A study on Pythagorean fuzzy soft topological spaces and continuous mappings

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Abstract. This paper is a study on Pythagorean fuzzy soft topological spaces and continuous mappings. Here, relevant topological notions are generalized into the new setting and proved some fundamental results. Pythagorean fuzzy soft mappings and continuous mappings are defined and illustrated with examples. Also, certain interesting properties of these mappings are obtained. Finally, we proved a couple of characterization theorems for Pythagorean fuzzy soft mappings.

Keywords: Pythagorean fuzzy soft sets, Topological spaces, Continuous mappings.

1. Introduction

Mathematical modeling of a real-life event is an incredibly difficult task. Many mathematical theories were developed to model an event with vagueness. Among these, the most important one is the invention of fuzzy sets (FS) by Zaheh [30]. FS handled vagueness with the help of the membership function. Atanassov [2] generalized FS by introducing intuitionistic fuzzy sets (IFS) and, then Yager [29] enhanced it by bringing out the Pythagorean fuzzy sets (PFS). PFS explains vagueness by the use of the membership function and non-membership function,

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where their square sum lies between 0 and 1. Molodtsov's soft set theory [8, 11] is another discovery that can describe fuzziness with the help of a parameter set. Later, hybridizing FS and soft set, fuzzy soft set (FSS) is proposed by Maji et.al. [16]. The same authors generalized FSS by introducing intuitionistic fuzzy soft sets (IFSS) [15]. Similarly, Peng [25] introduced the Pythagorean fuzzy soft set (PFSS) by combining PFS with the soft set. Since PFSS have a higher potential to capture fuzziness, many theories based on PFSS are on developing [3, 1, 4, 10, 15].

The classical topology has also been explored along with generalized sets mentioned above. Chang [8] proposed a suitable 'topological structure' on FS and since many unsolved problems in classical topology have solutions in this domain, it is a widely discussing area of research. Coker [9] introduced IFS topology, and Olgun et.al. [21] used it to define PFS topological structure. Shabir and Naz [28] studied soft topological spaces and according to them soft topological space is a parametrized family of topologies on the universal set. Later, Çağman et.al. [7] proposed a new topological structure on soft sets. Instead of considering a parametrized family of topologies, they used a collection of soft subsets to define the new topology. A nice survey of both approaches can be found in chapter 3 of the book [11]. After that many studies on soft topology are developed and that can be seen in [31, 32, 33, 34, 35]. Followed by Çağman's approach, a study on mappings and transformations on soft sets is carried out in [13, 17, 37, 38, 39]. Later, Roy and Samantha [27] considered FSS topological space and then a study on IFSS topological spaces is carried out in [6, 22, 14]. Subsequently Karata et.al. [12] defined IFSS mappings and IFSS continuous mappings. Recently Riaz et.al. [12] introduced the PFSS topological space and explained the multi-criteria decision-making problem as an application. Related studies on PFSS topology are done in [23, 24, 36]. Also, the notions of compactness, connectedness, continuous mappings and homeomorphisms are suitably defined and explored in all the above-mentioned generalized sets.

The notion of topology is used to indicate the relationship between spatial objects and their characteristics. There are real-life situations in which the boundaries of objects are vague. Using the techniques of topological structures considered on generalized sets one can analyze objects along with vagueness. Here we introduce a topological structure on PFSS. As PFSS is a parametrized family of PFSs, it is a very useful tool to model uncertainty from both parametric as well as membership points of view. Since PFSS can be considered as a single representation for a collection of certain related PFSs, it reduces the complexity of analysis and calculations, particularly in decision-making problems [10, 20]. Thus introducing a topological structure on PFSS has significant relevance and is of our interest.

In this paper, we have done a study on the topological structure of PFSS. Fundamental definitions of new notions and the corresponding examples are included. Some theorems related to interior and open sets are proved. We illustrated PFSS mappings with examples and certain properties of PFSS mapping are shown. In analogy with the classical settings, we identified a suitable definition for PFSS continuous mappings and characterized it in terms of PFSS open sets. The notion of PFSS continuous mappings can be used to compare two PFSS topological spaces and one can easily see that the collection of all PFSS with PFSS continuous mappings forms a category. It is also worth mentioning that our definition of PFSS continuous mapping generalizes IFSS continuous mapping. Hence it can be used to handle vagueness in many fields such as machine learning, data analysis, etc. as we have developed the continuous mappings from one topological space into another. As in [20], we can improve the algorithm for decision making with the help of topological space. So if we can identify the mapping corresponding to one set of parameters to another set and the corresponding mapping among the universal set, and if the mapping identified turns out to be continuous, then decision making in the latter situation is quite easy.

This paper is divided into three sections, besides the introduction. In section 2, we recall some basic definitions which are used in the entire discussion. Section 3 deals with PFSS topological spaces and the related topological notions. In section 4, we introduce PFSS mappings, PFSS continuous mappings and proved a couple of characterization theorems.

2. Preliminaries

This section addresses some basic definitions beneficial for entire discussions. Those are the main definitions connected with Pythagorean fuzzy soft sets and Pythagorean fuzzy soft topology. Unless otherwise specified, $\mho$ be the universal set, $\mathfrak{L}$ be parameter set, and $P(\mho)$ be power set of $\mho$.

2.1 Pythagorean fuzzy soft sets

Definition 2.1 ([29]). *A Pythagorean fuzzy set $\mathcal{P}$ on $\mho$ is defined as the set $\{(u, \mu_p(u), \nu_p(u)) : u \in \mho\}$ where $\mu_p : \mho \to [0,1]$ and $\nu_p : \mho \to [0,1]$ with $0 \leq \mu_p^2 + \nu_p^2 \leq 1$.*

Definition 2.2 ([18]). *The pair $(\mathcal{S}, \mathfrak{L})$ is said to be soft set over $\mho$, if $\mathcal{S}$ is a mapping from $\mathfrak{L}$ to $P(\mho)$. i.e., for each $\mathfrak{e} \in \mathfrak{L}$, $S(\mathfrak{e})$ is considered as the set of $\mathfrak{e}$-approximate elements of the soft set.*

Definition 2.3 ([25]). *A Pythagorean fuzzy soft set(PFSS) is defined as the pair $(\mathcal{P}, \mathfrak{L})$ where, $\mathcal{P} : \mathfrak{L} \to PFS(\mho)$ and $PFS(\mho)$ is the set of all Pythagorean fuzzy subsets of $\mho$.*

We are using the notation $0_{(\mho,\mathfrak{L})}$ for the PFSS with $\mu_p(u) = 0$ and $\nu_p(u) = 1$ corresponding to each $\mathfrak{e} \in \mathfrak{L}$ and $1_{(\mho,\mathfrak{L})}$ for the PFSS with $\mu_p(u) = 1$ and $\nu_p(u) = 0$ corresponding to each $\mathfrak{e} \in \mathfrak{L}$. For $\mho = \{u_1, u_2, \ldots, u_n\}$ and $\mathfrak{L} = \{\mathfrak{e}_1, \mathfrak{e}_2, \ldots \mathfrak{e}_m\}$

the PFSS $(\mathcal{P}, \mathfrak{L})$ is represented as

$$(\mathcal{P}, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}(\mathfrak{e}_1) & = & \{u_1/(a_{11}, b_{11}), u_2/(a_{12}, b_{12}), \dots u_n/(a_{1n}, b_{1n})\} \\ \mathcal{P}(\mathfrak{e}_2) & = & \{u_1/(a_{21}, b_{21}), u_2/(a_{22}, b_{22}), \dots u_n/(a_{2n}, b_{2n})\} \\ \cdot & = & \\ \cdot & = & \\ \cdot & = & \\ \mathcal{P}(\mathfrak{e}_m) & = & \{u_1/(a_{m1}, b_{m1}), u_2/(a_{m2}, b_{m2}), \dots u_n/(a_{mn}, b_{mn})\} \end{pmatrix},$$

where $(a_{ij}, b_{ij}) = (\mu_{\mathcal{P}_{(\mathfrak{e}_i)}}(u_j), \nu_{\mathcal{P}_{(\mathfrak{e}_i)}}(u_j))$ i.e., membership value and nonmembership value corresponding to parameter $\mathfrak{e}_i$ and universal element u_j.

The following definition is helpful to recollect various operations on PFSS.

Definition 2.4 ([20])**.** *Let $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ be two PFSSs on $\mho$. Then*

1. *$(\mathcal{P}_1, \mathfrak{L}_1)$ is subset of $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqsubseteq (\mathcal{P}_2, \mathfrak{L}_2)$ if:*
   - *(a) $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$;*
   - *(b) for each $\mathfrak{e} \in \mathfrak{L}_1$, $\mu_{\mathcal{P}_1(\mathfrak{e})}(u) \leq \mu_{\mathcal{P}_2(\mathfrak{e})}(u)$ and $\nu_{\mathcal{P}_1(\mathfrak{e})}(u) \geq \nu_{\mathcal{P}_2(\mathfrak{e})}(u)$, $\forall u \in \mho$.*
2. *The complement of $(\mathcal{P}, \mathfrak{L})$ i.e., $(\mathcal{P}, \mathfrak{L})^c = (\mathcal{P}^c, \mathfrak{L})$.*
3. *The union of $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqcup (\mathcal{P}_2, \mathfrak{L}_2) = (\mathcal{P}, \mathfrak{L}_1 \cup \mathfrak{L}_2)$ where, $\mathcal{P}(e) = \{\big(u, \max\{\mu_{\mathcal{P}_1}(u), \mu_{\mathcal{P}_2}(u)\}, \min\{\nu_{\mathcal{P}_1}(u), \nu_{\mathcal{P}_2}(u)\}\big) : u \in \mho\}$ for each $\mathfrak{e} \in \mathfrak{L}_1 \cup \mathfrak{L}_2$.*
4. *The intersection of $(\mathcal{P}_1, \mathfrak{L}_1)$ and $(\mathcal{P}_2, \mathfrak{L}_2)$ i.e., $(\mathcal{P}_1, \mathfrak{L}_1) \sqcap (\mathcal{P}_2, \mathfrak{L}_2) = (\mathcal{P}, \mathfrak{L}_1 \cap \mathfrak{L}_2)$ where, $\mathcal{P}(e) = \{\big(u, \min\{\mu_{\mathcal{P}_1}(u), \mu_{\mathcal{P}_2}(u)\}, \max\{\nu_{\mathcal{P}_1}(u), \nu_{\mathcal{P}_2}(u)\}\big) : u \in \mho\}$ for each $\mathfrak{e} \in \mathfrak{L}_1 \cap \mathfrak{L}_2$.*

2.2 Pythagorean fuzzy soft topology

Definition 2.5 ([26])**.** *Let $\mho$ be the universal set and $\mathfrak{L}$ be the set of parameters. A collection τ of Pythagorean fuzzy soft subsets of $\mho$ over $\mathfrak{L}$ is said to be PFSS topology if*

- *i. $0_{(\mho, \mathfrak{L})}$, $1_{(\mho, \mathfrak{L})} \in \tau$;*
- *ii. for any $(\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L}) \in \tau$, we have $(\mathcal{P}, \mathfrak{L}) \sqcap (\mathcal{G}, \mathfrak{L}) \in \tau$;*
- *iii. for any $(\mathcal{P}_i, \mathfrak{L})_{i \in I} \in \tau$, we have $\sqcup_{i \in I}(\mathcal{P}_i, \mathfrak{L}) \in \tau$, where I is any indexed set.*

Hereinafter the triptlet $(\mho, \tau, \mathfrak{L})$ refered PFSS topological space. Every element in τ is called an open PFSSs. Closed PFSSs is the complement of open PFSSs. Also, interior of a PFSS $(\mathcal{P}, \mathfrak{L})$ is the biggest open PFSS embedded in $(\mathcal{P}, \mathfrak{L})$ and closure of $(\mathcal{P}, \mathfrak{L})$ is the smallest closed PFSS containing $(\mathcal{P}, \mathfrak{L})$. Interior and closure of $(\mathcal{P}, \mathfrak{L})$ is denoted as $(\mathcal{P}, \mathfrak{L})^\circ$ and $\overline{(\mathcal{P}, \mathfrak{L})}$ respectively.

3. More results on PFSS topological space

In this section, some interesting theorems on PFSS topological space are explained. At first, the relationship between PFSS topological space and fuzzy soft bitopological space is explained. After that studies on interior and neighborhood are carried out and a characterization theorem of open sets is proved.

Now, $(\mho, \tau_1, \tau_2, \mathfrak{L})$ be known as fuzzy soft bitopological space [19] if the topologies $(\mho, \tau_1, \mathfrak{L})$ and $(\mho, \tau_2, \mathfrak{L})$ independently satisfy the axioms of fuzzy soft topological space. The following theorem says that corresponding to each PFSS topological space we get a fuzzy soft bitopological space.

Theorem 3.1. *Let* $(\mho, \tau, \mathfrak{L})$ *be a Pythagorean fuzzy soft topological spaces and* τ *is given by* $\tau = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_i, \mathfrak{L})_{i\in I}\}$ *where* I *is arbitrary indexed set. Then* $(\mho, \tau_1, \tau_2, \mathfrak{L})$ *is a fuzzy soft bitopological space on* $\mho$*, where* $\tau_1 = \{\widetilde{0}, \widetilde{1}, (\mu_{\mathcal{P}_i}, \mathfrak{L})_{i\in I}\}$ *and* $\tau_2 = \{\widetilde{0}, \widetilde{1}, (\nu_{\mathcal{P}_i}, \mathfrak{L})^{\complement}_{i\in I}\}$.

Proof. We have to prove that both τ_1 and τ_2 satisfy all the three axioms of fuzzy soft topology. $0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})} \in \tau$ gives that $\widetilde{0}, \widetilde{1} \in \tau_1$ and $\widetilde{0}, \widetilde{1} \in \tau_2$. Thus first condition is obtained.

Second condition is obtained from the fact that $(\mathcal{P}_1, \mathfrak{L}) \sqcap (\mathcal{P}_2, \mathfrak{L}) = (\mu_{\mathcal{P}_1\wedge\mathcal{P}_2}, \nu_{\mathcal{P}_1\vee\mathcal{P}_2}, \mathfrak{L})$. So, $(\mathcal{P}_1, \mathfrak{L}) \sqcap (\mathcal{P}_2, \mathfrak{L}) \in \tau$ gives $(\mu_{\mathcal{P}_1\wedge\mathcal{P}_2}, \mathfrak{L}) \in \tau_1$ and $(\nu_{\mathcal{P}_1\vee\mathcal{P}_2}, \mathfrak{L})^{\complement} = (\nu_{\mathcal{P}_1^{\complement}\wedge\mathcal{P}_2^{\complement}}, \mathfrak{L}) \in \tau_2$.

Let $J \subseteq I$ be any indexed set. Then, we have $\sqcup_{j\in J}(\mathcal{P}_j, \mathfrak{L}) = (\mu_{\vee_{j\in J}\mathcal{P}_j}, \nu_{\wedge_{j\in J}\mathcal{P}_j}, \mathfrak{L})$. So, $\sqcup_{j\in J}(\mathcal{P}_j, \mathfrak{L}) \in \tau$ gives $(\mu_{\vee_{j\in J}\mathcal{P}_j}, \mathfrak{L}) \in \tau_1$ and $(\nu_{\wedge_{j\in J}\mathcal{P}_j}, \mathfrak{L})^{\complement} = (\nu_{\vee_{j\in J}\mathcal{P}_j^{\complement}}, \mathfrak{L}) \in \tau_2$. Thus, $(\mho, \tau_1, \tau_2, \mathfrak{L})$ is a fuzzy soft bitopological spaces. □

The effect of Theorem 3.1 is from a given PFSS topology we can generate fuzzy soft bitopological space. But one can easily identify that converse of Theorem 3.1 is not valid always. But the following example says, even if ζ be a collection of PFSSs containing $0_{(\mho,\mathfrak{L})}$ and $1_{(\mho,\mathfrak{L})}$ with the property that $(\mho, \zeta_1, \mathfrak{L})$ and $(\mho, \zeta_2, \mathfrak{L})$ are fuzzy soft topological spaces, ζ need not be a PFSS topology.

Example 3.1. Let the universal set $\mho$ be $\{u_1, u_2\}$ and parameter set $\mathfrak{L} = \{\mathfrak{e}_1, \mathfrak{e}_2\}$. The collection of PFSS ζ is given by $\zeta = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_1, \mathfrak{L}), (\mathcal{P}_2, \mathfrak{L}), (\mathcal{P}_3, \mathfrak{L}), (\mathcal{P}_4, \mathfrak{L})\}$, where

$$(\mathcal{P}_1, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_1(\mathfrak{e}_1) & = & \{u_1/(0.7, 0.3), u_2/(0.2, 0.7)\} \\ \mathcal{P}_1(\mathfrak{e}_2) & = & \{u_1/(0.6, 0.6), u_2/(0.1, 0.7)\} \end{pmatrix},$$

$$(\mathcal{P}_2, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_2(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.4), u_2/(0.3, 0.6)\} \\ \mathcal{P}_2(\mathfrak{e}_2) & = & \{u_1/(0.7, 0.3), u_2/(0.4, 0.5)\} \end{pmatrix},$$

$$(\mathcal{P}_3, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_3(\mathfrak{e}_1) & = & \{u_1/(0.3, 0.3), u_2/(0.2, 0.6)\} \\ \mathcal{P}_3(\mathfrak{e}_2) & = & \{u_1/(0.2, 0.3), u_2/(0.1, 0.5)\} \end{pmatrix},$$

$$(\mathcal{P}_4, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_4(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.4), u_2/(0.5, 0.7)\} \\ \mathcal{P}_4(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.6), u_2/(0.7, 0.7)\} \end{pmatrix}.$$

It is clear that ζ is not a PFSS topology. Because $(\mathcal{P}_1, \mathfrak{L}) \sqcup (\mathcal{P}_2, \mathfrak{L})$ is not in ζ. But ζ_1 and ζ_2 is given by

$$\zeta_1 = \Big\{\widetilde{0}, \widetilde{1}, \begin{pmatrix} \mu_{\mathcal{P}_1(\mathfrak{e}_1)} & = & \{u_1/0.7, u_2/0.2\} \\ \mu_{\mathcal{P}_1(\mathfrak{e}_2)} & = & \{u_1/0.6, u_2/0.1\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_2(\mathfrak{e}_1)} & = & \{u_1/0.8, u_2/0.3\} \\ \mu_{\mathcal{P}_2(\mathfrak{e}_2)} & = & \{u_1/0.7, u_2/0.4\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_3(\mathfrak{e}_1)} & = & \{u_1/0.3, u_2/0.2\} \\ \mu_{\mathcal{P}_3(\mathfrak{e}_2)} & = & \{u_1/0.2, u_2/0.1\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_4(\mathfrak{e}_1)} & = & \{u_1/0.8, u_2/0.5\} \\ \mu_{\mathcal{P}_4(\mathfrak{e}_2)} & = & \{u_1/0.8, u_2/0.7\} \end{pmatrix} \Big\}$$

$$\zeta_2 = \Big\{\widetilde{0}, \widetilde{1}, \begin{pmatrix} \mu_{\mathcal{P}_1(\mathfrak{e}_1)} & = & \{u_1/0.7, u_2/0.3\} \\ \mu_{\mathcal{P}_1(\mathfrak{e}_2)} & = & \{u_1/0.4, u_2/0.3\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_2(\mathfrak{e}_1)} & = & \{u_1/0.6, u_2/0.4\} \\ \mu_{\mathcal{P}_2(\mathfrak{e}_2)} & = & \{u_1/0.7, u_2/0.5\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_3(\mathfrak{e}_1)} & = & \{u_1/0.7, u_2/0.4\} \\ \mu_{\mathcal{P}_3(\mathfrak{e}_2)} & = & \{u_1/0.7, u_2/0.5\} \end{pmatrix}, \begin{pmatrix} \mu_{\mathcal{P}_4(\mathfrak{e}_1)} & = & \{u_1/0.6, u_2/0.3\} \\ \mu_{\mathcal{P}_4(\mathfrak{e}_2)} & = & \{u_1/0.4, u_2/0.3\} \end{pmatrix} \Big\}.$$

We can easily verify that ζ_1 and ζ_2 are fuzzy soft topologies.

The following is the definition of neighborhood of a PFSS which is somewhat different from the classical one. Chang [8] gave this notion for fuzzy topological space and following this idea here we introduced the definition given below.

Definition 3.1. *Let $(X, \tau, \mathfrak{L})$ be PFSS topological space and $(\mathcal{P}, \mathfrak{L})$ be a PFSS. The PFSS $(N, \mathfrak{L})$ is said to be a neighbourhood of $(\mathcal{P}, \mathfrak{L})$ if there exist a $(\mathcal{G}, \mathfrak{L}) \in \tau$ such that $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{G}, \mathfrak{L}) \sqsubseteq (N, \mathfrak{L})$.*

Definition 3.2. *Let $(\mathcal{P}, \mathfrak{L})$ and $(\mathcal{G}, \mathfrak{L})$ be PFSS in the PFSS topological space $(X, \tau, \mathfrak{L})$ such that $(\mathcal{G}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Then $(\mathcal{G}, \mathfrak{L})$ serves as interior PFSS of $(\mathcal{P}, \mathfrak{L})$ if $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of $(\mathcal{G}, \mathfrak{L})$.*

The following example illustrates the definition of the neighborhood of a PFSS and interior PFSS.

Example 3.2. Consider the PFSS topology given below.

$\tau = \{0_{(\mho,\mathfrak{L})}, 1_{(\mho,\mathfrak{L})}, (\mathcal{P}_1, \mathfrak{L}), (\mathcal{P}_2, \mathfrak{L}), (\mathcal{P}_3, \mathfrak{L}), (\mathcal{P}_4, \mathfrak{L}), (\mathcal{P}_5, \mathfrak{L})\}$ is a PFSS topology on $\mho$, where the PFSSs $(\mathcal{P}_i, \mathfrak{L})$, $i = 1, 2, 3, 4, 5$ is given by

$$(\mathcal{P}_1, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_1(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.1), u_2/(0.9, 0.1), u_3/(0.3, 0.4)\} \\ \mathcal{P}_1(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.1), u_2/(0.8, 0.1), u_3/(0.6, 0.5)\} \end{pmatrix},$$

$$(\mathcal{P}_2, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_2(\mathfrak{e}_1) & = & \{u_1/(0.8, 0.1), u_2/(0.6, 0.2), u_3/(0.3, 0.7)\} \\ \mathcal{P}_2(\mathfrak{e}_2) & = & \{u_1/(0.4, 0.4), u_2/(0.8, 0.1), u_3/(0.6, 0.5)\} \end{pmatrix},$$

$$(\mathcal{P}_3, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_3(\mathfrak{e}_1) & = & \{u_1/(0.6, 0.3), u_2/(0.6, 0.2), u_3/(0.2, 0.7)\} \\ \mathcal{P}_3(\mathfrak{e}_2) & = & \{u_1/(0.4, 0.4), u_2/(0.6, 0.2), u_3/(0.5, 0.6)\} \end{pmatrix},$$

$$(\mathcal{P}_4, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_4(\mathfrak{e}_1) & = & \{u_1/(0.6, 0.3), u_2/(0.9, 0.1), u_3/(0.2, 0.4)\} \\ \mathcal{P}_4(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.1), u_2/(0.6, 0.2), u_3/(0.5, 0.6)\} \end{pmatrix},$$

$$(\mathcal{P}_5, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}_5(\mathfrak{e}_1) & = & \{u_1/(0.9, 0.1), u_2/(0.9, 0.0), u_3/(0.5, 0.2)\} \\ \mathcal{P}_5(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.0), u_2/(0.8, 0.1), u_3/(0.7, 0.2)\} \end{pmatrix}.$$

The PFSS $(\mathcal{P}, \mathfrak{L})$ and $(N, \mathfrak{L})$ is given by,

$$(\mathcal{P}, \mathfrak{L}) = \begin{pmatrix} \mathcal{P}(\mathfrak{e}_1) & = & \{u_1/(0.3, 0.5), u_2/(0.4, 0.3), u_3/(0.1, 0.8)\} \\ \mathcal{P}(\mathfrak{e}_2) & = & \{u_1/(0.2, 0.6), u_2/(0.6, 0.3), u_3/(0.2, 0.7)\} \end{pmatrix},$$

$$(N, \mathfrak{L}) = \begin{pmatrix} N(\mathfrak{e}_1) & = & \{u_1/(0.9, 0.0), u_2/(0.7, 0.1), u_3/(0.5, 0.5)\} \\ N(\mathfrak{e}_2) & = & \{u_1/(0.8, 0.3), u_2/(0.9, 0.1), u_3/(0.9, 0.3)\} \end{pmatrix}.$$

It can be seen that there exist a PFSS $(\mathcal{P}_2, \mathfrak{L}) \in \tau$ such that $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}_2, \mathfrak{L}) \sqsubseteq (N, \mathfrak{L})$. So $(N, \mathfrak{L})$ is a neighbourhood of the PFSS $(\mathcal{P}, \mathfrak{L})$ and thus $(\mathcal{P}, \mathfrak{L})$ is the interior PFSS of $(N, \mathfrak{L})$.

Theorem 3.2. *In a PFSS topological space, a PFSS $(\mathcal{P}, \mathfrak{L})$ is open if and only if each subsets of $(\mathcal{P}, \mathfrak{L})$ are interior PFSSs of $(\mathcal{P}, \mathfrak{L})$.*

Proof. Let $(\mathcal{P}, \mathfrak{L})$ be open PFSS and let $(\mathcal{G}, \mathfrak{L})$ be a subset of $(\mathcal{P}, \mathfrak{L})$. So we have $(\mathcal{P}, \mathfrak{L}) \in \tau$ such that $(\mathcal{G}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Thus each subsets of $(\mathcal{P}, \mathfrak{L})$ are interior PFSSs of $(\mathcal{P}, \mathfrak{L})$. Conversly, $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of each of its subsets so $(\mathcal{P}, \mathfrak{L})$ is the neighbourhood of itself. By definition there exist $(O, \mathfrak{L}) \in \tau$ with $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (O, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. Thus $(\mathcal{P}, \mathfrak{L}) = (O, \mathfrak{L})$ and so $(\mathcal{P}, \mathfrak{L})$ is open. □

Theorem 3.2 is a characterisation of PFSS open set using Definition 3.1. And so this theorem helps to identify the PFSS opens sets. The following theorem is a modification of Theorem 3.2 because it gives the relationship between the interior of a PFSS $(\mathcal{P}, \mathfrak{L})$ and interior PFSSs of $(\mathcal{P}, \mathfrak{L})$ and also says when a PFSS is open in terms of the interior of a PFSS.

Theorem 3.3. *The union of all interior PFSSs of $(\mathcal{P}, \mathfrak{L})$ gives the interior of $(\mathcal{P}, \mathfrak{L})$ and $(\mathcal{P}, \mathfrak{L})$ is open if and only if $(\mathcal{P}, \mathfrak{L}) = (\mathcal{P}, \mathfrak{L})^\circ$.*

Proof. Let $\mathscr{G} = \sqcup(\mathcal{G}, \mathfrak{L})$, where the union varies over all the interior PFSS of $(\mathcal{P}, \mathfrak{L})$. We will prove that $\mathscr{G}$ is the biggest open set embedded in $(\mathcal{P}, \mathfrak{L})$. Since $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of each interior PFSS $(\mathcal{G}, \mathfrak{L})$, $(\mathcal{P}, \mathfrak{L})$ is a neighbourhood of $\mathscr{G}$. Hence there exist $(O, \mathfrak{L}) \in \tau$ with $\mathscr{G} \sqsubseteq (O, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})$. But $(O, \mathfrak{L})$ also an interior PFSS of $(\mathcal{P}, \mathfrak{L})$ so $(O, \mathfrak{L}) \sqsubseteq \mathscr{G}$. Thus $\mathscr{G}$ is open and biggest open set embedded in $(\mathcal{P}, \mathfrak{L})$. Therefore $\mathscr{G} = (\mathcal{P}, \mathfrak{L})^\circ$. For $(\mathcal{P}, \mathfrak{L})$ is open, $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{P}, \mathfrak{L})^\circ$. Because, $(\mathcal{P}, \mathfrak{L})$ itself is interior PFSS of $(\mathcal{P}, \mathfrak{L})$. Thus $(\mathcal{P}, \mathfrak{L}) = (\mathcal{P}, \mathfrak{L})^\circ$. The converse part is straight forward from the definition of $(\mathcal{P}, \mathfrak{L})^\circ$. □

4. PFSS mappings and continuous mappings

This section includes the definition and properties of PFSS mapping and continuous mapping. From here on, the collection of all PFSS over $\mho$ under the parameter set $\mathfrak{L}$ is denoted as $PFS(\mho, \mathfrak{L})$. The following is the definition of PFSS mapping and some related features.

Definition 4.1. *Let $PFS(X, \mathfrak{L}_1), PFS(Y, \mathfrak{L}_2)$ be collection of all PFSSs over $(X, \mathfrak{L}_1)$ and $(Y, \mathfrak{L}_2)$ correspondingly. Also, let $\mathcal{J} : X \to Y$ and $\Omega : \mathfrak{L}_1 \to \mathfrak{L}_2$ be two mappings. Then*

(i) the PFSS mapping $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ is defined as, for $(\mathcal{P}, \mathfrak{L}_1) \in PFS(X, \mathfrak{L}_1)$, $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] = (\mathcal{J}(\mathcal{P}), \Omega(\mathfrak{L}_1))$ and $(\mathcal{J}(\mathcal{P}), \Omega(\mathfrak{L}_1)) \in$

$PFS(Y, \mathfrak{L}_2)$ and it is given by

$$\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y) = \begin{cases} \sup\limits_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}(\alpha)}(x), & \text{if } \mathcal{J}^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

and $\nu_{\mathcal{J}(\mathcal{P})(\beta)}(y) = \begin{cases} \inf\limits_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \nu_{\mathcal{P}(\alpha)}(x), & \text{if } \mathcal{J}^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise,} \end{cases}$ *, for each $\beta \in \mathfrak{L}_2$ and $y \in Y$*

(ii) *the inverse image under $\mathcal{J}_\Omega$, is defined as, for $(\mathcal{G}, \mathfrak{L}_2) \in PFS(Y, \mathfrak{L}_2)$, $\mathcal{J}_\Omega^{-1}[\mathcal{G}, \mathfrak{L}_2] = (\mathcal{J}^{-1}(\mathcal{G}), \Omega^{-1}(\mathfrak{L}_2)) \in PFS(X, \mathfrak{L}_1)$ and is given by $\mu_{\mathcal{J}^{-1}(\mathcal{G})(\alpha)}(x) = \mu_{G(\Omega(\alpha))}\mathcal{J}(x)$ and $\nu_{\mathcal{J}^{-1}(\mathcal{G})(\alpha)}(x) = \nu_{\mathcal{G}(\Omega(\alpha))}\mathcal{J}(x)$*

(iii) *$\mathcal{J}_\Omega$ is injective (surjective) mapping if $\mathcal{J}$ and Ω are injective (surjective) mappings*

(iv) *$\mathcal{J}_\Omega$ is identity mapping if $\mathcal{J}$ and Ω are identity mappings*

Remark 4.1. It can be easily obtained that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$ and $\mathcal{J}_\Omega^{-1}[\mathcal{G}, \mathfrak{L}_2]$ are PF-SSs. For $\mathcal{J}^{-1}(y) \neq \emptyset$,

$$\begin{aligned} \left(\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2 + \left(\nu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2 &= \left(\sup_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}(\alpha)}(x)\right)^2 + \left(\inf_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \nu_{\mathcal{P}(\alpha)}(x)\right)^2 \\ &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \mu^2_{\mathcal{P}(\alpha)}(x) + \inf_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \nu^2_{\mathcal{P}(\alpha)}(x) \\ &\leq \sup_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} (1 - \nu^2_{\mathcal{P}(\alpha)}(x)) + \inf_{\substack{\alpha \in \Omega^{-1}(\beta)\cap\mathfrak{L}_1, \\ x\in\mathcal{J}^{-1}(y)}} \nu^2_{\mathcal{P}(\alpha)}(x) \leq 1. \end{aligned}$$

For $\mathcal{J}^{-1}(y) = \emptyset$ we get, $\left(\mu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2 + \left(\nu_{\mathcal{J}(\mathcal{P})(\beta)}(y)\right)^2 = 1$. This is true for each $\beta \in \mathfrak{L}_2$ and $y \in Y$ implies that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \in PFS(Y, \mathfrak{L}_2)$. Similarly, it is obtained that $\mathcal{J}_\Omega^{-1}[\mathcal{G}, \mathfrak{L}_2] \in PFS(X, \mathfrak{L}_1)$.

Example 4.1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ are universal sets and $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $B = \{\beta_1, \beta_2\}$ are the parameter sets. $\mathcal{J} : X \to Y$ is given by $\mathcal{J}(x_1) = y_1$, $\mathcal{J}(x_2) = y_2$, $\mathcal{J}(x_3) = y_2$ and $\Omega : \mathfrak{L}_1 \to \mathfrak{L}_2$ is given by $\Omega(\alpha_1) = \beta_1$, $\Omega(\alpha_2) = \beta_1$, $\Omega(\alpha_3) = \beta_2$. Let $(\mathcal{P}, A) \in PFS(X, A)$ and $(\mathcal{G}, B) \in PFS(Y, B)$ which are given by

$$(\mathcal{P}, A) = \begin{pmatrix} \mathcal{P}(\alpha_1) & = & \{x_1/(0.8, 0.1), x_2/(0.6, 0.5), x_3/(0.7, 0.1)\} \\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \end{pmatrix}$$

and

$$(\mathcal{G}, B) = \begin{pmatrix} \mathcal{G}(\beta_1) & = & \{y_1/(0.1, 0.4), y_2/(0.6, 0.2), y_3/(0.5, 0.6)\} \\ \mathcal{G}(\beta_2) & = & \{y_1/(0.9, 0.2), y_2/(0.2, 0.1), y_3/(0.8, 0.3)\} \end{pmatrix}.$$

Now, the image of $(\mathcal{P}, A)$ under $\mathcal{J}_\Omega : PFS(X, A) \to PFS(Y, B)$ is given by,

$$\mathcal{J}_\Omega[\mathcal{P}, A] = (\mathcal{J}(\mathcal{P}), \Omega(A)) = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.7, 0.1), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.0), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}.$$

The inverse image of $(\mathcal{G}, B)$ under the PFSS mapping $\mathcal{J}_\Omega : PFS(X, A) \to PFS(Y, B)$ is given by

$$\mathcal{J}_\Omega^{-1}[\mathcal{G}, B] = (\mathcal{J}^{-1}(\mathcal{G}), \Omega^{-1}(\mathfrak{L}_2)) = \begin{pmatrix} \mathcal{P}(\alpha_1) & = & \{x_1/(0.1, 0.4), x_2/(0.6, 0.2), x_3/(0.6, 0.2)\} \\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.1, 0.4), x_2/(0.6, 0.2), x_3/(0.6, 0.2)\} \\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9, 0.2), x_2/(0.2, 0.1), x_3/(0.2, 0.1)\} \end{pmatrix}.$$

The following two theorems give some properties of PFSS mappings.

Theorem 4.1. *Let* $(\mathcal{P}_1, A_1), (\mathcal{P}_2, A_2) \in PFS(X, A)$ *and* $\mathcal{J} : X \to Y$ *and* $\Omega : A \to B$ *be two mappings. Then,*

1. $(\mathcal{P}_1, A_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}_1, A_1]]$, *the equality holds when* $\mathcal{J}_\Omega$ *is injective*
2. $(\mathcal{P}_1, A_1) \sqsubseteq (\mathcal{P}_2, A_2) \implies \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$
3. $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcup (\mathcal{P}_2, A_2)] = \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcup \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$
4. $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$

Proof. We only prove 3 & 4 remaining are trivial.

3. Let $(\mathcal{P}_1, A_1) \sqcup (\mathcal{P}_2, A_2) = (I, A_1 \cup A_2)$. Then,

$$\begin{aligned} \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcup \mathcal{J}_\Omega[\mathcal{P}_2, A_2] &= (\mathcal{J}(\mathcal{P}_1), \Omega(A_1)) \sqcup (\mathcal{J}(\mathcal{P}_2), \Omega(A_2)) \\ &= (J, \Omega(A_1) \cup \Omega(A_2)). \end{aligned}$$

Now, we will prove that $\mathcal{J}(I) = J$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cup \Omega(A_2)$.
Case 1. If $\mathcal{J}^{-1}(y)=\emptyset$ then $\mu_{\mathcal{J}(I)(\beta)}(y)=\mu_{J(\beta)}(y)=0$, $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)=1$.
Case 2. If $\beta \in \Omega(A_1) - \Omega(A_2)$ then it is obtained that there does not exist $\alpha \in A_2$ such that $\Omega(\alpha) = \beta$. i.e., $\Omega^{-1}(\beta) \cap (A_1 \cup A_2) = \Omega^{-1}(\beta) \cap (A_1 - A_2)$. That implies

$$\begin{aligned} \mu_{\mathcal{J}(I)(\beta)}(y) &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 \cup A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\ &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 - A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\ &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 - A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x) \\ &= \mu_{J(\beta)}(y). \end{aligned}$$

Likewise it is able to show that $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$.

Case 3. If $\beta \in \Omega(A_2) - \Omega(A_1)$ then analogous to case 2, it is obtained that $\mu_{\mathcal{J}(I)(\beta)}(y) = \mu_{J(\beta)}(y)$ and $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$.
Case 4. If $\beta \in \Omega(A_1) \cap \Omega(A_2)$ then

$$\begin{aligned}
\mu_{\mathcal{J}(I)(\beta)}(y) &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 \cup A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\
&= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_1 \,\cup\, \Omega^{-1}(\beta) \cap A_2, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\
&= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_1 - \Omega^{-1}(\beta) \cap A_2, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x) \\
&\quad \bigvee \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_2 - \Omega^{-1}(\beta) \cap A_1, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x) \\
&\quad \bigvee \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_1 \cap \Omega^{-1}(\beta) \cap A_2, \\ x \in \mathcal{J}^{-1}(y)}} \max\left\{\mu_{\mathcal{P}_1(\alpha)}(x), \mu_{\mathcal{P}_2(\alpha)}(x)\right\} \\
&= \max\left\{ \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_1, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x), \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_2, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x) \right\} \\
&= \mu_{J(\beta)}(y).
\end{aligned}$$

In similar way we can prove that $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y)$. Thus, from all the four possible cases it can be concluded that $\mathcal{J}(I) = J$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cup \Omega(A_2)$.
4. Let $(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2) = (I, A_1 \cap A_2)$. Then,

$$\begin{aligned}
\mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2] &= (\mathcal{J}(\mathcal{P}_1), \Omega(A_1)) \sqcap (\mathcal{J}(\mathcal{P}_2), \Omega(A_2)) \\
&= (J, \Omega(A_1) \cap \Omega(A_2)).
\end{aligned}$$

If $\mathcal{J}^{-1}(y) = \emptyset$ then $\mu_{\mathcal{J}(I)(\beta)}(y) = \mu_{J(\beta)}(y) = 0$, $\nu_{\mathcal{J}(I)(\beta)}(y) = \nu_{J(\beta)}(y) = 1$.
Otherwise,

$$\begin{aligned}
\mu_{\mathcal{J}(I)(\beta)}(y) &= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 \cap A_2), \\ x \in \mathcal{J}^{-1}(y)}} \mu_{I(\alpha)}(x) \\
&= \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap (A_1 \cap A_2), \\ x \in \mathcal{J}^{-1}(y)}} \max\left\{\mu_{\mathcal{P}_1(\alpha)}(x), \mu_{\mathcal{P}_2(\alpha)}(x)\right\} \\
&\leq \max\left\{ \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_1, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_1(\alpha)}(x), \sup_{\substack{\alpha \in \Omega^{-1}(\beta) \cap A_2, \\ x \in \mathcal{J}^{-1}(y)}} \mu_{\mathcal{P}_2(\alpha)}(x) \right\} \\
&= \mu_{J(\beta)}(y).
\end{aligned}$$

In the similar way we can prove that $\nu_{\mathcal{J}(I)(\beta)}(y) \geq \nu_{J(\beta)}(y)$ for all $y \in Y$ & $\beta \in \Omega(A_1) \cap \Omega(A_2)$. Thus $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] \sqsubseteq \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$ □

The following examples show that the equality need not always hold for 1 and 4.

Example 4.2. This example shows that $(\mathcal{P}, A)$ not always equal to $\mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$. From Example 4.1, we got

$$\mathcal{J}_\Omega[\mathcal{P}, A] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.7, 0.1), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.0), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}.$$

Now,

$$\mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]] = \begin{pmatrix} \mathcal{P}(\alpha_1) & = & \{x_1/(0.9, 0.0), x_2/(0.7, 0.1), x_3/(0.7, 0.1)\} \\ \mathcal{P}(\alpha_2) & = & \{x_1/(0.9, 0.0), x_2/(0.7, 0.1), x_3/(0.7, 0.1)\} \\ \mathcal{P}(\alpha_3) & = & \{x_1/(0.9, 0.0), x_2/(0.4, 0.1), x_3/(0.4, 0.1)\} \end{pmatrix}.$$

So, $(\mathcal{P}, A) \neq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$. But $(\mathcal{P}, A) \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{J}_\Omega[\mathcal{P}, A]]$.

Example 4.3. From example 4.1, consider

$$(\mathcal{P}_1, A_1) = \begin{pmatrix} \mathcal{P}_1(\alpha_1) & = & \{x_1/(0.8, 0.1), x_2/(0.6, 0.5), x_3/(0.7, 0.1)\} \\ \mathcal{P}_1(\alpha_2) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \\ \mathcal{P}_1(\alpha_3) & = & \{x_1/(0.9, 0.0), x_2/(0.2, 0.1), x_3/(0.4, 0.4)\} \end{pmatrix}, \text{ and}$$

$$(\mathcal{P}_2, A_2) = \begin{pmatrix} \mathcal{P}_2(\alpha_1) & = & \{x_1/(0.9, 0.1), x_2/(0.2, 0.5), x_3/(0.6, 0.6)\} \\ \mathcal{P}_2(\alpha_2) & = & \{x_1/(0.8, 0.1), x_2/(0.2, 0.5), x_3/(0.5, 0.4)\} \\ \mathcal{P}_3(\alpha_3) & = & \{x_1/(0.9, 0.1), x_2/(0.2, 0.2), x_3/(0.4, 0.4)\} \end{pmatrix}. \text{ Then,}$$

$$\mathcal{J}_\Omega[\mathcal{P}_1, A_1] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.7, 0.1), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.0), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}, \text{ and}$$

$$\mathcal{J}_\Omega[\mathcal{P}_2, A_2] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.1), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}.$$

$$\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.8, 0.1), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.2), y_3/(0.0, 1.0)\} \end{pmatrix}$$

$$\mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2] = \begin{pmatrix} \mathcal{J}(\mathcal{P})(\beta_1) & = & \{y_1/(0.9, 0.0), y_2/(0.6, 0.3), y_3/(0.0, 1.0)\} \\ \mathcal{J}(\mathcal{P})(\beta_2) & = & \{y_1/(0.9, 0.1), y_2/(0.4, 0.1), y_3/(0.0, 1.0)\} \end{pmatrix}$$

Thus $\mathcal{J}_\Omega[(\mathcal{P}_1, A_1) \sqcap (\mathcal{P}_2, A_2)] \neq \mathcal{J}_\Omega[\mathcal{P}_1, A_1] \sqcap \mathcal{J}_\Omega[\mathcal{P}_2, A_2]$.

Theorem 4.2. *Let* $(\mathcal{G}_1, B_1), (\mathcal{G}_2, B_2) \in PFS(Y, B)$ *and* $\mathcal{J} : X \to Y$ *and* $\Omega : A \to B$ *be two mappings. Then*

1. $\mathcal{J}_\Omega[\mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1]] \sqsubseteq (\mathcal{G}_1, B_1)$*, the equality holds when* $\mathcal{J}_\Omega$ *is surjective;*
2. $(\mathcal{G}_1, B_1) \sqsubseteq (\mathcal{G}_2, B_2) \implies \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqsubseteq \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$*;*
3. $\mathcal{J}_\Omega^{-1}[(\mathcal{G}_1, B_1) \sqcup (\mathcal{G}_2, B_2)] = \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqcup \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$*;*
4. $\mathcal{J}_\Omega^{-1}[(\mathcal{G}_1, B_1) \sqcap (\mathcal{G}_2, B_2)] = \mathcal{J}_\Omega^{-1}[\mathcal{G}_1, B_1] \sqcap \mathcal{J}_\Omega^{-1}[\mathcal{G}_2, B_2]$.

Proof. Proof is analogous to Theorem 4.1. □

Since continuity is an inevitable concept in the study of topological space, the next discussion is about the PFSS continuous mapping. Definition and two characterisation theorems of continuous mapping are given.

Definition 4.2. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces. A PFSS mapping $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ is said to be PFSS continuous if for each subsets $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and for any neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$ there exist a neighbourhood $(\mho, \mathfrak{L}_1)$ of $(\mathcal{P}, \mathfrak{L}_1)$ in a way $\mathcal{J}_\Omega[\mho, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$.*

The following two theorems are characterizing the PFSS continuous mappings.

Theorem 4.3. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces and $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ PFSS continuous iff for each subsets $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and for any neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$.*

Proof. Let $\mathcal{J}_\Omega$ is continuous. Consider a subset $(\mathcal{P}, \mathfrak{L}_1)$ of $(X, \mathfrak{L}_1)$ and a neighbourhood $(V, \mathfrak{L}_2)$ of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. By Definition 4.2 we have $(\mho, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$, where $(\mho, \mathfrak{L}_1)$ is neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. That is, $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq (\mho, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. Since $(\mho, \mathfrak{L}_1)$ is neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$, by Definition 3.1, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$.

Conversely, let $(\mathcal{P}, \mathfrak{L}_1)$ be subset of $(X, \mathfrak{L}_1)$ and $(V, \mathfrak{L}_2)$ be neighbourhood of $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. Then by assumption $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. From Definition 3.1, there exist $(O, \mathfrak{L}_1) \in \tau_1$ such that $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq (O, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. That implies $\mathcal{J}_\Omega[O, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$ and hence $\mathcal{J}_\Omega$ is continuous. □

Theorem 4.4. *Let $(X, \tau_1, \mathfrak{L}_1)$ and $(Y, \tau_2, \mathfrak{L}_2)$ be two PFSS topological spaces and $\mathcal{J}_\Omega : PFS(X, \mathfrak{L}_1) \to PFS(Y, \mathfrak{L}_2)$ PFSS continuous iff every open subset $(V, \mathfrak{L}_2)$ of $(Y, \mathfrak{L}_2)$, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$.*

Proof. Assume $\mathcal{J}_\Omega$ is continuous. Consider an open subset $(V, \mathfrak{L}_2)$ of $(Y, \mathfrak{L}_2)$ and let $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. i.e., $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (V, \mathfrak{L}_2)$. Since $(V, \mathfrak{L}_2)$ is open and from Theorem 3.2, there is a neighbourhood $(N, \mathfrak{L}_2)$ such that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (N, \mathfrak{L}_2) \sqsubseteq (V, \mathfrak{L}_2)$. We have $\mathcal{J}_\Omega$ is continuous so by Theorem 4.3, $\mathcal{J}_\Omega^{-1}[N, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$ and contained in $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. Since $(\mathcal{P}, \mathfrak{L}_1)$ is arbitrary subset of $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$, by Theorem 3.2, $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$.

On the other hand, let $\mathcal{J}_\Omega$ be a PFSS mapping from $(X, \tau_1, \mathfrak{L}_1)$ to $(Y, \tau_2, \mathfrak{L}_2)$. Also let $(\mathcal{P}, \mathfrak{L}_1)$ be subset of $(X, \mathfrak{L}_1)$ and $(V, \mathfrak{L}_2)$ be a neighbourhood $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1]$. Then by Definition 3.1 we obtain an open set $(O, \mathfrak{L}_2)$ so that $\mathcal{J}_\Omega[\mathcal{P}, \mathfrak{L}_1] \sqsubseteq (O, \mathfrak{L}_2) \sqsubseteq (V, \mathfrak{L}_2)$. By assumption $\mathcal{J}_\Omega^{-1}[O, \mathfrak{L}_2]$ is open subset of $(X, \mathfrak{L}_1)$. Thus we get $(\mathcal{P}, \mathfrak{L}_1) \sqsubseteq \mathcal{J}_\Omega^{-1}[O, \mathfrak{L}_2] \sqsubseteq \mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$. i.e., $\mathcal{J}_\Omega^{-1}[V, \mathfrak{L}_2]$ is a neighbourhood of $(\mathcal{P}, \mathfrak{L}_1)$. Hence Theorem 4.3 implies that $\mathcal{J}_\Omega$ is continuous. □

5. Conclusion

A study on the PFSS topological structure is carried out in this paper. We introduced a definition for PFSS continuous mappings which generalizes the existing definition for IFSS continuous mappings. Finally, we obtained a couple of characterization theorems for PFSS continuity. Subject to its real-life applications, PFSS is one of the developing areas and thus our future work will be analyzing the further topological properties of PFSS and studying the algebraic structures associated with a PFSS.

References

[1] Muhammad Akram, Ali Ghous, *Hybrid models for decision-making based on rough pythagorean fuzzy bipolar soft information,* Granular Computing, 5 (2020), 1-15.

[2] Krassimir T. Atanassov, *Intuitionistic fuzzy sets,* Intuitionistic fuzzy sets. Physica, Heidelberg, 1999, 1-137.

[3] T.M. Athira, Jacob John Sunil, Garg Harish, *Entropy and distance measures of pythagorean fuzzy soft sets and their applications,* Journal of Intelligent & Fuzzy Systems, 37 (2019), 4071-4084.

[4] T.M. Athira, Jacob John Sunil, Garg Harish, *A novel entropy measure of pythagorean fuzzy soft sets,* AIMS Mathematics, 5 (2020), 1050-1061.

[5] T.M. Athira, Jacob John Sunil, Kumar P. Rajish, *Incomplete pythagorean fuzzy soft sets,* AIP Conference Proceedings, Vol. 2261, no. 1, AIP Publishing LLC, 2020.

[6] Sadi Bayramov, Gunduz Cigdem, *Intuitionistic fuzzy topology on function spaces,* Ann. Fuzzy Math. Inform., 3 (2012), 19-30.

[7] Naim Çağman, Karatas Serkan, Enginoglu Serdar, *Soft topology,* Computers & Mathematics with Applications, 62 (2011), 351-358.

[8] Chin-Liang Chang, *Fuzzy topological spaces,* Journal of mathematical Analysis and Applications, 24 (1968), 182-190.

[9] Doğan Çoker, *An introduction to intuitionistic fuzzy topological spaces,* Fuzzy sets and Systems, 88 (1997), 81-89.

[10] Abhishek Guleria, Kumar Bajaj Rakesh, *On pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis,* Soft Computing, 23 (2019), 7889-7900.

[11] Jacob John Sunil, *Soft sets,* Soft Sets, Springer, Cham, 2021.

[12] Serkan Karata, Akdag Metin, *On intuitionistic fuzzy soft continuous mappings,* Journal of New Results in Science, 3 (2014), 55-70.

[13] Athar Kharal, B. Ahmad, *Mappings on soft classes,* New Mathematics and Natural Computation, 7 (2011), 471-481.

[14] Li, Zhaowen, Rongchen Cui, *On the topological structure of intuitionistic fuzzy soft sets,* Annals of fuzzy Mathematics and Informatics, 5 (2013), 229-239.

[15] P. Kumar Maji, Biswas Ranjit, A. Ranjan Roy, *Intuitionistic fuzzy soft sets,* Journal of Fuzzy Mathematics, 9 (2001), 677-692.

[16] Maji, Pabitra Kumar, R. K. Biswas, A. Roy, *Fuzzy soft sets,* (2001), 589-602.

[17] Majumdar, Pinaki, Syamal Kumar Samanta, *On soft mappings,* Computers & Mathematics with Applications, 60 (2010), 2666-2672.

[18] Dmitriy Molodtsov, *Soft set theory—first results,* Computers & Mathematics with Applications, 37 (1999), 19-31.

[19] Mukherjee, Prakash, Choonkil Park, *On fuzzy soft bitopological spaces,* Mathematics and Computer Sciences Journal, 10 (2015), 1-8.

[20] K. Naeem, M. Riaz, X. Peng, D. Afzal, *Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators,* Journal of Intelligent & Fuzzy Systems, 37 (2019), 6937-6957.

[21] Olgun, Murat, Mehmet Ünver, Şeyhmus Yardımcı, *Pythagorean fuzzy topological spaces,* Complex & Intelligent Systems, 5 (2019), 177-183.

[22] Ismail Osmanoglu, Tokat Deniz, *On intuitionistic fuzzy soft topology,* General Mathematics Notes, 19 (2013), 59-70.

[23] Taha Öztürk, *Weak structures on pythagorean fuzzy soft topological spaces,* Turkish Journal of Science, 5, 233-241.

[24] Taha Öztürk, Adem Yolcu, *Some structures on pythagorean fuzzy topological spaces,* Journal of New Theory, 33 (2020), 15-25.

[25] X.D. Peng, Y. Yang, J. Song, Y. Jiang, *Pythagorean fuzzy soft set and its application,* Computer Engineering, 41 (2015), 224-229.

[26] M. Riaz, K. Naeem, M. Aslam, D. Afzal, F.A. Ahmed Almahdi, S. Shaukat Jamal, *Multi-criteria group decision making with pythagorean fuzzy soft topology,* Journal of Intelligent & Fuzzy Systems Preprint, (2020), 1-18.

[27] Roy, Sanjay, T. K. Samanta, *A note on fuzzy soft topological spaces,* Annals of Fuzzy Mathematics and Informatics, 3 (2012), 305-311.

[28] Muhammad Shabir, Munazza Naz, *On soft topological spaces,* Computers & Mathematics with Applications, 61 (2011), 1786-1799.

[29] Ronald R. Yager, *Pythagorean fuzzy subsets,* 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS). IEEE, 2013.

[30] Lotfi A. Zadeh, *On fuzzy algorithms,* Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems, selected papers By Lotfi A Zadeh, 1996, 127-147.

[31] Ahmad, Bashir, Sabir Hussain, *On some structures of soft topology,* Mathematical Sciences, 6 (2012), 1-7.

[32] H. Hazra, P. Majumdar,S. K. Samanta, *Soft topology,* Fuzzy information and Engineering, 4 (2012), 105-115.

[33] Min, Won Keun, *A note on soft topological spaces,* Computers & Mathematics with Applications, 62 (2011), 3524-3528.

[34] K.V. Babitha, Jacob John Sunil, *Studies on soft topological spaces,* Journal of Intelligent & Fuzzy Systems, 28 (2015), 1713-1722.

[35] Sabir Hussain, *A note on soft connectedness,* Journal of the Egyptian Mathematical Society, 23 (2015), 6-11.

[36] Adem Yolcu, *On pythagorean fuzzy soft boundary,* Turkish Journal of Science, 5, 242-251.

[37] Oğuz Gülay, İlhan Icen, M. Habil Gursoy, *Actions of soft groups,* Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68 (2019), 1163-1174.

[38] Gülay, Oğuz, *Soft topological transformation groups,* Mathematics, 8 (2020), 1545.

[39] Gülay, Oğuz, *A new view on topological polygroups,* Turkish Journal of Science, 5 (2020), 110-117.

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