Orthogonality preserving maps on $B(H)$

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Abstract. In this paper, we study the linear maps on $B(H)$ preserving a class of operators in $B(H)$. We also introduce pointwise orthogonality on $B(H)$ and provide the conditions under which a linear mapping φ on $B(H)$ preserves pointwise orthogonality on $B(H)$. Finally, we prove that a Jordan $*-$homomorphism φ preserves OP operators in $B(H)$. Moreover, under additional condition of injectivity, φ preserves OP operators in both directions.

Keywords: preserving maps, orthogonality preserving operators, normal operators.

1. Introduction

In an inner product space, two elements x and y are called orthogonal and denoted by $x \perp y$ whenever the inner product $\langle x\ , y\rangle$ is zero. Now let $\|.\|$ be the induced norm by the inner product on the underlying space. It is straightforward to check that for each scalar λ

$$\|x\| \leq \|x + \lambda y\| \Leftrightarrow x \perp y.$$

This relation provided a good motivation to define orthogonality in normed linear spaces which became known as Birkhoff-James orthogonality so that two elements x and y in a normed linear space are called BJ orthogonal, denoted by

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$x \perp_B y$, whenever $\|x\| \leq \|x + \lambda y\|$, for every scalar λ. Other types of orthogonality have also been defined and studied in linear normed spaces, such as
Roberts orthogonality: $\|x - \lambda y\| = \|x + \lambda y\|$, for every scalar λ, ($x \perp_R y$);
Isosceles orthogonality: $\|x - y\| = \|x + y\|$, ($x \perp_I y$);
Pythagorean orthogonality: $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, ($x \perp_P y$).
Sikorska [12], Alonso and Benitez [1, 2] studied the properties of these orthogonalities and compared the relations between them. Among them, the most widely used is Birkhoff-James orthogonality which is nondegenerate ($x \perp_B x$ if and only if $x = 0$), homogenous ($x \perp_B y \Rightarrow (\lambda x \perp_B \mu y$ for all scalar $\lambda, \mu)$), non symmetric ($x \perp_B y$ does not imply $y \perp_B x$), and non additive ($x \perp_B y$ and $x \perp_B z$ does not imply $x \perp_B y + z$). Also, for every $x, y \in X$ there is an scalar λ such that $x \perp_B \lambda x + y$.

After introducing the orthogonal elements in inner product and normed spaces and studying their properties, mathematicians and researchers concentrated on the study of maps that preserve the property of the orthogonality between two spaces. Therefore, a wide field of study has been appeared under the title of "linear orthogonality preserver problems".

Let X, Y be two inner product spaces over the same field. An operator $T : X \to Y$ is said to be orthogonality preserving (OP in short), if for every $x, y \in X$, $x \perp y \Rightarrow Tx \perp Ty$. Chmieliński [5] characterized linear OP operators between inner product spaces. Actually, he proved that a linear operator $T : X \to Y$ is OP, if and only if it is a scalar multiple of an isometry(in the sense that there exists a scalar γ such that $\|Tx\| = \gamma\|x\|$, for all $x \in X$).

In the setting of real normed spaces Koldobsky [6] had already proved Chmieliński's theorem [5, Theorem 2.1] with the Birkhoff-James orthogonality, and then Blanco and Turnšek [4], generalized this theorem to complex normed spaces. Also recently, Ansari, Sanati and Kardel [3] proved that, for a Hilbert space H, a linear OP operator $T : H \to H$ is a scalar multiple of a unitary between Hilbert space H and $ran(T)$. Furthermore, different types of approximately orthogonality preserving mappings and their stability have been studied. For more details we refer to [5, 11, 14, 15, 16, 17].

In this paper, at first we study the linear maps on $B(H)$ that preserve a class of OP operators on $B(H)$ and then, we introduce the notion of pointwise orthogonality on $B(H)$ to studying pointwise orthoginality preservers.

Let H be an infinite-dimentional Hilbert space and $B(H)$ be the C^*-algebra of all bounded linear operators on H. The ideals of compact operators and finite rank operators in $B(H)$ is denoted by $K(H)$ and $F(H)$, respectively. An operator $T \in B(H)$ is said to be Fredholm if its image is closed and its kernel and co-kernel are both finite dimention. T is semi-Fredholm if its image is closed and its kernel or its co-kernel is finite dimention. The class of all Fredholm and semi-Fredholm operators are denoted by $FR(H)$ and $SF(H)$, respectively. For more details about Fredholm operators, see [9, 13]. The set of normal operators in $B(H)$ is also denoted by $N(H)$.

An operator $T \in B(H)$ is said to be generalized invertible if there is an operator $R \in B(H)$ such that $TRT = T$. The set of all generalized invertible operators in $B(H)$ is denoted by $G(H)$. Note that $T \in G(H)$ if and only if $ran(T)$ is closed [10].

We say that an operator $T \in B(H)$ is pointwise orthogonal to another operator $R \in B(H)$, denoted by $T \perp_p R$, if $\langle Tx, Rx \rangle = 0$ for all $x \in H$. Since pointwise orthogonality is based on inner product, it is nondegenerate, homogeneous, symmetric and additive on $B(H)$. This orthogonality is also continuous, for if $T_n \to T$ and $R_n \to R$ in $B(H)$ and $T_n \perp_p R_n$, for every $n \in \mathbb{N}$, then for every $x \in H$ we have,

$$\langle Tx, Rx \rangle = \lim_{n\to\infty} \langle T_n x, R_n x \rangle = 0,$$

and therefore $T \perp_p R$.

Here, we give two examples of operators which are pointwise orthogonal to each other.

Example 1.1. let M be a closed subspace of H. If $P : H \to M$ and $P^\perp : H \to M^\perp$ are the projections of H on M and $M^\perp$, respectively. Then $P \perp_p P^\perp$.

Example 1.2. If $T, R : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $T(x, y) = (x, -y)$ and $R(x, y) = (y, x)$. Then, for each $(x, y) \in \mathbb{R}^2$ we have,

$$\langle T(x, y), R(x, y) \rangle = \langle (x, -y), (y, x) \rangle = xy - yx = 0.$$

In the next section, we show that every non-zero OP operator in $B(H)$ is semi-Fredholm and every non-zero normal OP operator is Fredholm of index zero. Also, we concentrate on the linear mappings on $B(H)$ which preserve some classes of operators. For instance, we prove that a Jordan $*$-homomorphism φ on $B(H)$ preserves OP operators. Moreover, if the map φ is also one to one, then it preserves OP operators in both directions. In the last section, we compare the pointwise orthogonality on $B(H)$ with Birkhoff-James, Roberts, Isosceles, Pythagorean orthogonalities on $B(H)$ as a normed space. Finally, we provide the conditions under which a linear mapping φ preserves pointwise orthoginality on $B(H)$.

2. Orthogonality preserving maps on $B(H)$

The main theorem of this section is Theorem 2.2.

Proposition 2.1. *Let $\varphi : B(H) \to B(H)$ be a linear map which preserves the isometry maps of $B(H)$, then φ is an isometry on $OP(H)$.*

Proof. Let $T \in OP(H)$. We can write $T = \gamma U$, for some positive scalar γ and an isometry map U on H, and thus $\|T\| = \gamma$. On the other hand, since $\varphi(U)$ is an isometry we have, $\|\varphi(U)\| = 1$ so $\|\varphi(T)\| = \|\gamma\varphi(U)\| = \gamma\|\varphi(U)\| = \gamma$ and so $\|\varphi(T)\| = \|T\|$. □

Proposition 2.2. *Let T be a non-zero orthogonality preserving operator in $B(H)$, then T is semi-Fredholm. Moreover, if T is a normal operator, then it is Fredholm with zero index.*

Proof. Let $T \in OP(H)$. Since T is a multiple of an isometry, T is injective and $ran(T)$ is a closed subspace of H. Therefore, T is a semi-Fredholm operator. But we have,

$$\langle (TT^* - T^*T)x, x\rangle = \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle = \|T^*x\|^2 - \|Tx\|^2.$$

Now, if T belongs to $N(H)$, then $ker(T) = ker(T^*)$ and since $ker(T^*) = (ran(T))^{\perp}$ we obtain that, T is a Fredholm operator and $ind(T) = nul(T) - def(T) = dim(ker(T)) - dim(coker(T)) = 0$. □

The following Lemma plays an important role in expressing the relationship between compact and Fredholm operators.

Lemma 2.1. *Let $T \in B(H)$. Then the following are equivalent:*

- (i) *T is compact.*
- (ii) *For every $R \in FR(H)$, $R + T \in FR(H)$.*
- (iii) *For every $R \in SF(H)$, $R + T \in SF(H)$.*

Proof. This follows from [7, Lemma 2.2] and [8, Lemma 2.2]. □

Proposition 2.3. *Let $T \in B(H)$ be a non-zero normal OP operator, then for every compact operator R in $B(H)$ we have, $T + R$ is Fredholm with zero index.*

Proof. Let $T \in (OP(H) \cap N(H)) \setminus \{0\}$ and $R \in K(H)$, then Proposition 2.2 implies that $T \in FR(H)$ and $ind(T) = 0$. Now by Lemma 2.1 we have, $T + R \in FR(H)$. On the other hand, by using [9, Theorem 1.4.18], for each comapact operator R and Fredholm operator T, we obtain that $ind(T + R) = ind(T)$. Thus $T + R$ is a Fredholm operator of zero index. □

Theorem 2.1. *Let T be a non-zero orthogonality preserving operator in $B(H)$, then T^*T is invertible with inverse $\gamma^{-2}T^*T$, for some $\gamma > 0$.*

Proof of Theorem 2.1. Let x, y be two orthogonal elements of H. Since $\langle x, y\rangle = 0$ and T is OP, we have $\langle Tx, Ty\rangle = 0$ so $\langle x, T^*Ty\rangle = 0$. Therefore $\langle Tx, TT^*Ty\rangle = 0$, hence $\langle T^*Tx, T^*Ty\rangle = 0$, which implies that T^*T is OP and so it is a multiple of an isometry. Thus T^*T is injective.

Now, take $M = ran(T^*T)$. By using [3, Theorem 3.2] for T^*T, there is a unitary operator $U : H \to M$ such that $T^*T = \gamma U$ for some $\gamma > 0$ and $UU^* = id_M$, $U^*U = id_H$. Let U^+ be the adjoint of the operator U as an element of $B(H)$. Clearly, $U^+|_M = U^*$. Since $H = M \oplus M^{\perp}$, we can write for every $x \in H : x = T^*Ty + z$ for some $y \in H, z \in M^{\perp}$. Now we have

$$UU^+x = U(U^+(T^*Ty + z)) = U(U^+T^*Ty + U^+z). \tag{1}$$

We have $U^+z = 0$, because $z \in M^\perp = ran(T^*T)^\perp$ and $T^*T = \gamma U$, so $z \in ran(U)^\perp = ker(U^+)$. Now, equation (1) implies that

$$UU^+x = UU^+(T^*Ty) = UU^*(T^*Ty) = T^*Ty.$$

Hence, for every $x \in H$, we obtain that

$$UU^+x = \begin{cases} x, & x \in M, \\ 0, & x \in M^\perp. \end{cases}$$

On the other hand, $U^+Ux = U^*Ux = x$. Since $T^*T = \gamma U$ is normal, U is normal. Therefore, $UU^+x = U^+Ux$. But it is impossible unless, $M^\perp = 0$. Therefore, $ran(T^*T) = H$, i.e., T^*T is surjective.

So, T^*T is an invertible operator of $B(H)$. Moreover,

$$(\gamma^{-2}T^*T)(T^*T) = (T^*T)(\gamma^{-2}T^*T) = id_H.$$

The idea of proving the surjectivity of T^*T in Theorem 2.1, originates from [3, Theorem 3.3].

Theorem 2.2. *Let $\varphi : B(H) \to B(H)$ be a Jordan $*$-homomorphism on $B(H)$. Then:*

(i) $\varphi(OP(H)) \subseteq OP(H)$.

(ii) $\varphi(N(H) \cap OP(H)) \subseteq N(H) \cap OP(H)$ *and* $\varphi(N(H) \cap OP(H)) \nsubseteq F(H)$.

Moreover, if φ is one to one, then

(iii) φ preserves OP operators in both directions.

(iv) φ preserves normal OP operators in both directions.

Proof of Theorem 2.2. (i) Let $T \in OP(H)$ and $M = ran(T)$, then by [3, Theorem 3.2] there exists a positive scalar γ and a unitary map $U : H \to M$ such that $T = \gamma U$. Since $\varphi(U)^*\varphi(U) = \varphi(U^*U) = \varphi(id_H) = id_H$ we have,

$$\|\varphi(U)(x)\|^2 = \langle \varphi(U)(x), \varphi(U)(x)\rangle = \langle \varphi(U)^*\varphi(U)(x), x\rangle = \|x\|^2.$$

Therefore, $\varphi(U)$ is an isometry map of $B(H)$. On the other hand, $\varphi(T) = \gamma\varphi(U)$ which implies that $\varphi(T) \in OP(H)$.

(ii) Let $T \in N(H) \cap OP(H)$. By (i) we know that $\varphi(T)$ is OP. Also $\varphi(T)$ is normal, because

$$\varphi(T)^*\varphi(T) = \varphi(T^*T) = \varphi(TT^*) = \varphi(T)\varphi(T)^*.$$

Thus, $\varphi(T) \in N(H) \cap OP(H)$.

$\varphi(T)$ is surjective by a similar proof to what we saw in Theorem 2.1 but, since H is an infinite-dimentional space, $\varphi(T)$ can not be finite-rank.

(iii) From (i) we have, $\varphi(OP(H)) \subseteq OP(H)$. For the converse, suppose that $T \in B(H)$ such that $\varphi(T) \in OP(H)$. Then, there exists a positive scalar γ and a unitary map $V : H \to ran(\varphi(T))$ such that $\varphi(T) = \gamma V$. So $\varphi(T)^* = \gamma V^*$ and

$$\varphi(T^*T) = \varphi(T)^*\varphi(T) = \gamma^2 V^*V = \gamma^2 id_H = \gamma^2\varphi(id_H).$$

Hence, $T^*T - \gamma^2 id_H \in ker(\varphi)$ and, since φ is one to one we have, $T^*T = \gamma^2 id_H$. Now if $\langle x, y\rangle = 0$, then

$$\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle = \gamma^2 \langle x, y\rangle = 0.$$

Therefore, $T \in OP(H)$.

(iv) Using (ii) we have, $\varphi(N(H) \cap OP(H)) \subseteq N(H) \cap OP(H)$. Conversely, let $T \in B(H)$ such that $\varphi(T) \in N(H) \cap OP(H)$. By (iii), we obtain that T is OP. Since $\varphi(T)\varphi(T)^* = \varphi(T)^*\varphi(T)$ we have, $TT^* - T^*T \in ker(\varphi)$. So T is normal, because φ is injective.

Corollary 2.1. *Suppose that φ is a Jordan $*$-homomorphism on $B(H)$. If $T \in N(H) \cap OP(H)$, then $\varphi(T)^* \in OP(H)$.*

Proof. Suppose that $x \in H$. Then, we have

$$\langle (T^*T - TT^*)x, x\rangle = \langle T^*Tx, x\rangle - \langle TT^*x, x\rangle = \|Tx\|^2 - \|T^*x\|^2.$$

Since T is normal and OP we have, $\|T^*x\| = \|Tx\| = \gamma\|x\|$, for some $\gamma > 0$. So T^* is OP and by Theorem 2.2(i), $\varphi(T)^* = \varphi(T^*) \in OP(H)$. □

Remark 2.1. Suppose that $\varphi : B(H) \to B(H)$ is a linear map. Since $K(H) \cap SF(H) = \emptyset$, Proposition 2.2 implies that $K(H) \cap OP(H) = \{0\}$ and $F(H) \cap OP(H) = \{0\}$. These imply that $\varphi(K(H) \cap OP(H)) \subseteq K(H) \cap OP(H)$ and $\varphi(F(H) \cap OP(H)) \subseteq F(H) \cap OP(H)$, respectively. Moreover, if φ is one to one, then φ preserves compact OP and finite rank OP operators in both directions.

The following example shows that for preserving OP operators under φ, Jordan's condition can not be removed from the hypothesis of Theorem 2.2.

Example 2.1. Let $K \in K(H) \setminus \{0\}$ and let $\varphi : B(H) \to B(H)$ is defined by $\varphi(T) := KTK^*$, for each $T \in B(H)$. It is clear that the linear map φ preserves the adjoint on $B(H)$, but it is not necessarily Jordan homomorphism. Note that $id \in OP(H)$, but $\varphi(id) \in K(H)$ and $OP(H) \cap K(H) = \{0\}$. Consequently, $\varphi(id) \in OP(H)$ implies $KK^* = 0$, which means that $K = 0$. This is a contradiction. Therefore, $\varphi(OP(H)) \nsubseteq OP(H)$.

To prove the fact that (iii) implies injectivity of the linear map φ in Theorem 2.2, φ does not need to be Jordan $*$-homomorphism. We provide a condition under which the linear map φ preserving OP operators in both directions should be injective.

Proposition 2.4. *Let $\varphi : B(H) \to B(H)$ be a linear map. If φ preserves OP operators in both directions and $ker(\varphi) \subseteq K(H)$, then φ is injective.*

Proof. For all $B \in ker(\varphi)$ we have, $\varphi(B) \in OP(H)$. Since φ preserves OP operators in both directions, $B \in OP(H)$. On the other hand, $B \in K(H)$. Thus, $B \in OP(H) \cap K(H)$ and Remark 2.1 implies that $B = 0$ which means that φ is injective. □

3. Pointwise orthogonality on $B(H)$

Proposition 3.1. *The pointwise orthogonality implies:*

(i) *the Birkhoff-James orthogonality on $B(H)$.*

(ii) *Roberts, Isosceles, Pythagorean orthogonalities on $B(H)$.*

Proof. Let T, R be two operators in $B(H)$ and $T \perp_p R$.

(i) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ and H is a Hilbert space, we have $\|Tx\| \leq \|Tx+\lambda Rx\|$, for all scalar $\lambda \in \mathbb{C}$ and $x \in H$. Therefore, $\|T\| \leq \|T+\lambda R\|$ which means that $T \perp_B R$.

(ii) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we have,

$$\begin{aligned}\langle Tx - \lambda Rx, Tx - \lambda Rx\rangle &= \langle Tx + \lambda Rx, Tx + \lambda Rx\rangle \\ &= \langle Tx, Tx\rangle + |\lambda|^2 \langle Rx, Rx\rangle\end{aligned}$$

which implies that

$$\|Tx - \lambda Rx\|^2 = \|Tx + \lambda Rx\|^2 = \|Tx\|^2 + |\lambda|^2\|Rx\|^2.$$

Therefore, the following relations come true:

$$\|T - \lambda R\|^2 = \|T + \lambda R\|^2 = \|T\|^2 + |\lambda|^2\|R\|^2$$

and

$$\|T - R\|^2 = \|T + R\|^2 = \|T\|^2 + \|R\|^2.$$

Equivalently, $T \perp_R R$, $T \perp_I R$ and $T \perp_P R$. □

Lemma 3.1. *Let $T, R \in B(H)$. If $T \perp_p R$ and one of the following conditions:*

(i) *one of T^* or R^* belongs to $OP(H)$;*

(ii) $TR^* = R^*T$;

holds, then $T^* \perp_p R^*$.

Proof. If we assume that $T^* \in OP(H)$ there is a unitary map $U : H \to ran(T^*)$ such that $T^* = \gamma\, U$ for some $\gamma > 0$ so $TT^* = \gamma^2 id_H$. On the other hand, since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we have $\langle TT^*x, RT^*x\rangle = 0$. So, $\left\langle \gamma^2 x, RT^*x\right\rangle = 0$. Therefore, $\langle R^*x, T^*x\rangle = 0$ which means $T^* \perp_p R^*$.

By condition (ii), since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we obtain that $\langle R^*Tx, x\rangle = 0$. Thus, $\langle TR^*x, x\rangle = 0$ and so $\langle R^*x, T^*x\rangle = 0$ which implies that $T^* \perp_p R^*$. □

Now, we define the set A as follows:

$$A = \{(T, R) \in B(H) \times B(H)|\ T * RT* = T*\}.$$

A is non empty, because for every generalized invertible element $T \in B(H)$, there is an operator $R \in B(H)$ such that $TRT = T$. It is straightforward to check that $(T, R^*) \in A$. It is also clear that A is self adjoint.

Theorem 3.1. *Let* $T, R \in B(H)$ *and* $\varphi : B(H) \to B(H)$ *be a linear map which preserves the adjoints. Then:*

(i) *If* $T \perp_p R$ *and* $(T, R) \in A$, *then* $\varphi(T) \perp_p \varphi(R)$ *and* $(\varphi(T), \varphi(R)) \in A$.

Moreover, if φ *is one to one, then:*

(ii) $T \perp_p R$ *and* $(T, R) \in A$ *if and only if* $\varphi(T) \perp_p \varphi(R)$ *and* $(\varphi(T), \varphi(R)) \in A$.

Proof of Theorem 3.1. Take $A = \{(T, R) \in B(H) \times B(H)|\ \mathrm{T^*RT^*} = \mathrm{T^*}\}$.

(i) Since for every $x \in H$, $\langle Tx, Rx\rangle = 0$ we obtain that $\langle TT^*x, RT^*x\rangle = 0$ so $\langle T^*x, T^*RT^*x\rangle = 0$. Therefore, for every $x \in H$

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle T^*x, T^*RT^*x\rangle = 0,$$

which implies that $\|T^*\| = 0$ and thus $T^* = 0$. Now since φ preserves the adjoint on $B(H)$ we have $\varphi(T)^* = 0$, so $(\varphi(T), \varphi(R)) \in A$. Also

$$\langle \varphi(T)x, \varphi(R)x\rangle = \langle x, \varphi(T^*)\varphi(R)x\rangle = 0.$$

Hence, $\varphi(T) \perp_p \varphi(R)$.

(ii) By (i), if $T \perp_p R$ and $(T, R) \in A$, then $\varphi(T) \perp_p \varphi(R)$ and $(\varphi(T), \varphi(R)) \in A$. Conversly, suppose that $\varphi(T) \perp_p \varphi(R)$, $(\varphi(T), \varphi(R)) \in A$. By the similar proof in (i), we obtain that $\varphi(T)^* = 0$. Since φ is one to one, $T^* = 0$. Thus, $(T, R) \in A$ and

$$\langle Tx, Rx\rangle = \langle x, T^*Rx\rangle = 0,$$

which means $T \perp_p R$.

Remark 3.1. All pointwise orthogonal members $T, R \in B(H)$ satisfy $\langle Tx, Rx\rangle = 0$ $(x \in H)$. Therefore, $T \perp_p T^*RT$, $R \perp_p R^*TR$, $T^* \perp_p T^*RT^*$, $R^* \perp_p R^*TR^*$.

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