Matrix methods for the up and down Steenrod squares

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Abstract. Let $\mathrm{P}(n)$ be the polynomial algebra which is a graded left module over the Steenrod algebra. The divided power algebra $\mathrm{DP}^d(n)$ is defined as the Hopf dual of $\mathrm{P}^d(n)$. The dual of the Steenrod square Sq^k is the linear map $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$, called the down Steenrod square, defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where $v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big)$ for $f \in \mathrm{P}^d(n)$. In this article we consider the down Steenrod squares and establish some results about. Further, we show that there is some periodic calculation on the down Steenrod squares hanging on the limited leading period. Finally, using this fact, we exhibit a matrix method for manipulating the down Steenrod squares.

Keywords: Steenrod algebra, down Steenrod squares, divided power algebra.

1. Introduction

Consider the polynomial algebra $\mathrm{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n] = \bigoplus_{d\geq 0} \mathrm{P}^d(n)$, viewed as a graded left module over the Steenrod algebra $\mathcal{A}_2$ at the prime 2. The grading is by the homogeneous polynomials $\mathrm{P}^d(n)$ of degree d in the n variables $x_1, \ldots, x_n$ of grading 1. The algebra $\mathrm{P}(n)$ realizes the cohomology of the product of n copies of infinite real projective spaces.

The Steenrod algebra $\mathcal{A}_2$ is briefly defined as the graded algebra over the finite field $\mathbb{F}_2$, generated by symbols Sq^k, called the up Steenrod squares, in grading k, for $k \geq 0$, subject to Adem relations and the condition $Sq^0 = 1$ [4, 7]. In classical texts, the word 'up' is not written for the Steenrod squares Sq^k. However, we use this word since the dual of the Steenrod squares Sq^k, called the 'down' Steenrod squares Sq_k, are considered in this article. The latter word

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was first used in [7] for the 'going down' integral Steenrod square SQ_2 (see also [6]).

In Definition 2.1, the divided power algebra $\mathrm{DP}^d(n)$ is defined as the Hopf dual of $\mathrm{P}^d(n)$. By the down Steenrod square (Definition 2.2) it is meant the dual of $Sq^k : \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$ which is the linear map $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$ defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where

$$v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big),$$

for $f \in \mathrm{P}^d(n)$. Some properties of the Sq_k, especially the dual version of Cartan formula for these operations, are discussed in Section 2. Moreover, the Steenrod kernel $\mathrm{K}^d(n)$ is defined as the set of elements $v \in \mathrm{DP}^d(n)$ such that $Sq_k(v) = 0$ for all $k > 0$. Thus, the Steenrod kernel $\mathrm{K}^d(n)$, as a subspace of $\mathrm{DP}^d(n)$, is the transpose dual of the cohit sub-module $\mathrm{Q}^d(n)$ of $\mathrm{P}^d(n)$. The dual of the hit problem is to determine $\mathrm{K}^d(n)$. For more information about the hit and dual hit problems we cite [3, 5]. The hit and dual hit problems are not in the scope of this article.

In Section 3, we show that to manipulate the down operations Sq_k, it suffices to know only the $Sq_k(v_1^{(d_1)} \cdots v_n^{(d_n)})$ for limited values of d_i and for higher values the calculations are periodically repeated. Using this fact, in the last Section 4, certain matrices are presented and a matrix method for computing the down Steenrod squares Sq_k is exhibited. This method may be used for computer calculations of the down Steenrod squares. Also, the idea beside can be experienced for the Steenrod power operations in the mod-p Steenrod algebra $\mathcal{A}_p$ for any odd prime p.

2. Preliminaries and some properties of down squares

In this section, we review some preliminaries of the up and down Steenrod squares and establish some properties of the down squares. First, recall some fundamental concepts of the up Steenrod squares from [5, Section 1.1].

The study of Steenrod squares is often started from the total Steenrod square $\mathrm{Sq} : \mathrm{P}(n) \to \mathrm{P}(n)$ acting on the variable x_i, $1 \le i \le n$, by $\mathrm{Sq}(x_i) = x_i + x_i^2$ and $\mathrm{Sq}(1) = 1$. Then, for $k, d \ge 0$, the up Steenrod square Sq^k is defined to be the restriction of Sq to $\mathrm{P}^d(n)$ and projecting on $\mathrm{P}^{d+k}(n)$. Therefore, Sq is the formal sum $\mathrm{Sq} = \sum_{k\ge 0} Sq^k$ of its graded parts. The basic property of the up Steenrod square $Sq^k \colon \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$ is explained in the next result.

Proposition 2.1. *For all $x \in \mathrm{P}^1(n)$, $Sq^k(x^d) = \binom{d}{k} x^{d+k}$. In particular, $Sq^k(x^k) = x^{2k}$.*

The up Steenrod squares work out on products of cohomology classes $f, g \in \mathrm{P}(n)$ by Cartan formula [1].

$$Sq^k(fg) = \sum_{i+j=k} Sq^i(f) Sq^j(g).$$

The next fundamental result shows why Sq^k is called a 'square' operation.

Proposition 2.2. *For a homogeneous element* $f \in \mathrm{P}(n)$ *we have* $Sq^k(f) = f^2$ *if* $k = \deg(f)$ *and* $Sq^k(f) = 0$ *if* $k > \deg(f)$.

The following corollary is immediately concluded.

Corollary 2.1. *Let* $k = d_1 + \cdots + d_n$. *Then,*

$$Sq^k(x_1^{d_1} \cdots x_n^{d_n}) = x_1^{2d_1} \cdots x_n^{2d_n}.$$

For a definition of 'down' Steenrod operators, the concept of the divided power algebra is needed.

Definition 2.1. For $n > 0$ and $d \geq 0$, the linear dual of the $\mathbb{F}_2$-vector space $\mathrm{P}^d(n)$ is denoted by $\mathrm{DP}^d(n) = \mathrm{Hom}(\mathrm{P}^d(n), \mathbb{F}_2)$. The divided power algebra is defined as the infinite sum $\mathrm{DP}(n) = \sum_{d \geq 0} \mathrm{DP}^d(n)$ for which the grading and product is defined as follows.

Denote by $v_1, \ldots, v_n$ the basis of $\mathrm{DP}^1(n)$ dual to the basis $x_1, \ldots, x_n$ of $\mathrm{P}^1(n)$, so that $\langle v_i, x_j \rangle$ is 1 if $i = j$ and is 0 otherwise. The d-monomial $v_1^{(d_1)} \cdots v_n^{(d_n)}$ in $\mathrm{DP}(n)$ is defined as the dual of the monomial $x_1^{d_1} \cdots x_n^{d_n}$ in $\mathrm{P}(n)$, where the parenthesized exponents are called divided powers. For any $v \in \mathrm{DP}^1(n)$, put $v^{(0)} = 1$, the identity map of $\mathbb{F}_2$ which is also the identity element of $\mathrm{DP}(1)$. Let $v^{(1)} = v$ have degree 1 and define the degree of the d-monomial $v_1^{(d_1)} \cdots v_n^{(d_n)}$ as $d = d_1 + \cdots + d_n$. The degree of a d-polynomial in $\mathrm{DP}(n)$ is defined naturally.

Now, we introduce a product on $\mathrm{DP}(n)$. We start with one variable $v_1 = v$ and define the product of d-monomials in $\mathrm{DP}(1)$ by

$$v^{(r)} v^{(s)} = \binom{r+s}{r} v^{(r+s)}, \tag{1}$$

where the binomial coefficient is taken mod 2. For a field of characteristic 0 in place of $\mathbb{F}_2$, the corresponding situation is $v^{(r)} = v^r/r!$ since $v^{(r)} v = (r+1) v^{(r+1)}$ by (1). That is why $v^{(r)}$ is called the r-th divided power of v.

For $d > 0$, define $\mathrm{bin}(d) = \{2^{d_1}, \ldots, 2^{d_r}\}$, where $d = 2^{d_1} + \cdots + 2^{d_r}$ with distinct terms and $\mathrm{bin}(0) = \emptyset$. Then, the product (1) can be written as follows.

$$v^{(r)} v^{(s)} = \begin{cases} v^{(r+s)}, & \text{if } \mathrm{bin}(r) \cap \mathrm{bin}(s) = \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For a nonzero d-polynomial $u = a_0 + a_1 v + a_2 v^{(2)} + \cdots + a_s v^{(s)}$ in $\mathrm{DP}(1)$, with $a_s \neq 0$, if $a_0 = 0$ then $u^2 = 0$ by the product formula (2).

Now, the product in $\mathrm{DP}(n)$ can be defined on d-monomials by using either of the equivalent product formulas (1) or (2), for each variable v_i, and is extended for d-polynomials by linearity. This product is commutative and associative and $\mathrm{DP}(n)$ is a graded algebra over $\mathbb{F}_2$ for all $n \geq 1$. In fact, $\mathrm{DP}(n)$ is the Hopf dual of $\mathrm{P}(n)$ [5, Section 9.1].

Definition 2.2. The down Steenrod square $Sq_k : \mathrm{DP}^{d+k}(n) \to \mathrm{DP}^d(n)$ is defined to be the linear dual of the up Steenrod square $Sq^k : \mathrm{P}^d(n) \to \mathrm{P}^{d+k}(n)$. More precisely, Sq_k is defined by $Sq_k(u) = v$ for $u \in \mathrm{DP}^{d+k}(n)$, where

$$v(f) = \big(Sq_k(u)\big)(f) = u\big(Sq^k(f)\big),$$

for $f \in \mathrm{P}^d(n)$. Since Sq_k lowers degree by k, it is called the 'down' square operation in opposite to the 'up' square operation Sq^k which ups the degree. In the bilinear literature, Sq_k is redefined by

$$\langle Sq_k(u), f\rangle = \langle u, Sq^k(f)\rangle.$$

The identity property of Sq_0 is now apparent. In other words, Sq_0 is the identity homomorphism. The algebra generated by the down squares Sq_k is isomorphic to the opposite algebra $\mathcal{A}_2^{\mathrm{op}}$, and not to $\mathcal{A}_2$ itself since

$$\langle Sq_a Sq_b(u), f\rangle = \langle Sq_b(u), Sq^a(f)\rangle = \langle u, Sq^b Sq^a(f)\rangle.$$

Due to the duality essence, we started the study of down Steenrod squares with the graded parts Sq_k. The total down square $\mathrm{Sq}_* : \mathrm{DP}(n) \to \mathrm{DP}(n)$, the dual of the total up square Sq, is the linear map $\mathrm{Sq}_*(u) = \sum_{k\geq 0} Sq_k(u)$, where $u \in \mathrm{DP}(n)$. Therefore, $\langle \mathrm{Sq}_*(u), f\rangle = \langle u, \mathrm{Sq}(f)\rangle$ for $u \in \mathrm{DP}(n)$ and $f \in \mathrm{P}(n)$.

In the sequel, we assume v and all its subscripts are in $\mathrm{DP}^1(n)$. The only indecomposable single monomials in $\mathrm{P}(n)$ are x_i, $1 \leq i \leq n$, and we saw that $\mathrm{Sq}(x_i) = x_i + x_i^2$. However, by the product formula (2), the indecomposable single d-monomials in $\mathrm{DP}(n)$ are $v^{(2^t)}$ for all $t \geq 0$ and if $\mathrm{bin}(d) = \{2^{t_1}, \ldots, 2^{t_s}\}$ then $v^{(d)} = v^{(2^{t_1})} \cdots v^{(2^{t_s})}$. Now, $\mathrm{Sq}_*(v^{(2^t)}) = v^{(2^t)} + \sum_{i=0}^{t-1} v^{(2^t - 2^i)}$.

The next result is the basic property of the down Steenrod operations [5, Proposition 9.3.6].

Proposition 2.3. *For all $v \in \mathrm{DP}^1(n)$,*

$$Sq_k(v^{(d)}) = \binom{d-k}{k} v^{(d-k)}.$$

In particular, $Sq_k(v^{(2k)}) = v^{(k)}$.

Proposition 2.3 is the dual version of Proposition 2.1. The particular case in Proposition 2.3 is somehow the dual of the first part of Proposition 2.2 for one variable. The dual of the last part of Proposition 2.2 is the following.

Corollary 2.2. *For $u \in \mathrm{DP}(n)$, $Sq_k(u^{(d)}) = 0$ if $k > d/2$.*

An immediate conclusion is dual to Corollary 2.1.

Corollary 2.3. *Let $k = d_1 + \cdots + d_n$. Then,*

$$Sq_k(v_1^{(2d_1)} \cdots v_n^{(2d_n)}) = v_1^{(d_1)} \cdots v_n^{(d_n)}.$$

It is proved [5, Proposition 1.4.11] that $\binom{a}{b} = 0$ if and only if $\mathrm{bin}(b) \nsubseteq \mathrm{bin}(a)$. This determines the d-monomials in the null image of Sq_k.

Corollary 2.4. *For $k > 0$, $Sq_k(v^{(d)}) = 0$ if and only if* $\mathrm{bin}(k) \nsubseteq \mathrm{bin}(d-k)$.

The following corollaries are immediately deduced from Corollary 2.4.

Corollary 2.5. *For any non-negative integers t and m,*

$$Sq_{2^t-1}\Big(v^{\big(m2^t+2(2^t-1)\big)}\Big) = v^{\big((m+1)2^t-1\big)}.$$

Corollary 2.6. *Let $1 \leq k < 2^t$. Then,*

$$Sq_k(v^{(2^t+r)}) = 0,$$

where $r = k, \dots, 2k-1$.

Corollary 2.7. *For any $t > 0$,*

$$Sq_{2^t}(v^{(2^{t+1}+2^t+r)}) = 0,$$

where $r = 0, \dots, 2^t - 1$.

Corollary 2.7 determines zeros of $Sq_k(v^{(d)})$ when k is a 2-power. The next corollary indicates the same determination when d is a 2-power.

Corollary 2.8. *Let $1 \leq i \leq t-2$, $t \geq 3$. Then, for $2^i < k < 2^{i+1}$,*

$$Sq_k(v^{(2^t)}) = 0.$$

Cartan formula for divided power algebras enables us to calculate the action of Sq_k on the products of elements in $\mathrm{DP}(n)$ [5, Proposition 9.3.4].

Theorem 2.1 (Cartan formula for $\mathrm{DP}(n)$)**.** *For $k \geq 0$ and any $u, w \in \mathrm{DP}(n)$, we have*

$$Sq_k(uw) = \sum_{i+j=k} Sq_i(u)Sq_j(w). \tag{3}$$

For simplicity, we refer to (3) as Cartan formula. A d-monomial of the form $v_1^{(2^{j_1}-1)} \cdots v_r^{(2^{j_r}-1)}$ for some positive powers is called a d-spike. The down operations annihilates the d-spikes.

Proposition 2.4. *Let $k > 0$. Then, $Sq_k(v^{(2^j-1)}) = 0$ for any positive integer j. In general, for any d-spike $u = v_1^{(2^{j_1}-1)} \cdots v_r^{(2^{j_r}-1)}$, $Sq_k(u) = 0$.*

Proof. We prove the one-variable case. The case $k > 2^{j-1}$ follows from Proposition 2.3 since $d = 2^j - 1 < 2^j = 2 \cdot 2^{j-1} < 2k$. For $k \leq 2^{j-1}$, we have $\mathrm{bin}(k) \nsubseteq \mathrm{bin}(2^{j-1} - k)$ and the result is established. The general statement is obtained from Cartan formula. □

$i=1$						
s	2		3		4	
The d-spike:	$u^{(1)}v^{(3)}$		$u^{(1)}v^{(7)}$		$u^{(1)}v^{(15)}$	
for p:	0	1	0	1	0	1
is the Sq_3 of:	$u^{(1)}v^{(6)}$	$u^{(2)}v^{(5)}$	$u^{(1)}v^{(10)}$	$u^{(2)}v^{(9)}$	$u^{(1)}v^{(18)}$	$u^{(2)}v^{(17)}$

$i=2$								
s	3				4			
The d-spike:	$u^{(3)}v^{(7)}$				$u^{(3)}v^{(15)}$			
for p:	0	1	2	3	0	1	2	3
is the Sq_3 of:	$u^{(3)}v^{(10)}$	$u^{(4)}v^{(9)}$	$u^{(5)}v^{(8)}$	$u^{(6)}v^{(7)}$	$u^{(3)}v^{(18)}$	$u^{(4)}v^{(17)}$	$u^{(5)}v^{(16)}$	$u^{(6)}v^{(15)}$

$i=3$				
s	4			
The d-spike:	$u^{(7)}v^{(15)}$			
for p:	0	1	2	3
is the Sq_3 of:	$u^{(7)}v^{(18)}$	$u^{(8)}v^{(17)}$	$u^{(9)}v^{(16)}$	$u^{(10)}v^{(15)}$

Table 1: Proposition 2.6 for $t = 2$.

Definition 2.3. The Steenrod kernel $\mathrm{K}(n) = \sum_{d\geq 0} \mathrm{K}^d(n)$, where $\mathrm{K}^d(n)$ is the set of elements $v \in \mathrm{DP}^d(n)$ such that $Sq_k(v) = 0$ for all $k > 0$.

Proposition 2.4 says that d-spikes are always contained in the Steenrod kernel. We know that spikes cannot appear in the image of any Sq^k for $k > 0$. However, due to the duality nature, d-spikes are always in the image of some Sq_k. In corollary 2.5 (for $m = 0$) we experienced this in one variable, which says that d-spikes may be obtained from special down operators with spike gradings. The next result, which is trivial by Corollary 2.4, shows a template for d-spikes in one variable.

Proposition 2.5. *Let $v \in \mathrm{DP}^1(n)$. Then, for any $t > 0$,*

$$Sq_i(v^{(2^t-1+i)}) = v^{(2^t-1)},\ i = 1, 2, \ldots, 2^t - 1.$$

Conversely, any d-spike is obtained in this way.

Proposition 2.5 is extended for two variables.

Proposition 2.6. *For $t > 0$ let $\ell = 2^t - 1$, $s \geq t$, and $i \geq 1$. Then, for $s > i$,*

$$u^{(2^i-1)}v^{(2^s-1)} = Sq_\ell\big(u^{(2^i-1+p)}v^{(2^s-1+\ell-p)}\big),$$

where $p = 0, 1, \ldots, 2^i - 1$ if $i < t$ and $p = 0, 1, \ldots, \ell$ otherwise.

Proof. The zero case $p = 0$ is trivial. For positive p the result comes from the one-variable case in Proposition 2.5 and Corollary 2.2 using Cartan formula. □

Example 2.1. Take $t = 2$. Then, Table 1 applies Proposition 2.6 for $i = 1, 2, 3$.

Proposition 2.6 creates the 2-variable d-spikes with distinct exponents. For equal exponents we have the following result with the proof discussion as above.

Proposition 2.7. *For $t > 0$ let $\ell = 2^t - 1$ and $s \geq t$. Then,*

$$u^{(2^s-1)}v^{(2^s-1)} = Sq_\ell\big(u^{(2^s-1+r)}v^{(2^s-1+\ell-r)}\big),$$

where $r = 0, 1, \ldots, 2^{t-1} - 1$.

Example 2.2. Again, take $t = 2$. Then, Table 2 is an example of Proposition 2.7 for some values of s.

As seen in Examples 2.1 and 2.2, the methods in Propositions 2.6 and 2.7 lead often to repeated d-spikes. Contrary to the one-variable case in Proposition 2.5, the converses of Propositions 2.6 and 2.7 are not always true. For instance, the d-spike $u^{(3)}v^{(7)}$ may be created as

$$u^{(3)}v^{(7)} = Sq_8\big(u^{(4)}v^{(14)}\big) = Sq_8\big(u^{(5)}v^{(13)}\big) = Sq_8\big(u^{(6)}v^{(12)}\big).$$

Also, we can obtain $u^{(7)}v^{(7)}$ by $Sq_6\big(u^{(7)}v^{(13)}\big)$ or $Sq_6\big(u^{(5)}v^{(11)}\big)$.

For more variables than two, however, the pre-image d-monomials should have all but two variables in power of spikes. If this happens then, due to Proposition 2.4, we will have the same forms as in Propositions 2.6 and 2.7 except for the number of spike-power variables. For instance, compare the following relations with Table 1 (middle one).

$$\begin{aligned} u^{(3)}v^{(15)}w^{(63)} &= Sq_3\big(u^{(3)}v^{(18)}w^{(63)}\big) = Sq_3\big(u^{(4)}v^{(17)}w^{(63)}\big) \\ &= Sq_3\big(u^{(5)}v^{(16)}w^{(63)}\big) = Sq_3\big(u^{(6)}v^{(15)}w^{(63)}\big). \end{aligned}$$

The existence of the above condition in more than two variables is necessary; otherwise, non-d-spike terms may occur. For example,

$$Sq_3\big(u^{(2)}v^{(2)}w^{(4)}\big) = u^{(1)}v^{(1)}w^{(3)} + u^{(1)}v^{(2)}w^{(2)}.$$

All the above computations in more than one variable are up to permutation of variables.

3. Periodic calculations

For the multisets $\boldsymbol{d} = (d_1, d_2, \dots, d_n)$ and $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ of non-negative integers, we shall adopt the following abbreviated notations.

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{d}} &= x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}, \\ \boldsymbol{x}^{\boldsymbol{m}2^t} &= x_1^{m_1 2^t} x_2^{m_2 2^t} \cdots x_n^{m_n 2^t}, \\ \boldsymbol{x}^{\boldsymbol{m}2^t+\boldsymbol{d}} &= x_1^{m_1 2^t + d_1} x_2^{m_2 2^t + d_2} \cdots x_n^{m_n 2^t + d_n}. \end{aligned} \tag{4}$$

where $t > 0$. To handle with the up operations Sq^k, it suffices to know only the values of the $Sq^k(x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n})$ with $d_1, d_2, \dots, d_n < 2^t$ and $1 \le k < 2^t$, for some $t > 0$ [2, Corollary 3.3].

s	2		3		4	
The d-spike:	$u^{(3)}v^{(3)}$		$u^{(7)}v^{(7)}$		$u^{(15)}v^{(15)}$	
for r:	0	1	0	1	0	1
is the Sq_3 of:	$u^{(3)}v^{(6)}$	$u^{(4)}v^{(5)}$	$u^{(7)}v^{(10)}$	$u^{(8)}v^{(9)}$	$u^{(15)}v^{(18)}$	$u^{(16)}v^{(17)}$

Table 2: Proposition 2.7 for $t = 2$.

Theorem 3.1. *Let* $1 \leq d_i < 2^t$, $1 \leq i \leq n$, *and let* $0 \leq k < 2^t$. *Then*

$$Sq^k(\boldsymbol{x}^{\boldsymbol{m}2^t+\boldsymbol{d}}) = \boldsymbol{x}^{\boldsymbol{m}2^t} Sq^k(\boldsymbol{x}^{\boldsymbol{d}}).$$

We establish the dual statements of the general Theorem 3.1. To this end, we start by one variable.

Proposition 3.1. *Let* $2^{t-1} \leq k \leq 2^t - 1$, $t > 0$. *Let also* $d = 2k + \ell$, *where* $\ell = 0, 1, \ldots, 2^t - 1$. *Then, for any non-negative integer* m,

$$Sq_k(v^{(m2^t+d)}) = v^{(m2^t)} Sq_k(v^{(d)}).$$

Remark 3.1. Corollary 2.5 shows the establishment of Proposition 3.1 for double of d-spikes. In fact, given any non-negative integer t,

$$Sq_{2^t-1}\left(v^{\left(m2^t+2(2^t-1)\right)}\right) = v^{(m2^t)} Sq_{2^t-1}\left(v^{\left(2(2^t-1)\right)}\right) = v^{\left((m+1)2^t-1\right)}, m \geq 0.$$

Proof of Theorem 3.1. We prove the result for $m = 1$, that is,

$$Sq_k(v^{(2^t+d)}) = v^{(2^t)} Sq_k(v^{(d)}).$$

Cartan formula, using the product formula (2), is applied.

$$Sq_k(v^{(2^t+d)}) = Sq_k(v^{(2^t)} v^{(d)}) = \sum_{i+j=k} Sq_i(v^{(2^t)}) Sq_j(v^{(d)}).$$

Appealing to Corollaries 2.2 and 2.8, the range of summation is reduced.

$$Sq_k(v^{(2^t+d)}) = v^{(2^t)} Sq_k(v^{(d)}) + \sum_{p=0}^{t-1} Sq_{2^p}(v^{(2^t)}) Sq_{k-2^p}(v^{(d)}). \tag{5}$$

Take $0 \leq p \leq t-1$. Then, $Sq_{2^p}(v^{(2^t)}) = v^{2^t - 2^p}$. We have $\text{bin}(2^t - 2^p) = \{2^{p+1}, \ldots, 2^t\}$. So, regarding the product formula (2), the necessary condition for the term $Sq_{2^p}(v^{(2^t)}) Sq_{k-2^p}(v^{(d)})$ to be nonzero is that $\text{bin}(2^t - 2^p) \cap \text{bin}(d - (k - 2^p)) = \emptyset$. This condition is true whenever $d - (k - 2^p) < 2^p$ which is impossible. Therefore, all terms of the summation in (5) are zero and the result follows for $m = 1$. For $m > 1$, the proof is similar.

Example 3.1.

$$Sq_1(v^{(18)}) = Sq_1(v^{(8\cdot 2+2)}) = v^{(8\cdot 2)} Sq_1(v^{(2)}) = v^{(16)} v^{(1)} = v^{(17)},$$
$$Sq_4(v^{(33)}) = Sq_4(v^{(3\cdot 2^3+9)}) = v^{(3\cdot 2^3)} Sq_4(v^{(9)}) = v^{(24)} v^{(5)} = v^{(29)},$$
$$Sq_{22}(v^{(85)}) = Sq_{22}(v^{(2^5+53)}) = v^{(2^5)} Sq_{22}(v^{(53)}) = v^{(32)} v^{(31)} = v^{(63)}.$$

The following main result extends Proposition 3.1 and establishes the dual of Theorem 3.1. We adopt the same notation as in (4).

Theorem 3.2. *For $t > 0$, let $2^{t-1} \leq k \leq 2^t - 1$. Then,*

$$Sq_k(\boldsymbol{v}^{(\boldsymbol{m}2^t+\boldsymbol{d})}) = \boldsymbol{v}^{(\boldsymbol{m}2^t)} Sq_k(\boldsymbol{v}^{(\boldsymbol{d})}),$$

where for each $1 \leq i \leq n$, d_i is $2k + \ell$ for $n = 1$ and is $k + \ell$ for $n > 1$, and $\ell = 0, 1, \ldots, 2^t - 1$.

Proof. The result for $n = 1$ is Proposition 3.1. We prove the result for $n = 2$; that is,

$$Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) = v_1^{(m_1 2^t)} v_2^{(m_2 2^t)} Sq_k(v_1^{(d_1)} v_2^{(d_2)}). \tag{6}$$

Cartan formula implies

$$Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) = \sum_{i+j=k} Sq_i(v_1^{(m_1 2^t+d_1)}) Sq_j(v_2^{(m_2 2^t+d_2)}).$$

For positive i, j, consider the general term

$$Sq_i(v_1^{(m_1 2^t+d_1)}) Sq_j(v_2^{(m_2 2^t+d_2)}). \tag{7}$$

Let $2^{r-1} \leq i \leq 2^r - 1$ and $2^{s-1} \leq j \leq 2^s - 1$, where r, s are positive and $\leq t$ if $k = 2^{t-1}$ and are $\leq t - 1$ if $k > 2^{t-1}$. Write (7) as

$$Sq_i(v_1^{(n_r 2^r+d_1)}) Sq_j(v_2^{(n_s 2^s+d_2)}),$$

where $n_r 2^r = m_1 2^t$ and $n_s 2^s = m_2 2^t$. We have

$$\begin{aligned} &Sq_k(v_1^{(m_1 2^t+d_1)} v_2^{(m_2 2^t+d_2)}) \\ &\qquad = v_1^{(m_1 2^t+d_1)} Sq_k(v_2^{(m_2 2^t+d_2)}) + Sq_k(v_1^{(m_1 2^t+d_1)}) v_2^{(m_2 2^t+d_2)} \\ &\qquad\quad + \sum_{\substack{i+j=k \\ i,j>0}} Sq_i(v_1^{(n_r 2^r+d_1)}) Sq_j(v_2^{(n_s 2^s+d_2)}). \end{aligned}$$

But, using Proposition 3.1, the summation equals

$$\begin{aligned} &\sum_{\substack{i+j=k \\ i,j>0}} v_1^{(n_r 2^r)} v_2^{(n_s 2^s)} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}) \\ &= v_1^{(m_1 2^t)} v_2^{(m_2 2^t)} \sum_{\substack{i+j=k \\ i,j>0}} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}). \end{aligned}$$

The result (6) is now followed. For $n > 2$, the proof goes the same lines. Finally, in the case where $n \geq 2$, we show that each d_i, $1 \leq i \leq n$, varies from k to $k + 2^t - 1$. The endpoint is trivial by the selection of k. We claim that $Sq_k(\boldsymbol{v}^{(\boldsymbol{d})}) = 0$ if all $d_i < k$. The claim

$$Sq_k(v_1^{(d_1)} v_2^{(d_2)}) = \sum_{i+j=k} Sq_i(v_1^{(d_1)}) Sq_j(v_2^{(d_2)}) = 0.$$

is true for $n = 2$ by Corollary 2.2. The value of $Sq_k(\boldsymbol{v}^{(\boldsymbol{d})})$ for $n > 2$ is calculated recursively from two variables. Therefore, the claim holds by Cartan formula. □

Remark 3.2. Propositions 2.6 and 2.7 applies, somehow, Theorem 3.2 for the cases where $s > i + 1$. This can be seen in Tables 1 and 2.

Example 3.2.

$$\begin{aligned} Sq_3(u^{(10)}v^{(14)}) &= Sq_3(u^{(2^2+6)}v^{(2\cdot 2^2+6)}) = u^{(2^2)}v^{(2\cdot 2^2)}Sq_3(u^{(6)}v^{(6)}) \\ &= u^{(4)}v^{(8)}\big(u^{(3)}v^{(6)} + u^{(6)}v^{(3)}\big) = u^{(7)}v^{(14)} + u^{(10)}v^{(11)}; \end{aligned}$$

$$\begin{aligned} Sq_8(u^{(32)}v^{(32)}w^{(32)}) &= Sq_8(u^{(3\cdot 2^3+8)}v^{(3\cdot 2^3+8)}w^{(3\cdot 2^3+8)}) \\ &= u^{(24)}v^{(24)}w^{(24)}Sq_8(u^{(8)}v^{(8)}w^{(8)}) \\ &= u^{(24)}v^{(24)}w^{(24)}\big(u^{(4)}v^{(4)}w^{(8)} + u^{(8)}v^{(4)}w^{(4)} + u^{(4)}v^{(8)}w^{(4)} \\ &\quad + u^{(6)}v^{(6)}w^{(4)} + u^{(4)}v^{(6)}w^{(6)} + u^{(6)}v^{(4)}w^{(6)}\big) \\ &= u^{(28)}v^{(28)}w^{(32)} + u^{(32)}v^{(28)}w^{(28)} + u^{(28)}v^{(32)}w^{(28)} \\ &\quad + u^{(30)}v^{(30)}w^{(28)} + u^{(28)}v^{(30)}w^{(30)} + u^{(30)}v^{(28)}w^{(30)}. \end{aligned}$$

4. Matrix methods

In this section, we exhibit some matrix methods to manipulate the up and down Steenrod operations. First, the up Steenrod operation is considered.

Definition 4.1. For the monomial $x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, the string function pow is defined by

$$\mathrm{pow}(x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}) = r_1\, r_2 \dots r_n,$$

where $r_1\, r_2 \dots r_n$ is supposed as a string. This definition is extended naturally for the polynomial

$$f = x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n} + \cdots + x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$$

in $\mathrm{P}(n)$ additively.

$$\mathrm{pow}(f) = r_1\, r_2 \dots r_n + \cdots + s_1\, s_2 \dots s_n.$$

For example, $\mathrm{pow}(x^2y^5 + x^4y^3) = 2\,5 + 4\,3$.

Note that we use only the correspondence feature of this function, not the algebraic nature. The motivation for this definition is nothing but simplification.

Definition 4.2. For a positive integer k, by the up Steenrod k-matrix it is meant the square matrix $[\mathcal{U}^k]$ of size $k+1$ with the string entries

$$[\mathcal{U}^k]_{ij} = \mathrm{pow}\left(Sq^{i-1}(x^j)\right),$$

for $i, j = 1, 2, \dots, k+1$. It is noteworthy that, due to the string nature, this definition is free of variable; that is,

$$[\mathcal{U}^k]_{ij} = \mathrm{pow}\left(Sq^{i-1}(x^j)\right) = \mathrm{pow}\left(Sq^{i-1}(y^j)\right).$$

Denote by $[\mathcal{U}^k]^t$ the transpose of $[\mathcal{U}^k]$, and $[\mathcal{U}^k]^r$ the matrix obtained from $[\mathcal{U}^k]$ by reversing the order of rows. In other words,

$$([\mathcal{U}^k]^t)_{ij} = [\mathcal{U}^k]_{ji} = \mathrm{pow}\left(Sq^{j-1}(x^i)\right),$$
$$([\mathcal{U}^k]^r)_{ij} = [\mathcal{U}^k]_{(k+2-i)j} = \mathrm{pow}\left(Sq^{k+1-i}(x^j)\right).$$

For instance,

$$[\mathcal{U}^3] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 4 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}, \ [\mathcal{U}^3]^t = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 3 & 4 & 5 & 6 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \ [\mathcal{U}^3]^r = \begin{bmatrix} 0 & 0 & 6 & 0 \\ 0 & 4 & 5 & 0 \\ 2 & 0 & 4 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

A large size $[\mathcal{U}^{32}]$ is depicted in Figure 1, where dots indicate zeroes. In the sequel, following a similar simplification as in Definition 4.1, the string matrices defined in Definition 4.2 are multiplied together to produce another string matrices. More precisely, instead of the matrix multiplication

$$\begin{bmatrix} x^2 & x^5 \\ x^3 & x^8 \end{bmatrix} \begin{bmatrix} y^4 & y^2 \\ y^2 & y^9 \end{bmatrix} = \begin{bmatrix} x^2y^4 + x^5y^2 & x^2y^2 + x^5y^9 \\ x^3y^4 + x^8y^2 & x^3y^2 + x^8y^9 \end{bmatrix}, \tag{8}$$

we will apply

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 2\,4 + 5\,2 & 2\,2 + 5\,9 \\ 3\,4 + 8\,2 & 3\,2 + 8\,9 \end{bmatrix}, \tag{9}$$

which is, in fact,

$$\begin{bmatrix} \mathrm{pow}(x^2y^4 + x^5y^2) & \mathrm{pow}(x^2y^2 + x^5y^9) \\ \mathrm{pow}(x^3y^4 + x^8y^2) & \mathrm{pow}(x^3y^2 + x^8y^9) \end{bmatrix}.$$

There is no confusion in the matrix multiplication (9) since the two matrices on the left sides of the equations (8) and (9) due to different variables.

The next result indicates a matrix method to calculate of the up Steenrod operations in the two variables x and y.

Theorem 4.1. *Let k be a positive integer. Then,*

$$\mathrm{pow}\left(Sq^k(x^iy^j)\right) = \left([\mathcal{U}^k]^t\,[\mathcal{U}^k]^r\right)_{ij}.$$

```
1 2 3 4 5 6 7 8 9 1011121314151617181920212223242526272829303132
2 . 4 . 6 . 8 . 10. 12. 14. 16. 18. 20. 22. 24. 26. 28. 30. 32.
. 4 5 . . 8 9 . . 1213. . 1617. . 2021. . 2425. . 2829. . 3233.
. . 6 . . . 10. . . 14. . . 18. . . 22. . . 26. . . 30. . . 34.
. . . 8 9 1011. . . . 16171819. . . . 24252627. . . . 32333435.
. . . . 10. 12. . . . . 18. 20. . . . . 26. 28. . . . . 34. 36.
. . . . . 1213. . . . . . 2021. . . . . . 2829. . . . . . 3637.
. . . . . . 14. . . . . . . 22. . . . . . . 30. . . . . . . 38.
. . . . . . . 1617181920212223. . . . . . . . 3233343536373839.
. . . . . . . . 18. 20. 22. 24. . . . . . . . . 34. 36. 38. 40.
. . . . . . . . . 2021. . 2425. . . . . . . . . . 3637. . 4041.
. . . . . . . . . . 22. . . 26. . . . . . . . . . . 38. . . 42.
. . . . . . . . . . . 24252627. . . . . . . . . . . . 40414243.
. . . . . . . . . . . . 26. 28. . . . . . . . . . . . . 42. 44.
. . . . . . . . . . . . . 2829. . . . . . . . . . . . . . 4445.
. . . . . . . . . . . . . . 30. . . . . . . . . . . . . . . 46.
. . . . . . . . . . . . . . . 32333435363738394041424344454647.
. . . . . . . . . . . . . . . . 34. 36. 38. 40. 42. 44. 46. 48.
. . . . . . . . . . . . . . . . . 3637. . 4041. . 4445. . 4849.
. . . . . . . . . . . . . . . . . . 38. . . 42. . . 46. . . 50.
. . . . . . . . . . . . . . . . . . . 40414243. . . . 48495051.
. . . . . . . . . . . . . . . . . . . . 42. 44. . . . . 50. 52.
. . . . . . . . . . . . . . . . . . . . . 4445. . . . . . 5253.
. . . . . . . . . . . . . . . . . . . . . . 46. . . . . . . 54.
. . . . . . . . . . . . . . . . . . . . . . . 4849505152535455.
. . . . . . . . . . . . . . . . . . . . . . . . 50. 52. 54. 56.
. . . . . . . . . . . . . . . . . . . . . . . . . 5253. . 5657.
. . . . . . . . . . . . . . . . . . . . . . . . . . 54. . . 58.
. . . . . . . . . . . . . . . . . . . . . . . . . . . 56575859.
. . . . . . . . . . . . . . . . . . . . . . . . . . . . 58. 60.
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6061.
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 62.
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Figure 1: The up Steenrod matrix $[\mathcal{U}^{32}]$

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1 2 3 4 5 6 7 8 9 1011121314151617181920212223242526272829303132333435363738394041424344454647484950515253545556575859606162
. 1 . 3 . 5 . 7 . 9 . 11. 13. 15. 17. 19. 21. 23. 25. 27. 29. 31. 33. 35. 37. 39. 41. 43. 45. 47. 49. 51. 53. 55. 57. 59. 61
. . . 2 3 . . 6 7 . . 1011. . 1415. . 1819. . 2223. . 2627. . 3031. . 3435. . 3839. . 4243. . 4647. . 5051. . 5455. . 5859.
. . . . . 3 . . . 7 . . . 11. . . 15. . . 19. . . 23. . . 27. . . 31. . . 35. . . 39. . . 43. . . 47. . . 51. . . 55. . . 59
. . . . . . . 4 5 6 7 . . . . 12131415. . . . 20212223. . . . 28293031. . . . 36373839. . . . 44454647. . . . 52535455. . .
. . . . . . . . . 5 . 7 . . . . . 13. 15. . . . . 21. 23. . . . . 29. 31. . . . . 37. 39. . . . . 45. 47. . . . . 53. 55. .
. . . . . . . . . . . 6 7 . . . . . . 1415. . . . . . 2223. . . . . . 3031. . . . . . 3839. . . . . . 4647. . . . . . 5455.
. . . . . . . . . . . . . 7 . . . . . . . 15. . . . . . . 23. . . . . . . 31. . . . . . . 39. . . . . . . 47. . . . . . . 55
. . . . . . . . . . . . . . . 8 9 101112131415. . . . . . . . 2425262728293031. . . . . . . . 4041424344454647. . . . . . .
. . . . . . . . . . . . . . . . . 9 . 11. 13. 15. . . . . . . . . 25. 27. 29. 31. . . . . . . . . 41. 43. 45. 47. . . . . .
. . . . . . . . . . . . . . . . . . . 1011. . 1415. . . . . . . . . . 2627. . 3031. . . . . . . . . . 4243. . 4647. . . . .
. . . . . . . . . . . . . . . . . . . . . 11. . . 15. . . . . . . . . . . 27. . . 31. . . . . . . . . . . 43. . . 47. . . .
. . . . . . . . . . . . . . . . . . . . . . . 12131415. . . . . . . . . . . . 28293031. . . . . . . . . . . . 44454647. . .
. . . . . . . . . . . . . . . . . . . . . . . . . 13. 15. . . . . . . . . . . . . 29. 31. . . . . . . . . . . . . 45. 47. .
. . . . . . . . . . . . . . . . . . . . . . . . . . . 1415. . . . . . . . . . . . . . 3031. . . . . . . . . . . . . . 4647.
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15. . . . . . . . . . . . . . .31. . . . . . . . . . . . . . . 47
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16171819202122232425262728293031. . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17. 19. 21. 23. 25. 27. 29. 31. . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1819. . 2223. . 2627. . 3031. . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19. . . 23. . . 27. . . 31. . . . . . . . . . . .
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. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31
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. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
```

Figure 2: The down Steenrod matrix $[\mathcal{D}_{64}]$

Proof. For $i, j = 1, 2, \ldots, k+1$, by Cartan formula, we have

$$\begin{aligned} Sq^k(x^i y^j) &= \sum_{r+s=k} Sq^r(x^i) Sq^s(y^j) \\ &= \sum_{m=1}^{k+1} Sq^{m-1}(x^i) Sq^{k+1-m}(y^j). \end{aligned}$$

On the other hand, by Definition 4.2,

$$([\mathcal{U}^k]^t)_{im} = \text{pow}\big(Sq^{m-1}(x^i)\big),\ ([\mathcal{U}^k]^r)_{mj} = \text{pow}\big(Sq^{k+1-m}(y^j)\big).$$

The result is now follows from Definition 4.1. □

The above-mentioned matrix method may be extended inductively.

Definition 4.3. Consider the monomial $x_{i_1}^{d_1} \cdots x_{i_n}^{d_n} \in \mathrm{P}^d(n)$, where $d_1 + \cdots + d_n = d$.

For a positive integer k, the up Steenrod $k(i_1, \ldots, i_n)$-matrix $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]$ of size $k+1$ is defined as the following square matrix with string entries.

$$[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{pq} = \text{pow}\left(Sq^{p-1}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right),$$

where $p, q = 1, 2, \ldots, k+1$. For each single variable x_{i_j}, we have the free of variable up Steenrod k-matrix $[\mathcal{U}^k]$ as in Definition 4.2. The transpose $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^t$, and the row-reversed $[\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^r$ is defined as before.

$$\begin{aligned} ([\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^t)_{pq} &= [\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{qp} = \text{pow}\left(Sq^{q-1}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{p})\right), \\ ([\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]^r)_{pq} &= [\mathcal{U}^k_{i_1,\ldots,i_{n-1}|i_n}]_{(k+2-p)q} = \text{pow}\left(Sq^{k+1-p}(x_{i_1}^{d_1} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right). \end{aligned}$$

The next result extends Theorem 4.1. The proof is similar.

Theorem 4.2. *Let k be a positive integer and $n > 2$. Let the $Sq^k(x_{i_1}^{d_1} \cdots x_{i_n}^{d_n})$ has already been calculated by the above-mentioned matrix method for the values less than n. Take the variables $x_{i_2}, \ldots, x_{i_{n-1}}$ and fix them. Then,*

$$\text{pow}\left(Sq^k(x_{i_1}^{d_1} x_{i_2}^{d_2} \cdots x_{i_{n-1}}^{d_{n-1}} x_{i_n}^{q})\right) = \left([\mathcal{U}^k_{i_1}]^t\, [\mathcal{U}^k_{i_2,\ldots,i_{n-1}|i_n}]^r\right)_{pq}.$$

Now, we concentrate on the down Steenrod operations. The next definition is analogous to Definition 4.1

Definition 4.4. For the d-monomial $v_1^{(r_1)} v_2^{(r_2)} \cdots v_n^{(r_n)}$, we correspond a string $r_1\, r_2 \ldots r_n$ written as

$$\text{pow}(v_1^{(r_1)} v_2^{(r_2)} \cdots v_n^{(r_n)}) = r_1\, r_2 \ldots r_n.$$

This definition is extended naturally for sums of d-monomials.

Definition 4.5. For a positive integer k, the down Steenrod k-matrix $[\mathcal{D}_k]$ of size $k+1$ is defined as

$$[\mathcal{D}_k]_{ij} = \text{pow}\left(Sq_{i-1}(v^{(j)})\right),$$

for $i, j = 1, 2, \ldots, k+1$. The matrix $[\mathcal{D}_k]^{tc}$ is obtained by reversing the order of the columns of the transpose $[\mathcal{D}_k]^t$ of $[\mathcal{D}_k]$. More precisely,

$$([\mathcal{D}_k]^{tc})_{ij} = ([\mathcal{D}_k]^t)_{i(k+2-j)} = ([\mathcal{D}_k])_{(k+2-j)i} = \text{pow}\left(Sq_{k+1-j}(v^{(i)})\right).$$

For example,

$$[\mathcal{D}_3] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ [\mathcal{D}_3]^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 3 & 2 & 0 \end{bmatrix}, \ [\mathcal{D}_3]^{tc} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

Figure 2 illustrates a large size $[\mathcal{D}_{64}]$, where dots stand for zeroes. To calculate the down Steenrod operations in two variables the next result is applied.

Theorem 4.3. *Let k be a positive integer. Then,*

$$\text{pow}\left(Sq_k(u^{(i)}v^{(j)})\right) = \left([\mathcal{D}_k]^{tc}\,[\mathcal{D}_k]\right)_{ij}.$$

Proof. The proof goes along the same lines as the proof of Theorem 4.1 with a little difference. By Cartan formula, we have

$$\begin{aligned} Sq_k(u^{(i)}v^{(j)}) &= \sum_{r+s=k} Sq_r(u^{(i)})Sq_s(v^{(j)}) \\ &= \sum_{m=1}^{k+1} Sq_{k+1-m}(u^{(i)})Sq_{m-1}(v^{(j)}), \end{aligned}$$

for $i, j = 1, 2, \ldots, k+1$. By Definition 4.5 we have

$$([\mathcal{D}_k]^{tc})_{im} = \text{pow}\big(Sq_{k+1-m}(u^{(i)})\big),\ ([\mathcal{D}_k])_{mj} = \text{pow}\big(Sq_{m-1}(v^{(j)})\big).$$

The result is now clear by Definition 4.4. □

Definition 4.5 is extended in a similar way as Definition 4.3. In particular, the down Steenrod $k(i_1, \ldots, i_n)$-matrix $[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]$ of size $k+1$ is defined as follows.

$$[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]_{pq} = \text{pow}\left(Sq_{p-1}(v_{i_1}^{(d_1)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(q)})\right),$$

for $p, q = 1, 2, \ldots, k+1$. The matrix $[\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]^{tc}$ is defined as before.

$$([\mathcal{D}_k^{i_1,\ldots,i_{n-1}|i_n}]^{tc})_{pq} = \text{pow}\left(Sq_{k+1-q}(v_{i_1}^{(d_1)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(p)})\right).$$

Now, the extended Theorem analogous to Theorem 4.2 can be stated.

Theorem 4.4. *Given the integers $k > 0$ and $n > 2$, let the $Sq_k(v_{i_1}^{(d_1)} \cdots v_{i_n}^{(d_n)})$ has already been calculated by the above-mentioned matrix method for the values less than n. Take the variables $v_{i_2}, \ldots, v_{i_n}$ and fix them. Then,*

$$\text{pow}\left(Sq_k(v_{i_1}^{(p)} v_{i_2}^{(d_2)} \cdots v_{i_{n-1}}^{(d_{n-1})} v_{i_n}^{(q)})\right) = \left([\mathcal{D}_k^{i_2,\ldots,i_{n-1}|i_n}]^{tc} [\mathcal{D}_k^{i_1}]\right)_{pq}.$$

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Accepted: April 6, 2021