G)$ for any $x \in N(u) \cap R$. Now, if $\alpha(N(u) \cap R) \geq 2$, then $\alpha(G) \geq 2(3)+2+|\{v\}| = 9$. And, if $\alpha(N(u) \cap R) = 1$, then $N[u] \cap R$ is K_4 and $\alpha(N(N[u] \cap R) \cap (R-(N[u] \cap R))) \geq 2$. Therefore, $\alpha(G) \geq 2(3) + 2 + |\{u\}| = 9$. The proof is done. □

Lemma 2.5. *If G contains K_1+C_{s-3} and does not contain K_{s-2} and K_1+P_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1,\ldots,u_{s-3}\}$ and v be the vertex set of P_{s-3} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$, $i = 1,\ldots s-3$. By (1) $|U_i| \geq 2$. If $U_i \cap U_j \neq \phi$ for some $i = a$ and $j = b$, then $U_i \cap U_j = \phi$ for $1 \leq i < j \leq s-3$ and $\{i,j\} \neq \{a,b\}$. Now, by (2) if $x \in U_a \cap U_b$, then x is isolated in U_a and in U_b. Since G does not contain $K_1 + P_{s-2}$, then by (1) $|(U_a \cup U_b) \setminus \{x\}| \geq 2$. Moreover, by (2) $uw \notin E(G)$ for any $u \in U_i$ and $w \in U_j$, or $u \in N(U_i) \cap R$ and $w \in N(U_j) \cap R$, $i = 1,\ldots s-3$. Note that there are at least four independent vertices, y_1, y_2, y_3, and y_4 with $y_i \in U_i$, $y_i \neq x$ and $|N(y_i) \cap R| \geq s-3$. Also, these points exist even if $U_i \cap U_j = \phi$, $1 \leq i < j \leq s-3$. Since G doesn't contain K_{s-2}, then $\alpha(N[y_i] \cap R) \geq 2$, $i = 1,2,3,4$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{4}(N[y_i] \cap R) \cup \{v\}) \geq 9$, where v is the vertex of K_1. If $U_i \cap U_j = \phi$, then with a similar arguments as above we have $\alpha(G) \geq 9$. The proof is done. □

Lemma 2.6. *If G contains K_1+P_{s-3} and doesn't contain K_1+P_{s-2}, K_1+C_{s-3} and K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1,\ldots,u_{s-3}\}$ and v be the vertex set of P_{s-3} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and $U_i = N(u_i) \cap R$, $i = 1,\ldots s-3$. By (1) $|U_i| \geq 2$. We consider two cases :

Case 2.6.1. *$x_i x_j \in E(G)$ for some $x_i, x_j \in N(v) \cap R$. Suppose that $x_1 x_2 \in E(G)$. Now, since G doesn't contain $K_1 + P_{s-2}$, then $x_i \notin U_j$ where $i = 1,2$ and $j = 1,\ldots,s-3$ except for $j = 3$ when $s = 8$.* □

Proof. For $s = 7$, $xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $x \neq y$ and $1 \leq i < j \leq 4$. Thus, by (2) there exist $y_i \in U_i$, $1 \leq i < j \leq s-3$, such that $\{y_1, y_2, \ldots, y_{s-3}\}$ is an independent set. Also, by (2) $|U_i \cap U_j \cap U_k| \leq 1$ for any

$\{i \neq j \neq k\} \in \{1, \ldots, s-3\}$. Therefore, by (1) $|N(y_i) \cap R| \geq s-3$ for at least three vertices of $\{y_1, \ldots, y_{s-3}\}$. Moreover, by (2) $N(y_i) \cap N(y_j) \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N(y_i)$ and $y \in N(y_j)$; $1 \leq i < j \leq 4$. Also, $x_1 \notin N(y_i)$, $N(x_1) \cap N(y_i) \cap R = \phi$ and $ab \notin E(G)$ for any $a \in N(y_i) \cap R$ and $b \in N[x_1] \cap R$, $1 \leq i \leq 4$. By (1) $|N[x_1] \cap R| \geq s-1$ and hence $\alpha(N[x_1] \cap R) \geq 2$. Now, since G doesn't contain K_{s-2} and $|N[y_i] \cap R| \geq s-2$ for at least 3 values of i, then $\alpha(N(y_i) \cap R) \geq 2$ for at least three values of i. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{4}(N(y_i) \cap R) \cup (N[x_1] \cap R)) \geq 3(2) + 1 + 2 = 9$.

For $s = 8$, we consider two subcases:

Case 2.6.1.1. *$x_i u_3 \in E(G)$ for some $i = 1, 2$.* □

Proof. By (2) $U_i \cap U_j = \phi$ for $i < 3$ and $j > 3$ and $U_i \cap (N[x_j] \cap R) = \phi$ for $1 \leq i \leq 5$, $i \neq 3$ and $1 \leq j \leq 2$. Also, by (2) $|U_i \cap U_j \cap U_k| \leq 1$ for any $\{i \neq j \neq k\} \in \{1, \ldots, s-3\}$ and hence $|N[U_i] \cap R| \geq 6$ for at least three values of $i \neq 3$, say $i = 1, 2, 4$. Since G does not contain K_{s-2}, then $\alpha(N(U_i) \cap R) \geq 2$; for $i = 1, 2, 4$ and $\alpha(N[\{x_1, x_2\}] \cap R) \geq 2$.

By (2) $xy \notin E(G)$ for any $x \in (N[U_i] \cap R)$ and $y \in ((N[U_j] \cup N[x_1, x_2]) \cap R)$, $i = 1, 2, 4, 5$, $j = 1, 2, 4, 5$ and $i \neq j$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1, i\neq 3}^{5}(N[U_i] \cap R) \cup (N[x_1, x_2] \cap R)) \geq 3(2) + 1 + 2 = 9$.

Case 2.6.1.2. *$x_i u_3 \notin E(G)$; $i = 1, 2$.* □

Proof. Two subsubcases are considered:

Case 2.6.1.2.1. *$U_3 \cap N(x_i) \neq \phi$ for some $i = 1, 2$.* □

Proof. By (2) $U_i \cap (N(x_j) \cap R) = \phi$, $1 \leq i \leq 5$, $i \neq 3$, $j = 1, 2$. Also, $U_i \cap U_j = \phi$ and $N(U_i) \cap N(U_j) \cap R = \phi$, $1 \leq i < j \leq 5$ and hence $|N[U_i] \cap R| \geq 7$ and $\alpha(N[U_i] \cap R) \geq 2$.

Moreover, $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in N[U_j] \cap R$, $1 \leq i < j \leq 5$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1, i\neq 3}^{5}(N[U_i] \cap R) \cup \{x_1\}) \geq 4(2) + 1 = 9$.

Case 2.6.1.2.2. *$U_3 \cap N(x_i) \cap R = \phi$,$i = 1, 2$.* □

Proof. By (2) $U_j \cap N(x_i) \cap R = \phi$ and $(N(U_j) \cap R) \cap (N(x_i) \cap R) = \phi$, $j = 1, \ldots, 5$, $i = 1, 2$. Moreover, $xy \notin E(G)$ for any $x \in N[U_i] \cap R$ and $y \in ((N[U_j] \cup N[x_1, x_2]) \cap R)$. Also, $|N(U_i) \cap R| \geq 5$ for at least 3 values of i, say i = 1,2, 3, and hence $\alpha(N(U_i) \cap R) \geq 2$, $1 \leq i \leq 3$, and $\alpha(N(x_1) \cap R) \geq 2$. Therefore, $\alpha(G) \geq \alpha(\cup_{i=1}^{3}(N[U_i] \cap R) \cup (N[x_1] \cap R) \cup \{b\})) \geq 9$, where $b \in (N(U_4) \cap R)$ such that $|N[U_4] \cap R| \leq 5$.

Case 2.6.2. *$x_i x_j \notin E(G)$ for any $x_i, x_j \in N(v) \cap R$.* □

Proof. Note that $\alpha(N(v))$ is the minimum when $d(v) = s-1$ which will be consider in the proof. Since G does not contain $K_1 + C_{s-3}$ and $K_1 + P_{s-2}$, then $k = \{x_1, x_2, u_1, u_{s-3}\}$ is an independent set. Since k is independent, then $|N(k)| \geq 4(s-2) + 1$ as, otherwise, G contains a 9-element independent set.

Therefore, $|N(k) \setminus (K_1 + P_{s-3} \setminus \{k\})| \geq 4s - 7 - (s-4) = 3s - 3 \geq R(C_{s-3}, K_6)$. Moreover, since between any two points of k there is a path of order 3 and G doesn't contain $K_1 + C_{s-3}$, then $N(k) \cap R$ doesn't contain C_{s-3} and hence $\alpha((N(k) \cap R) \cup \{v\}) \geq 7$, say $W = \{w_1, \ldots, w_6, v\}$ is an independent set. Also, since G does not contain $K_1 + C_{s-3}$, then by (2) $N(W) \cap (G \setminus N[v])$ doesn't contain C_{s-3}. Note that, $|N(W)| \geq 7(s-2)+1$ and $|N(W) \setminus N(v)| \geq 6s-12$. Now, since $R(C_{s-3}, K_8) \leq 6s-12$, then $\alpha(N(W) \setminus N[v]| \geq 8$ and hence $\alpha(N(W) \cup \{v\}) \geq 9$.

These cases complete the proof. □

Remark 1. Suppose that G doesn't contain K_{s-2} and $K_1 + P_{s-3}$. If G contains $K_4 = \{x_1, x_2, x_3, x_4\}$, then

$$\alpha(N(x_1, x_2, x_3)) \geq \begin{cases} 3, & \text{if } s = 7 \\ 2, & \text{if } s = 8 \end{cases}$$

.

Proof. The proof is clear for $s = 8$ because G does not contain K_{s-2}. For $s = 7$, let $R = V(G) \setminus V(K_4)$. Since G doesn't contain $K_1 + P_{s-3}$, then $N(x_i) \cap N(x_j) \cap R = \phi$, $1 \leq i < j \leq 4$. By(1) $|N(x_i)| \geq 3$, $i = 1, 2, 3$. By(2), if $xy \in E(G)$ for some $x \in N(x_1) \cap R$ and $y \in N(x_2) \cap R$, then x is isolated in $N(x_1) \cap R$ and y is isolated in $N(x_2) \cap R$. Therefore, $\alpha(N(x_1) \cup N(x_2)) \geq 2$ and $\alpha(N(x_3)) \geq 1$. By(2) $xy \notin E(G)$ for any $x \in N(x_3)$ and $y \in N(x_i); i = 1, 2$, therefore, $\alpha(N(x_1, x_2, x_3) \cap R) \geq 3$. If $xy \notin E(G)$ for any $x \in N(x_i)$ and $y \in N(x_j); 1 \leq i < j \leq 3$, then $\alpha(N(x_1, x_2, x_3) \cap R) \geq 3$. The proof is done. □

Remark 2. *If G doesn't contain K_{s-2} and $K_1 + P_{s-3}$ and $\{u, v\}$ are independent vertices that are adjacent to the vertex of K_1, then $\alpha(N(u, v) \setminus K_1) \geq 3$.*

Proof. Since u and v are independent, then $|N(u, v)| \geq 2s - 3$ and hence $|N(u, v) \setminus K_1| \geq 2s - 4$. Since $R(C_{s-3}, K_3) = 2s - 7 \leq 2s - 4$, then $\alpha(N(u, v) \setminus K_1) \geq 3$. The proof is done. □

Lemma 2.7. *If G contains K_{s-3} and doesn't contain $K_1 + P_{s-3}$ and K_{s-2}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-3}\}$ be the vertex set of K_{s-3}, $R = G - U$ and $U_i = R \cap N(u_i)$, $i = 1, \ldots, s-3$. Since G doesn't contain $K_1 + P_{s-3}$, then $U_i \cap U_j = \phi; 1 \leq i < j \leq s-3$. By (1) $\lceil U_i \geq 3$, $i = 1, \ldots, s-3$. Also, by (2) $R \cap N(U_i) \cap N(U_j) = \phi$ and $xy \notin E(G)$ for any $x \in N(U_i) \cap R$ and $y \in N(U_j) \cap R$, $1 \leq i < j \leq s-3$. To complete the proof we consider two cases:

Case 2.7.1. *$xy \in E(G)$ for some $x \in U_i$ and $y \in U_j$, $1 \leq i < j \leq s-3$.* □

Proof. Without loss of generality, assume that $x \in U_1$ and $y \in U_2$. By (2) x is isolated in U_1 and y is isolated in U_2, therefore $\alpha(U_1 \cup U_2) \geq 3$. Also, by (2) $st \notin E(G)$ for any $s \in U_i$ and $t \in U_j$, $i = 3, 4$, $1 \leq j \leq 4$ and $i \neq j$. Now, by Remarks 1 and 2 , $\alpha((N[U_3] \ldots \cup N[U_{s-3}]) \cap R) \geq 6$. Therefore, $\alpha(G) \geq \alpha(U_1 \cup U_2 \cup (N[U_3 \ldots \cup U_{s-3}] \cap R)) \geq 3 + 6 = 9$.

Case 2.7.2. *$xy \notin E(G)$ for any $x \in U_i$ and $y \in U_j$, $1 \leq i < j \leq s-3$.* □

Proof. By Remarks 1 and 2, $\alpha(N[U_i] \cap R) \geq 3$ for all $1 \leq i \leq s-3$, therefore, $\alpha(G) \geq 9$.

The proof is done. □

Remark 3. *Suppose that G contains $K_1 + P_4$ and $|N(P_4) \setminus K_1| \geq 10$. If G doesn't contain $C_8, K_1 + P_5$ and K_5, then $\alpha(N(P_4) \setminus K_1) \geq 4$.*

Proof. If $N(P_4) \setminus K_1$ contains path of order 4, say L, then since G doesn't contain C_8 the vertices of L are adjacent to one vertex in $V(P_4)$. Also, since G doesn't contain $K_1 + P_5$ and C_8, then the vertices of L and $(N(P_4) \setminus K_1) \setminus L$ are disjoint. Therefore, $\alpha(N(P_4) \setminus K_1) \geq \alpha(L) + \alpha((N(P_4) \setminus K_1) \setminus L) \geq 2 + 2 = 4$. If $N(P_4 \setminus K_1)$ doesn't contain P_4, then $\alpha(N(P_4) \setminus K_1) \geq \lceil \frac{|N(P_4) \setminus K_1|}{3} \rceil \geq 4$. The proof is done. □

Remark 4. *Suppose that G contains $K_1 + P_3$ and $|N(P_3) \setminus K_1| \geq 9$. If G doesn't contain $C_7, K_1 + P_4$ and K_4, then $\alpha(N(K_1 + P_3)) \geq 4$.*

Proof. Since G doesn't contain C_7 and K_4, then $N(P_3) \setminus K_1$ doesn't contain C_3. Now, since $R(C_3, K_4) = 9$, then $\alpha(N(P_3) \setminus K_1) \geq 4$. The proof is done □

Lemma 2.8. *If G contains $K_1 + P_{s-4}$ and doesn't contain K_{s-3} and $K_1 + P_{s-3}$, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \ldots, u_{s-4}\}$ and v be the vertex set of P_{s-4} and K_1, respectively. Also, let $R = G - (U \cup \{v\})$ and let $U_i = N(u_i) \cap R$, $i = 1, \ldots, s-4$. By (1) $|U_i| \geq 3$, $i = 1, \ldots, s-4$. To complete the proof, we consider two cases:

Case 2.8.1. *$x_i x_j \in E(G)$ for some $x_i, x_j \in N(v) \cap R$, say $x_1, x_2 \in E(G)$.* □

Proof. By (2) and since G doesn't contain $K_1 + P_{s-3}$, then $x_i \notin U_j$ for all $i = 1, 2$ and $j = 1, \ldots, s-4$. Also, by (2) $N(U_j) \cap N(x_i) \cap R = \phi$ and $xy \notin E(G)$ for any $x \in N(U_j) \cap R$, $y \notin U_j$ and $y \in (N(x_i)) \cap R)$ for $i = 1, 2$ and $j = 1, \ldots, s-4$. Now, two subcases are considered:

Case 2.8.1.1. *$U_j \cap N(x_i) \cap R = \{u\}$ for some $i = 1, 2$ where $j = 2$ for $s = 7$ and where $j = 2$ or 3 for $s = 8$.* □

Proof. Without loss of generality, assume that $u \in U_2 \cap N(x_2) \cap R$, then by (2) $U_i \cap U_j = \phi$ for $1 \leq i < j \leq s-4$, except for $\{i, j\} = \{1, 2\}$ when $s = 8$. Moreover, $U_j \cap N(x_i) \cap (R \setminus \{u\}) = \phi$ for $i = 1, 2$, $j = 1, \ldots, s-4$ and $\{i, j\} \neq \{1, 2\}$. Also, by (2) $xy \notin E(G)$ for any $x \in U_k$ and $y \in (U_j \cup N(x_i)) \cap R$ for all $i = 1, 2$, $k, j = 1, \ldots, s-4$ and $j \neq k$. Therefore, there are two disjoint independent sets $Y \subseteq N(P_{s-4}) \setminus K_1$ and $X \subseteq N(x_1) \cap (R - \{u\})$. Note that Y has at least three independent vertices, say $W = \{w_1, w_2, w_3\}$, with each vertex adjacent to different u_i, where $i \neq 2$ for $s = 8$. By (1) $|N(w_i) \cap R| \geq s-4$ and $|X| \geq s-4$, $i = 1, 2, 3$. Since G doesn't contain K_{s-3}, then $\alpha(G) \geq \alpha(\cup_{i=1}^{3} N[w_i] \cup X \cup \{u\}) \geq 9$.

Case 2.8.1.2. *$U_j \cap N(x_i) \cap R = \phi$ for all $i = 1, 2$ and $j = 1, \dots, s-4$.* □

Proof. Since G doesn't contain K_{s-3}, then $\alpha(U) = 2$, and hence $|N(U)| \geq 2s-3$ and $|N(U) \cap R| \geq s+2$. By (2) there are two disjoint independent sets $Y \subseteq N(U) \cap R$ and $X \subseteq N(x_1, x_2) \cap R$. Since G doesn't contain K_{s-3} and $K_1 + P_{s-3}$, then $|X| \geq 2$, with two vertices in X adjacent to x_1 and x_2, and by Remarks 3 and 4, $|Y| \geq 4$. Also, by (2) $N(X) \cap (R \setminus \{x_1, x_2\})$ and $N(Y) \cap R$ are disjoint. Note that $N(Y) \cap R \geq 3s-2$ and $N(X) \cap (R \setminus \{x_1, x_2\}) \geq s+2$. With a similar argument as in Remarks 3 and 4, we get $\alpha(N(X)(R \setminus \{x_1, x_2\})) \geq 4$. Also, since $R(C_{s-3}, K_5) \leq 3s-2$ and G doesn't contain C_s and $K_1 + C_{s-3}$, then $\alpha(N(Y) \cap R) \geq 5$. Therefore, $\alpha(G) \geq 5 + 4 = 9$.

Case 2.8.2. *$x_i x_j \notin E(G)$ for all $x_i, x_j \in N(v) \cap R$.* □

Proof. In the proof, we consider the case when $|N(v)| = s-1$. Note that $\alpha(N(v)) = 5$, say $W = \{w_1, \dots, w_5\}$ is an independent set. Therefore, $|N(N(v)) \cap R| \geq 4s-4 \geq R(C_{s-3}, K_7)$. Since G does not contain C_s and $K_1 + C_{s-3}$, then $\alpha((N(W) \cap R) \cup \{v\}) \geq 8$, say $H = \{h_1, \dots, h_7, v\}$ is an independent set. Note that, $|N(H) \setminus (P_{s-4} \cup N(v))| \geq 7s-14 \geq R(C_{s-3}, K_8)$. Therefore, $\alpha((N(H) \setminus N(v)) \cup \{v\}) \geq 9$.

The proof is done. □

Lemma 2.9. *If G contains K_{s-4} and doesn't contain $K_1 + P_{s-4}$ and K_{s-3}, then G contains a 9-element independent set.*

Proof. Let $U = \{u_1, \dots, u_{s-4}\}$ be the vertex set of K_{s-4}, $R = G - U$ and $U_i = N(u_i) \cap R$. Since G doesn't contain $K_1 + P_{s-4}$, then $U_i \cap U_j = \phi$, $1 \leq i < j \leq s-4$. By (2) $\alpha(\cup_{i=1}^{s-4} U_i) \geq 6$. We consider the case when $\alpha(\cup_{i=1}^{s-4} U_i) = 6$, say $\{y_1, \dots, y_6\}$ is an independent set. Since $R(K_4, K_3) = 9$ and $R(K_5, K_3) = 14$, then $|N(y_1, \dots, y_6)| \geq 49-8$ for $s = 7$ and $|N(y_1, \dots, y_6)| \geq 57-13 = 44$ for $s = 8$. Thus, $|N(y_1, \dots, y_6) \setminus K_{s-4}| \geq 38$ for $s = 7$ and $|N(y_1, \dots, y_6) \setminus K_{s-4}| \geq 40$ for $s = 8$. Since $R(C_4, K_9) = 30$ and $R(C_5, K_9) \leq 36$, then $\alpha(N(y_1, \dots, y_6) \cap R) \geq 9$. The proof is done. □

Lemma 2.10. *If G doesn't contain K_{s-4} and $K_1 + P_{s-4}$, then G contains a 9-element independent set.*

Proof. Since $R(K_3, K_9) = 36$, then $\alpha(G) \geq 9$ for $s = 7$.

Whereas for $s = 8$, since $\delta(G) \geq 7$, then $\alpha(N(v)) \geq 4$. Here we consider the case when $\alpha(N(v)) = 4$, say $\{y_1, \dots, y_4\}$ is an independent set. Since, $R(K_1 + P_4, K_5) = 19$, then $|N(y_1, \dots, y_4) \setminus \{v\})| \geq 34$. Also, since $R(C_5, K_8) = 29$, then $\alpha(N(y_1, \dots, y_4) \setminus \{v\}) \geq 8$, say $W = \{w_1, \dots, w_8\}$ is an independent set. Moreover, $|N(W)| \geq 49$ and $|N(W) \setminus (\{y_1, \dots, y_4\} \cup \{v\})| \geq 44$. Since, $R(C_5, K_9) \leq 36$, then $\alpha(G) \geq 9$. The proof is done.

□

Proof of Theorem 2.1. Let G be a graph of order $8s-7$, where $s = 7, 8$. Suppose that G does not contain C_s. If G contains K_{s-4} then by Lemma 2.9 G contains a 9-element independent set. If G does not contain K_{s-4}, then by Lemma 2.10 G contains a 9-element independent set. Therefore, G contains either C_s, $s = 7, 8$, or a 9-element independent set. Thus, $R(C_s, K_9) = 8s - 7$, $s = 7, 8$. □

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Accepted: June 28, 2021