On the nullity of Nijenhuis torsion of a vector valued 1-form

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Abstract. On a differentiable manifold, the data of a vector valued 1-form L with a constant rank, and the nullity of Nijenhuis torsion define a foliation. We study on the scalar forms the cohomology of a derivation with degree 1 associated to L. The obtained results of this study generalize those of Willmore. The vector fields by which Lie derivative of L is null, constitute a Lie subalgebra of infinitesimal automorphisms of the foliation. We study the Chevalley-Eilenberg cohomology space associated to L when L is a product structure and this study is different from the one made by Lehmann-Lejeune who studied the same cohomology in case of L as a tangent structure. As far as Lie algebra defined by any vector valued 1-form L is concerned, the derivations are not always adjoint linear applications of vector fields.

Keywords: differentiable manifold, vector 1-form, Lie algebra, Nijenhuis torsion.

1. Introduction

We consider M a smooth manifold, paracompacte, connected with $(n+m)$ dimension. All objects are supposed to be C^∞ class. In the first place, we will study the d_L-cohomology on the scalar forms in which some of the results have been mentioned in [1]. We will use in this study the theory of Frölicher-Nijenhuis [5] and [9] which shows in particular that all derivation of degree 1 on the scalar forms that commutes (in the sense of the graduated rings) with the exterior differentiation d, is of the form d_L, where L is a vector valued 1-form determined only by derivation. If $L = I$, I being the identity vector valued 1-form, d_I equals to d. The cohomologic study of d_L naturally assumes that the square of d_L is zero ($d_L^2 = 0$). To put it differently, the torsion of Nijenhuis of L is null.

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Willmore [22] has proved that if L is regular then the d_L-cohomology is isomorphic to that of de Rham. Whereas if L is singular, Poincaré's Lemma is false. In addition, E. Ayassou demonstrated in [2] that if L is the tangent structure then all form in the kernel of L^*, which is d_L-closed, is globally d_L-exact. These properties derive essentially from the nullity of the square of d_L. The nullity of d_L^2 implies the nullity of the Nijenhuis torsion of L. It is assumed in the following section that the manifold M is provided with a vector valued 1-form L of rank m and Nijenhuis torsion zero. The nullity of the Nijenhuis torsion of L defines on M a foliation such that the image space of L corresponds to the tangent space to leaves. We prove Poincaré Lemma for the L-semi-basic forms which, if L is regular, are the forms on M. The d_L-cohomology on any leave is isomorphic to that of de Rham. For homogeneous forms, we adapt a method used by J. Klein and A. Voutier [12]. The use of a projetable vector valued 1-form makes it possible to compare the cohomologies between certain manifolds. We write, in our formalism, the classical derivations d' and d'' on foliated manifolds and on complex manifolds. We give in [10] some versions for the L-semi-basic forms, of Fröbenius' theorem and a result of H. Jacobowitz's work. In the second place, our study consists in working on the cohomology of Chevalley-Eilenberg of $\mathfrak{A}_L$ where $\mathfrak{A}_L$ denotes the set of vector fields by which the Lie derivative of L is zero. F. Takens in [21] has proved that any derivation of the Lie algebra of the vector fields $\chi(M)$ on a manifold M is inner. A. Lichnerowicz [15] considered the Lie algebra of the infinitesimal automorphisms of a foliation $\mathcal{L}_{\mathcal{F}}$ on a manifold and showed in particular that, whatever the foliation, any derivation of $\mathcal{L}_{\mathcal{F}}$ is inner. As for the considered manifold (M, L) , $\mathfrak{A}_L$ is a subalgebra of $\mathcal{L}_{\mathcal{F}}$. If L is the vector valued 1-form of the transverse bundle to a foliation ($rank\,(L) = n$ and $L^2 = 0$), Lehmann-Lejeune in [14] proved that the derivations of $\mathfrak{A}_L$ are adjoint linear applications of the normalizer of $\mathfrak{A}_L$ in $\chi(M)$. One can wonder the derivation behaviors of $\mathfrak{A}_L$ for any vector valued 1-form. If we denote by $\overline{\chi(M)}$ the complete bearing on the tangent bundle TM of the vector fields $\chi(M)$ on M, the first cohomology of the normalizer of $\overline{\chi(M)}$ in $\chi(TM)$ is of dimension 1, whereas that of $\overline{\chi(M)}$ is zero. If L is a tangent structure, $\overline{\chi(M)}$ is a subalgebra of $\mathfrak{A}_L$; we can demonstrate by using the result on $\overline{\chi(M)}$, that any derivation of $\mathfrak{A}_L$ comes from its normalizer - it is a result of Lehmann-Lejeune's work in [14] in this particular case - and that any derivation of the normalizer of $\mathfrak{A}_L$ is interior. If L is a connection in the sense of Grifone [8] ($L^2 = I$) then $\mathfrak{A}_L$ is isomorphic to $\chi(M) \times \chi(\mathbb{R}^n)$, therefore, any derivation of $\mathfrak{A}_L$ is interior; it is also a result of [16] by a different approach. Finally, we give an example of a vector valued 1-form of null Nijenhuis torsion admitting non-adjoint derivations of the vector fields. For Chevalley-Eilenberg cohomology space studies in other cases, the results can be found in [16], [17], [18] and [19].

2. d_L-Cohomologies

2.1 Preliminaries

Let M be a differentiable manifold, TM as the tangent bundle to M, $\mathcal{F}(M)$ the $\mathbb{R}$-algebra of differentiable functions on M, $\phi^p(M)$ the $\mathcal{F}(M)$-module of scalar p-forms on M and $\phi^0(M) = \mathcal{F}(M)$; $\psi^1(M)$ the $\mathcal{F}(M)$-module of 1-vector forms on M; $\psi^0(M)$ or $\chi(M)$ the $\mathcal{F}(M)$-module of vector fields on M, and $\phi(M)$, $\psi(M)$ or simply ϕ, ψ if there is no possible confusion on the corresponding graduated rings. The formalism of Frölicher-Nijenhuis will be the fundamental tool of this work (see [5] and [9]).

Definition 2.1 ([5], [9]). *Let's consider $L \in \psi^1$. We define the endomorphism L^* of ϕ by:*

- *a)* $L^* f = f$ *if* $f \in \phi^0$
- *b)* $(L^*\omega)\left(X^1, X^2, \ldots, X^p\right) = \omega\left(LX^1, LX^2, \ldots, LX^p\right)$ *if* $\omega \in \phi^p$ *and* $X^1, \ldots, X^p \in \psi^0$.

Definition 2.2 ([5], [9]). *Let's assume $\omega \in \phi^p$ and $L \in \psi^l$. We define the element $i_L\omega$ of ϕ^{p+l-1} by:*

$$
\begin{aligned}
&(i_L\omega)\left(X^1, X^2, \ldots, X^{p+l-1}\right) \\
&= \frac{1}{(p-1)!l!} \sum_{\alpha} sgn(\alpha)\, \omega\left(L\left(X^{\alpha_1}, \ldots, X^{\alpha_l}\right), X^{\alpha_{l+1}}, \ldots, X^{\alpha_{p+l-1}}\right),
\end{aligned}
$$

for all $X^1, X^2, \ldots, X^{p+l-1} \in \psi^0$; the summation above being taken from the group of permutations $\{1, 2, \ldots, p+l-1\}$. As for $\omega \in \phi^0$, $i_L\omega$ is null by definition. The application $i_L : \omega \longmapsto i_L\omega$ is a derivation of degree $l-1$ on ϕ.

Definition 2.3 ([5], [9]). *The commutator of two derivations $[i_L, d] = i_L d - (-1)^l d i_L$ defines a derivation of degree l, noted d_L. In particular, if I is the vector valued 1-form identity then $d_I = d$ and d_X is Lie derivative with respect to X, which will be noted in the following by θ_X, for $X \in \psi^0$.*

Definition 2.4 ([5], [9]). *Let's consider $K \in \psi^k$ and $L \in \psi^l$. We define $[K, L]$, element of ψ^{k+l}, by $[d_K, d_L] = d_{[K,L]}$. In particular, if $L \in \psi^1$, we have $[d_L, d_L] = 2d_L^2 = d_{[L,L]}$.*

The bracket $N_L = \frac{1}{2}[L, L]$ is called Nijenhuis torsion of L. We have the following formula:

$$
\frac{1}{2}[L, L](X, Y) = [LX, LY] + L^2[X, Y] - L[LX, Y] - L[X, LY]
$$

for all $X, Y \in \psi^0$.

As a result, the study of Poincaré's Lemma on the derivation d_L, L being any vector valued 1-form, supposes the nullity of Nijenhuis torsion of L that is, $[LX, LY] = L[LX, Y] + L[X, LY] - L^2[X, Y]$.

If L is of constant rank in a neighborhood U of a point z_0 of M, according to the Fröbenius theorem, the distribution which at all z of U associates $L_z T_z$, is completely integrable.

Let M be a paracompact, differentiable manifold of $(n+m)$-dimension with a vector valued 1-form L of rank $m \geq 1$ and Nijehnuis torsion null. Thus L defines on M a completely integrable distribution $\mathfrak{D}$ with m dimension. The structure of foliated manifold by L can be defined by an overlap of open sets $\mathfrak{U}$ of M and by the data, in each $U \in \mathfrak{U}$, of a coordinate system $\left(x^1, \ldots, x^n, y^1, \ldots, y^m\right)$ such that $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}$ form a tangent local basis to the leaves. Note $a, b, \ldots$ etc, the indexes ranging from 1 to n and $i, j, \ldots$ etc, those ranging from 1 to m. For U, U' of $\mathfrak{U}$, with $U \cap U' \neq \emptyset$, we have $x^{b'} = x^{b'}\left(x^a\right)$ and $y^{j'} = y^{j'}\left(x^a, y^i\right)$. Then, the distribution $\mathfrak{D}$ is defined by equations $dx^a = 0$.

2.2 Generalization of Willmore's theorem

Proposition 2.1. *Let's assume $X \in \psi^0$ and $L \in \psi^1$. Then $i_X L^* = L^* i_{LX}$.*

Proof. Taking into account the definitions of the inner product i_X and the endomorphism L^* in the Definition 2.1, we can write successively:

$$\begin{aligned}\left(i_X L^* \omega\right)\left(X^1, \ldots, X^{p-1}\right) &= \left(L^* \omega\right)\left(X, X^1, \ldots, X^{p-1}\right) \\ &= \omega\left(LX, LX^1, \ldots, LX^{p-1}\right) \\ &= \left(L^* i_{LX} \omega\right)\left(X^1, \ldots, X^{p-1}\right),\end{aligned}$$

where $X^1, \ldots, X^{p-1} \in \psi^0$ and $\omega \in \phi^p$. □

Proposition 2.2. *If $[L, L] = 0$ then $L^* d = d_L L^*$.*

Proof. Let's consider $\omega \in \phi^p$ and $X^0, \ldots, X^p \in \psi^0$, the symbol $\widehat{\ }$ means omission. We explain the expression $d_L L^*$:

$$\left(d_L L^* \omega\right)\left(X^0, \ldots, X^p\right) = \left(\left(i_L d - d i_L\right) L^* \omega\right)\left(X^0, \ldots, X^p\right). \tag{1}$$

By using formulas of derivations i_L and d, the equation (1) becomes

$$\begin{aligned}&\sum_{0 \leq u \leq p} (-1)^u LX^u.\omega\left(LX^0, \ldots, \widehat{LX^u}, \ldots, LX^p\right) \\ &+ \sum_{0 \leq u < v \leq p} (-1)^{u+v} \omega\left(L\left[LX^u, X^v\right] + L\left[X^u, LX^v\right]\right. \\ &\left.-L^2\left[X^u, X^v\right], LX^0, \ldots, \widehat{LX^u}, \ldots, \widehat{LX^v}, \ldots, LX^p\right) \\ &= \sum_{0 \leq u \leq p} (-1)^u LX^u.\omega\left(LX^0, \ldots, \widehat{LX^u}, \ldots, LX^p\right) \\ &+ \sum_{0 \leq u < v \leq p} (-1)^{u+v} \omega\left(\left[LX^u, LX^v\right], LX^0, \ldots, \widehat{LX^u}, \ldots, \widehat{LX^v}, \ldots, LX^p\right)\end{aligned}$$

$$-\sum_{0\leq u<v\leq p}(-1)^{u+v}\,\omega\{\frac{1}{2}\left[L,L\right]\left(X^u,X^v\right),LX^0,\ldots,\widehat{LX^u},\ldots,\widehat{LX^v},\ldots,LX^p\}.$$

This last equality is due to the formula of 2.4. If Nijenhuis torsion of L is null then $\left(d_L L^*\omega\right)\left(X^0,\ldots,X^p\right)=\left(L^*d\omega\right)\left(X^0,\ldots,X^p\right)$. □

Remark 2.1. If L is reversible then

$$L^*dL^{*-1}=d_L. \tag{2}$$

It is the result which makes it possible to obtain immediately Willmore's theorem (cf. [22]). Indeed, if $d_L\omega=0$ then $dL^{*-1}\omega=0$. By applying Poincaré's Lemma, there is $\pi\in\phi^{p-1}$ such that $L^{*-1}\omega=d\pi$. In other words, taking into account equation (2), we obtain $\omega=L^*d\pi=d_L L^*\pi$.

Proposition 2.3. *For all z of the manifold M, there is an open-set U containing z and an endomorphism K of $\phi\left(U\right)$ verifying:*

1. *For $p\geq1$, $K\left(\phi^p\left(U\right)\right)\subset\phi^{p-1}\left(U\right)$,*
2. *For every p-form ω on U, there is a p-form ω_0 determined by ω such that $\omega=Kd\omega+dK\omega+\omega_0$ and $i_L\omega_0=0$;*
3. *If $L^*\omega=0$ then $L^*K\omega=0$.*

For the demonstration, we use the following classic result:

Lemma 2.1 ([7])**.** *We consider J_h, with $h=0,1$, the injections of $\mathbb{R}^{n+m}$ in $\mathbb{R}^{n+m}\times\mathbb{R}$ defined by $J_h\left(z\right)=\left(z,h\right)$. There is a linear application $k:\phi\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\longrightarrow\phi\left(\mathbb{R}^{n+m}\right)$ having the following properties:*

1. *$k\left(\phi^p\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\right)\subset\phi^{p-1}\left(\mathbb{R}^{n+m}\right)$, $p\geq1$;*
2. *$dk+kd=J_1^*-J_0^*$.*

For understanding the next, we will give the definition of k. We define the application $k:\phi\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)\longrightarrow\phi\left(\mathbb{R}^{n+m}\right)$ by:

- $kf=0$ if $f\in\mathcal{F}\left(\mathbb{R}^{n+m}\times\mathbb{R}\right)$,
- $k\alpha=0$ if $\alpha=\lambda_{A,I}dx^A\wedge dy^I$, A and I denote the multi-indexes;
- $k\alpha=\int_0^1\lambda_{A,I}dtdx^A\wedge dy^I$ if $\alpha=\lambda_{A,I}dt\wedge dx^A\wedge dy^I$.

Proof of the Proposition 2.3. Let U be an open set containing z, which is the compatible chart domain of M. We denote by H the differentiable application of $U\times\mathbb{R}$ on U defined by $H\left(x^a,y^i,t\right)=\left(x^a,ty^i\right)$. HJ_1 is the identical application of U, and HJ_0 the application which has $\left(x^a, y^i\right) \longrightarrow (x^a, 0)$. Then we have $(HJ_1)^* \omega = \omega$. By setting $\omega = \lambda_{A,I} dx^A \wedge dy^I$, we obtain

$$(HJ_0)^* \omega = \begin{cases} 0, & \text{if } I \neq \emptyset \\ \lambda_{A,\emptyset}(x, 0)\, dx^A, & \text{otherwise.} \end{cases}$$

So, by considering the previous Lemma, we can write $\omega = (J_1^* - J_0^*)\, H^*\omega + (HJ_0)^* \omega = dkH^*\omega + kdH^*\omega + (HJ_0)^* \omega$.

As $dH^* = H^*d$ we obtain $\omega = dkH^*\omega + kH^*d\omega + (HJ_0)^* \omega$. Pose $K\omega = kH^*\omega$ and $\omega_0 = (HJ_0)^* \omega$, then ω_0 thus defined is zero or of the form $\lambda_A(x, 0)\, dx^A$, so $i_L\omega_0 = 0$, and we have the following expression:

$$\omega = dK\omega + Kd\omega + \omega_0.$$

If $L^*\omega = 0$, ω is then of the form $\omega = \lambda_{A,I} dx^A \wedge dy^I$ with $A \neq \emptyset$. According to the definition of K and for such ω, $L^*K\omega = 0$. □

Proposition 2.4. *Let's consider $\omega \in \phi^p(U)$ with $p \geq 1$. If $L^*\omega$ is d_L-closed, we have $\pi \in \phi^{p-1}(U)$ such that $L^*\omega = d_L(L^*\pi)$.*

Proof. According to 2. of the Proposition 2.3, ω can be written $\omega = dK\omega + Kd\omega + \omega_0$. Since $L^*\omega_0 = 0$ then $L^*\omega = L^*Kd\omega + L^*dK\omega$. By applying Proposition 2.2, namely $L^*d = d_L L^*$, we have $L^*d\omega = d_L L^*\omega = 0$. The last equality is due to the hypothesis on $L^*\omega$. And considering 3. of the Proposition 2.3, $L^*Kd\omega = 0$. Finally, according to the Proposition 2.2 we have $L^*\omega = d_L(L^*K\omega) = d_L(L^*\pi)$ by assuming $\pi = K\omega$. □

Definition 2.5. *A p-form ω is said to be L-semi-basic if for any vector fields X in the kernel of L, the interior product $i_X\omega$ is zero.*

Remark 2.2. If L is a tangent structure, an L-semi-basic form is, in the usual sense, semi-basic.

Proposition 2.5. *The following two relationships are equivalent:*

1. *ω is an L-semi-basic p-form,*
2. *There is a p-form π such that $\omega = L^*\pi$.*

Proof. 2. $\Longrightarrow$ 1. For that, we just need to see if X is into the kernel of L, according to Proposition 2.1, we have $i_X L^* = L^* i_{LX} = 0$.

1. $\Longrightarrow$ 2. With the appropriate coordinate system, $L^*dx^a = 0,\ a = 1, \ldots, n$; and $L^*dy^i = d_L y_i,\ i = 1, \ldots, m$ form a base of the orthogonal space at the kernel of L. Let's represente this kernel by N, the dual by N^*, the orthogonal by $N^\perp$; then $N_z^\perp \oplus N_z^* = \phi_z^1$ with a base of ϕ_z^1 adapted to the direct sum, the semi-basic condition with respect to L is translated by $\omega = \lambda_{\emptyset,\ I} d_L y^I = L^*\left(\lambda_{\emptyset,\ I} dy^I\right) = L^*\pi$ by assuming $\pi = \lambda_{\emptyset,\ I} dy^I$. □

Theorem 2.1. *Let $\mathcal{F}$ be the set of functions on M, $\mathcal{F}_L$ the set of constant functions on the leaves, ϕ_L^p the set of L-semi-basic p-forms. So, for every point z of M, there is an open U containing z such that the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}_L(U) \longrightarrow \mathcal{F}(U) \xrightarrow{d_L} \phi_L^1(U) \xrightarrow{d_L} \dots \xrightarrow{d_L} \phi_L^{m-1}(U) \xrightarrow{d_L} \phi_L^m(U) \xrightarrow{d_L} 0.$$

Proof of Theorem 2.1. Let U be a domain of the adapted chart to foliation and $\omega \in \phi_L^p(U)$. According to Proposition 2.5 there is $\alpha \in \phi^p(U)$ such that $\omega = L^*\alpha$. If ω is d_L-closed, by Proposition 2.4, there is $\pi \in \phi^{p-1}$ such that $\omega = d_L(L^*\pi)$. As indicated in Proposition 2.5, $L^*\pi \in \phi_L^{p-1}$. In addition, ω being L-semi-basic, $d_L\omega$ is also L-semi-basic, according to Propositions 2.1 and 2.4, $d_L\omega = d_L L^*\alpha = L^*d\alpha$. To finish the cohomological resolution, we just need to note if the dimension of $\phi_L^1(U)$ is equal to m and $d_L f = 0$ is equivalent to $f \in \mathcal{F}_L$.

Theorem 2.2. *On every leave, the cohomology of d_L is isomorphic to that of de Rham.*

Proof of Theorem 2.2. Indeed, a leaf S can be considered as a submanifold of M. The application $L^* : \phi^p(S) \longrightarrow \phi_L^p(S)$ is an isomorphism. In applying Proposition 2.2 we find the result.

2.3 Homogeneous forms

In this section, we assume that the leaves of L are the fibers of a vector fibration of M. The group of homotheties on U: $H_t\left(x^a, y^i\right) = \left(x^a, e^t y^i\right)$ with $t \in \mathbb{R}$ defines a Lie group to a parameter of transformations on M. Let's consider C as a vector field that generates this group. If S stands for a vector field such that $LS = C$, then $i_S\omega$ depends only on C, for any L-semi-basic form. In other words, $i_S\omega = i_{S'}\omega$ if $LS = LS' = C$. This derives from Propositions 2.1 and 2.5.

Definition 2.6. *A scalar form ω is said to be homogeneous of degree r, if the Lie derivative of ω with respect to C is such that $\theta_C\omega = r\omega$.*

A vector form K is said to be homogeneous of degree k if $[C, K] = kK$.

Theorem 2.3. *If L is homogeneous of degree l, any p-form ω with $p \geq 1$, L-semi-basic, homogeneous of degree $r \neq lp$ which is d_L-closed, is d_L-exacte and $(r - lp)\,\omega = d_L i_S\omega$, where S designates a vector field on M such that $LS = C$.*

Remark 2.3. This theorem gives a version of Theorem 2.1 for the forms globally defined in this particular case, and generalizes a result of J. Klein and A. Voutier, see [12], namely: If J is the vector valued 1-form defining the tangent structure of M, then $(r + p)\,\omega = d_J i_S\omega$.

To prove Theorem 2.3, we will use the following Lemma:

Lemma 2.2. *$L[S,L] = lL$.*

Proof. From (2.4) and the relation $LS = C$, we have $\frac{1}{2}[L,L](S,X) = [C,LX] - L[C,X] + L^2[S,X] - L[S,LX]$ for all $X \in \psi^0$. Now, according to the formulas of [5], we have $[C,L]X = [C,LX] - L[C,X]$ and $[S,L]X = [S,LX] - L[S,X]$; then nullity of the Nijenhuis torsion implies $L[S,L] = [C,L]$. But L is homogeneous of degree l by hypothesis. Hence $L[S,L] = lL$. □

Demonstration of Theorem 2.3. We have the following formula (cf. [5] p.352)

$$i_S d_L \omega + d_L i_S \omega = \theta_C \omega - i_{[S,L]}\omega. \tag{3}$$

Since ω is homogeneous of degree r, then $\theta_C \omega = r\omega$. Moreover ω is L-semi-basic, according to Proposition 2.5 there is $\pi \in \phi$ such that $\omega = L^*\pi$. Thus, from the formula 2.2 and the Lemma, we can write

$$\begin{aligned} &i_{[S,L]}\omega\left(X^1,\ldots,X^p\right) \\ &= \sum_{u=1}^{p} \pi\left(LX^1,\ldots,L[S,L]X^u,\ldots,LX^p\right) = lpL^*\pi\left(X^1,\ldots,X^p\right), \end{aligned}$$

for all $X^1,\ldots,X^p \in \psi^0$.

Finally, the expression (3) becomes $i_S d_L \omega + d_L i_S \omega = (r - lp)\,\omega$.

If $d_L\omega = 0$ then $d_L i_S \omega = (r - lp)\,\omega$. □

Remark 2.4. The result of Theorem 2.3 is not true if $r - lp = 0$. Indeed, $\mathbb{R}^2$ the manifold and ω the following form on $T\mathbb{R}^2 \setminus \{0\}$: $\omega = \frac{y^1dx^1 - y^2dx^2}{(y^1)^2 + (y^2)^2}$. It is clear that ω is C^∞ on $T\mathbb{R}^2 \setminus \{0\}$, homogeneous of degree -1 and d_j-closed, where J denotes the tangent structure. It is well known that ω does not admit a primitive on $T\mathbb{R}^2 \setminus \{0\}$.

2.4 Fröbenius' theorem on L-semi-basics Pfaff system

We present here a version of Fröbenius' theorem with the d_L derivation.

Definition 2.7. *The L-semi-basic Pfaff system of rank $s \le m$, $(\omega^u)_{u=1,\ldots,s}$ is said to be completely integrable in an open U provided that there is on U s independent differentiable functions f^u such that $\omega^u = \lambda^u_v d_L f^v$, $v = 1,\ldots,s$; where λ^u_v are the differentiable functions on U and elements of a regular matrix.*

We note that s must be less than or equal to m in order to get such property, otherwise the system $(\omega^u)_{u=1,\ldots,s}$ would be bound.

Theorem 2.4. *If an L-semi-basic Pfaff system is fully integrable into $U \subset M^{n+m}$, it passes through each generic point $z \in U$ an integral manifold of dimension $m - s$ defined in a neighborhood of this point which is a submanifold of M^{n+m}.*

Proof of Theorem 2.3. The leaves of foliation may be considered as submanifolds of M. Let S be a submanifold passing through $z \in U$, the restriction of $L^* : \phi^1(S) \longrightarrow \phi^1_L(U)$ is an isomorphism. We consider $\tilde{L^*}$ this isomorphism, $\tilde{L^*}^{-1}\omega^u = \lambda^u_v dy^v$ is a completely integrable system. According to the classical Fröbenius theorem, it passes through z an integral variety of dimension $m - s$ which is a submanifold of the submanifold S of M. This submanifold is also an integral variety of ω^u. Indeed, if we consider (W, g) as an integral variety of $\tilde{L^*}^{-1}\omega^u$, we have: $g^*\tilde{L^*}^{-1}\omega^u = 0$ on W, that is, $\lambda^u_v \circ gd\,(y^v \circ g) = 0$ on W. So, we can write $\tilde{L^*}g^*\tilde{L^*}^{-1}\omega^u = \lambda^u_v \circ gd_L\,(y^v \circ g) = g^*\omega^u = 0$ on W.

Hence the result.

Theorem 2.5. *A necessary and sufficient condition for an L-semi-basic Pfaff system to be completely integrable in the neighborhood of a generic point, is that in a neighborhood of this point the system is d_L-closed, that is*

$$d_L\omega^u \wedge \omega^1 \wedge \cdots \wedge \omega^s = 0,\ u = 1, \ldots, s.$$

The proof is implied directly from the classical Fröbenius theorem via isomorphism $\tilde{L^*}$ previously used.

2.5 Projection of a vector valued 1-form

In this section, we will presente some results on a projectable vector valued 1-form and establish homotopic links for the d_L-cohomology; the study of projection of a vector valued 1-form is done in [6], [13]. Let μ be a differentiable application of a manifold M in N, μ_* the tangent linear application, μ^* the cotangent linear application.

Definition 2.8. *A vector l-form L on M is projectable provided that there is a vector l-form L' on N such that*

$$\mu_* L_z\left(X^1, \ldots, X^l\right) = L'_{\mu(z)}\left(\mu_* X^1, \ldots, \mu_* X^l\right),$$

for all $X^1, \ldots, X^l \in T_z(M)$ and for any $z \in M$.

Proposition 2.6 ([6])**.** *Let $i : M \longrightarrow N$ be an application of rank equals to the dimension of M, and L' a vector valued 1-form on N. There is a unique vector valued 1-form L on M which projects on L' if and only if $L'(X) \in i_*(TM)$ for all $X \in i_*(TM)$.*

Proposition 2.7 ([6])**.** *Let $\mathfrak{p} : M \longrightarrow N$ be a fibration to the connected bundles. A vector valued 1-forme L is projectable if and only if $L(Y)$ is a vertical field and $\theta_Y L$ has vertical values for all vertical field Y.*

Proposition 2.8. *If L is a projectable vector valued 1-form, then we have $\mu^* d_{\mu_* L} = d_L \mu^*$ and $\mu_*[L, L] = [\mu_* L, \mu_* L]$ where $\mu_* L$ is the projection of L by μ.*

Proof. The proof is obtained from Theorem 1, p.233 of [6]. □

Theorem 2.6. *Let L be a μ-projectable vector valued 1-form, H_L the algebra of d_L-cohomology of L-semi-basic forms, μ^* a homomorphism of algebras of $H_{\mu_* L}(N) \longrightarrow H_L(M)$.*

Proof of Theorem 2.4. The Proposition 2.8 proves that the manifold N is $\mu_* L$-foliated, that the image of $d\mu_* L$-closed form is a d_L-closed form, and that the image of $d\mu_* L$-exact form is d_L-exact form. Furthermore, by doing a simple calculation, we also have $\mu^*\left((\mu_* L)^* \omega\right) = L^*\left(\mu^* \omega\right)$, for all $\omega \in \phi(N)$. Hence the result.

2.6 Derivations having the types d' and d'' on a foliated manifold

Let M be a paracompact differentiable manifold of $n+m$ dimension, $\mathcal{D}$ an integrable distribution of m dimension on M. The variety M thus defined is foliated. By giving on M a riemannian metric, we can define a distribution $\mathcal{D}^{\perp}$ orthogonal to $\mathcal{D}$. The variety M is therefore provided with an almost produced structure Γ (cf. [20]). We assume $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$ where I designates the identical application; h and v are the projectors of respectively ranks n and m. The distribution $\mathcal{D}^{\perp}$ is completely integrable if and only if Nijenhuis torsion of Γ is zero. In other words, $[h,h] = [v,v] = 0$. It is assumed in the following that Nijenhuis torsion of Γ is zero.

Proposition 2.9. $h^* d_v = 0$ *and* $v^* d_h = 0$.

Proof. This proposition derives from the properties of the projectors h and v, namely: $h \circ v = 0$, $v \circ h = 0$ and $[h,h] = [v,v] = 0$. Since they play symmetrical roles, all we need to do is to calculate $h^* d_v$. We assume ω as a scalar p-form. According to the definition of the derivation d_v, we have $h^* d_v \omega = h^*\left(i_v d - d i_v\right)\omega$. From relation $v \circ h = 0$ we obtain $h^* i_v d\omega = 0$. What is to be calculated next is $h^* d i_v \omega$. By doing a simple calculation, we get:

$$\begin{aligned} &\left(h^* d i_v \omega\right)\left(X^0, \ldots, X^p\right) \\ &= \sum_{0 \le u < t \le p} (-1)^{u+t}\, i_v \omega \Big(\left[hX^u, hX^t\right], hX^0, \ldots, \widehat{hX^u}, \ldots, \widehat{hX^t}, \ldots, hX^p\Big), \end{aligned}$$

for all $X^0, \ldots, X^p \in \psi^0$. Since Nijenhuis torsion of h is zero and $h^2 = h$ we have

$$\left[hX^u, hX^t\right] = h\left[hX^u, X^t\right] + h\left[X^u, hX^t\right] - h\left[X^u, X^t\right].$$

Hence, taking into account the relation $v \circ h = 0$ we get $h^* d i_v \omega = 0$. □

Proposition 2.10. $v^* d_v = d_v v^*$ *and* $h^* d_h = d_h h^*$.

Proof. From Proposition 2.9 and the relation $h = I - v$, we have

$$v^* d_{I-v} = 0 = v^* d - v^* d_v.$$

According to Proposition 2.2, we have $v^*d = d_v v^*$, that is $v^*d_v = d_v v^*$. It is the same for $h^*d_h = d_h h^*$. □

Theorem 2.7. *Let consider v a projector on M of constant rank m and of Nijenhuis torsion null. Any p-form ω with $p \geq 1$ which is d_v-closed, is locally d_v-exact if and only if $(I-v)^*\omega = 0$. Especially if $p > n$, any p-form, that is, d_v-closed is locally d_v-exact.*

Proof of Theorem 2.5. The condition is necessary. Indeed, if ω is a p-form such that $\omega = d_v\pi$, with $\pi \in \phi^{p-1}$, according to Proposition 2.9, we have $(I-v)^*\omega = h^*d_v\pi = 0$. The condition is sufficient. With the coordinates system of the section 2.1 and knowing that $dx^a = d_h x^a,\ d_v y^i = dy^i$, ω can be written: $\omega = \lambda_{A,I} d_v y^I \wedge d_h x^A$, where A, I stand for multi-indexes such that $|A| + |I| = p$. The condition $h^*\omega = 0$ means $|I| \geq 1$. Since $d_v(d_h x^a) = 0$, the condition d_v-closed ω becomes $d_v\omega = d_v\left(\lambda_{A,I} d_v y^I\right) \wedge d_h x^A = 0$. It implies that $d_v\left(\lambda_{A,I} d_v y^I\right) = 0$. According to Proposition 2.4 there is a π' form such that $d_v\pi' = \lambda_{A,I} d_v y^I$. So, the expression of ω becomes $\omega = d_v\pi' \wedge d_h x^A = d_v\left(\pi' \wedge d_h x^A\right)$. If $p > n$, it is immediate to note that $h^*\omega = 0$.

Theorem 2.8. *Let ω be a scalar p-form $(p \geq 1)$ d_v-closed. There is locally a scalar $(p-1)$-form π and a p-form ω' such that*

1. $\omega = d_v\pi + \omega'$;
2. *ω' a basic p-form of the foliation defined by the distribution $\mathcal{D}$.*

Proof of Theorem 2.6. On a suitable open U, any p-form ω can be written $\omega = \lambda_A dx^A + \mu_{B,I} dx^B \wedge d_v y^I$ where A, B, I denote the multi-indexes such that $|A| = p, |B| + |I| = p$ and $|I| \geq 1$.

If ω is d_v-closed, we have $d_v(\lambda_A) \wedge dx^A + d_v\left(\mu_{B,I} dx^B \wedge d_v y^I\right) = 0$. That implies $d_v(\lambda_A) = 0$ and $d_v\left(\mu_{B,I} dx^B \wedge d_v y^I\right) = 0$. In other words, $\lambda_A dx^A$ is a constant p-form on the leaves defined by the distribution $\mathcal{D}$, and $\mu_{B,I} dx^B \wedge d_v y^I$ is a p-form which makes the conditions of Theorem 2.7 satisfying. Consequently, by assuming $\omega' = \lambda_A dx^A$, there is π a $(p-1)$-form such that $\omega = d_v\pi + \omega'$.

Remarks 2.1.

1. The derivations $d_v,\ d_h$ are classical derivations $d',\ d''$ on foliated manifolds. The obtained results make it possible to study the scalar forms without having to introduce the notion of scalar forms of type (p, q).

2. Let M be a complex paracompact differentiable manifold and F be an almost complex structure $\left(F^2 = -I\right)$. The projectors associated with F are $P = \frac{1}{2}(I - iF)$ and $Q = \frac{1}{2}(I + iF)$ where $i = \sqrt{-1}$. The nullity of Nijenhuis torsion of F is equivalent to $[P, P] = [Q, Q] = 0$.

   As for the derivations d_P and d_Q, we get results similar to those of the derivations d_v and d_h. The classical derivations d' and d'' are found on complex manifolds.

2.7 Some types of d_L-cohomologies

Proposition 2.11. *On an L-foliated manifold M, the following conditions are supposed to be satisfactory:*

1. *There is a Pfaff system of rank n, (θ^a), $a = 1, \dots, n$ such that, (θ^a), $a = 1, \dots, n$ and $(d_L y^i)$, $i = 1, \dots, m$ form a basis of $\phi^1(U)$;*
2. *the 1-forms (θ^a), $a = 1, \dots, n$ are d_L-closed.*

Then for all p-form ω $(p \geq 1)$ on M, d_L-closed, there is locally a $(p-1)$-form π and a p-form ω' such that

i) $\omega = d_L \pi + \omega'$;

ii) ω' is a p-form generated by (θ^a) on the constant functions to the leaves.

Proof. This is a version of Theorem 2.8 in its general cases. The demonstration is done with the same principle. □

Proposition 2.12. *If the kernel of L noted N is completely integrable, then for all p-form ω $(p \geq 0)$ and for any $X^i \in N$, we have $d_L\omega\left(X^0, \dots, X^p\right) = 0$.*

Proof. For that, we just calculate $d_L\omega\left(X^0, \dots, X^p\right)$ for all $X^0, \dots, X^p \in N$ and we find

$$d_L\omega\left(X^0, \dots, X^p\right)$$
$$= -\sum_{0 \leq i < j \leq n} (-1)^{i+j}\, \omega\left(L\left[X^i, X^j\right], X^0, \dots, \widehat{X^i}, \dots, \widehat{X^j}, \dots, X^p\right).$$

If the kernel is fully integrable, we get $d_L\omega\left(X^0, \dots, X^p\right) = 0$. □

Theorem 2.9. *Let M be an L-foliated manifold. If L verifies one of the following two conditions:*

1. *L is regular on the leaves;*
2. *The image of L equals to the kernel of L.*

Then all p-form ω $(p \geq 1)$ that is d_L-closed is locally d_L-exact if and only if $\omega\left(X^1, \dots, X^p\right) = 0$ for any fields $X^1, \dots, X^p$ in the kernel of L.

Proof of Theorem 2.7. If L confirms one of the two above conditions, the kernel of L is completely integrable according to the nullity of Nijenhuis torsion of L. In this case $d_L\pi\left(X^1, \dots, X^p\right)$ is null for all $(p-1)$-form π and for all fields $X^1, \dots, X^p$ in the kernel of L as stated in Proposition 2.12. In addition, there are functions $f^1, \dots, f^n$ such that df^a, $a = 1, \dots, n$ and $(d_L y^i)$, $i = 1, \dots, m$ form a local basis of $\phi^1(M)$ and that $d_L df^a = 0$. Hence the result by considering Proposition 2.11.

Theorem 2.10. *Let M be an L-foliated manifold such that L is regular on the leaves. The algebra of d_L-cohomology is locally isomorphic to that of the basic forms.*

Proof of Theorem 2.8. With the coordinates system in Section 2.1, we note that $\left(dx^a, d_L y^i\right)$, $a = 1, \ldots, n$ and $i = 1, \ldots, m$ form a basis of $\phi^1(U)$. Since $d_L(dx^a) = 0$, as indicated in Proposition 2.11, we get the result we've been searching for.

Proposition 2.13. *If the leaves of L are simply connected and at the same time fibers of a fibration of M, then all L-semi-basic p-form $(p \geq 1)$ which is d_L-closed is generally d_L-exact.*

Proof. This is a consequence of Theorems 2.1 and 2.2, because the derivation of d_L acts only on the fibers, which makes it possible to paste again the pieces by means of the unit partition. □

Corollary 2.1. *Let M be a paracompact differentiable manifold, Γ a connection of Grifone $\left(\Gamma^2 = I\right)$ on M with null curvature, v the vertical projector. For all p-form ω on TM, d_v-closed, with $p \geq 1$, there is a $(p-1)$-form π and a p-form ω' such that*

1. $\omega = d_v \pi + \omega'$
2. $\omega' = \mathfrak{p}^* \omega''$ *where* $\omega'' \in \phi^p(M)$, $\mathfrak{p}$ *is the projection of* $TM \longrightarrow M$

The cohomology space $H^p_{d_v}(TM)$ is therefore isomorphic to $\phi^p(M)$.

Proof. This is a version of Theorem 2.8 for globally defined forms, because the leaves are fibers of the tangent space to M and the basic forms are the elements of $\mathfrak{p}^* \phi(M)$. □

Corollary 2.2 ([2])**.** *Let J be the vector valued 1-form defining on TM the natural structure of tangent variety. Any p-form ω on TM $(p \geq 1)$ is exact if and only if $d_J \omega = 0$ and $J^* \omega = 0$.*

Proof. This is a version of Theorem 2.9 for the tangent structure J. Here, the condition of Proposition 2.12 is translated by $J^* \omega = 0$. □

Theorem 2.11. *Let ω be a d_J-closed p-form, with $p \geq 1$. There is a $(p-1)$-form π and a p-form ω' such that*

1. $\omega = d_J \pi + \omega'$
2. $J^* \omega' = \mathfrak{p}^* \omega''$ *where* $\omega'' \in \phi^p(M)$.

The cohomology space $H^p_{d_J}(TM)$ is therefore isomorphic to $\phi^p(M)$.

Proof of Theorem 2.9. This is a consequence of Proposition 2.11 and Theorem 2.9.

2.8 Poincaré's lemma for $d_L\omega = F(z,\omega)$

In this section, we provide in d_L-derivations a version of Jacobowitz's theorem on the existence of local solutions for non-linear problems having the following types: $d\omega = F(\omega)$ (cf. [10]). Let (M, L) be an L-foliated manifold, ϕ_L^p the set of L-semi-basic p-forms on M with $p \geq 0$, F a differentiable application of $\phi_L^p \longrightarrow \phi_L^{p+1}$, $G_L^p(z)$ the set of germs of L-semi-basic p-forms at the point z, M' a submanifold of M. For all forms α and $\beta \in \phi_L^p$, we denote by $\alpha_{|_{M'}} = \beta_{|_{M'}}$ if $i^*\alpha = i^*\beta$, where i means the canonical injection of M' in M.

Definition 2.9. *We say that F satisfies the condition of compatibility $(\star)$ at point z, if $d_L F(\omega) = 0$ at z for all $\omega \in G_L^p(z)$ such that $(d_L\omega - F(\omega))(z) = 0$.*

Theorem 2.12. *We suppose that F verifies $(\star)$ for all point z of M. Let M' be a submanifold of M passing through a point z_0. If $\omega_0 \in G_L^p(z_0)$ satisfies $d_L\omega_{0|_{M'}} = F(\omega_0)_{|_{M'}}$, then there is $\omega \in G_L^p(z)$ such that $d_L\omega = F(\omega)$, and $\omega = \omega_0$ on M' ($\omega(z) = \omega_0(z)$ for all $z \in M'$).*

Corollary 2.3. *If $\omega_0 \in \phi_L^p(z_0)$ and F verifies $(\star)$ at neighborhood of z_0, so there is $\omega \in G_L^p(z)$ such that $d_L\omega = F(\omega)$ and $\omega(z_0) = \omega_0$.*

The Corollary 2.3 is inferred from Theorem 2.12 by posing $M' = \{z_0\}$.

Particular cases 2.1.

1. *If F does not depend on ω, that is, if $F(\omega) = \beta$, where β is an L-semi-basic $(p+1)$-form, the condition $(\star)$ becomes $d_L\beta = 0$. Thus by adding an initial condition, we get the Theorem 2.1.*

2. *For $p = 0$, we obtain a version of Fröbenius' theorem.*

Proof of Theorem 2.10. Let U be a compatible chart of the L-foliated manifold M, z a point of U. We have $z = (x, y)$, where y means the coordinates of z on the leaves. An element β of $G_L^p(z)$ is written: $\beta \underset{U}{=} \beta_I(x,y)\, d_L y^I$, I being a multi-index $(i_1, \ldots, i_p)$ with $1 \leq i_1, \ldots, i_p \leq m$ and $|I| = p$. Similarly, if $F(z,\beta)$ is L-semi-basic, we have $F(z,\beta) \underset{U}{=} F_J(x,y,\beta)\, d_L y^J$, with $|J| = p+1$. Each F_J is a function of coefficients of the form β and the point z of M: $F_J\left(x^1, \ldots, x^n, y^1, \ldots, y^m, \beta_{K_1}, \ldots, \beta_{K_p}\right) \underset{U}{=} F_J\left(\beta_K d_L y^K\right)$, $|K| = p$. Since the derivation d_L works only on the variables $\left(y^i\right)$, $i = 1, \ldots, m$, that is, for all function $f \in \mathcal{F}(M)$, $d_L f(x,y) = \frac{\partial f}{\partial y^i}(x,y)\, d_L y^i$, the components (x^a), $a = 1, \ldots, n$ play the role of parameters. Thus, our study can be reduced to the results of Jacobowitz's work with parameters, see [10].

Proposition 2.14. *Let F be a differentiable application of $\phi^p(M)$ in $\phi^{p+1}(M)$, with $p \geq 0$. If L confirms one of the following two conditions:*

1. *L is regular on the leaves;*

2. *the image of L equals to the kernel of L;*

then we have Poincaré's lemma for $d_L\omega = F(\omega)$, if F verifies the condition of compatibility $(\star)$ and $F(\beta)\left(X^0, \ldots, X^p\right) = 0$ for all p-form β and for all vector fields $X^0, \ldots, X^p$ in the kernel of L.

Proof. We have seen during the demonstration of Theorem 2.9 that for such L, there are functions $f^1, \ldots, f^n$ such that df^a, $a = 1, \ldots, n$ and $d_L y^i$, $i = 1, \ldots, m$ form a local basis of $\phi^1(M)$ and that $d_L(df^a) = 0$. It is possible to write $\beta = \beta_{I,A} d_L y^I \wedge df^A$ for all p-form β , where I, A designate multi-indexes $I \subset [1, m]$, $A \subset [1, n]$ such that $|I| + |A| = p$. Similarly, we have $F(z, \beta) = F_{J,B}(z, \beta)\, d_L y^J \wedge df^B$ with $|J| + |B| = p+1$. The condition $F(\beta)\left(X^0, \ldots, X^p\right) = 0$ for any p-form β and for all vector fields $X^0, \ldots, X^p$ in the kernel of L means that the length of J, $|J| \geq 1$. Since $d_L(df^a) = 0$, $a = 1, \ldots, n$, the relation $d_L F(z, \beta) = 0$ is equivalent to $d_L\left(F_{J,B}(z, \beta)\, d_L y^J\right) = 0$.

Similarly, the relation

$$(d_L\beta - F(\beta)) = 0$$

implies $\left(d_L\left(\beta_{I,A} d_L y^I\right) - F_{J,A} d_L y^J\right)(z) = 0$. Thus, the condition of compatibility $(\star)$ for F is translated by the condition $(\star)$ for L-semi-basic forms. Hence the result, by taking Theorem 2.12 into account. □

Remark 2.5. If L confirms the conditions of Proposition 2.11, it is obvious from the demonstration of Proposition 2.14, to have a similar result.

Particular cases 2.2. Let $F : \phi^p(M) \longrightarrow \phi^{p+1}(M)$, G_z^p be the set of germes of p-forms at point z.

1. F satisfies to the condition of compatibility at point z for a projector v, if $d_v F(\omega) = 0$ at point z for all $\omega \in G_z^p$ such that $(d_v\omega - F(\omega))(z) = 0$ and if $(I - v)^* F(\omega) = 0$ for all $\omega \in G_z^p$.

2. If L and its image space equals to the kernel of L then the latter condition becomes $L^* F(\omega) = 0$ for all $\omega \in G_z^p$.

3. If F does not depend on ω, we obtain Theorem 2.9 by adding an initial condition.

The Proposition 2.14, according to Remark 2.5 generates an extension of Jacobowitz's theorem for the classical derivations having the types d' and d''.

3. Chevalley-Eilenberg cohomologies defined by vector valued 1-form

3.1 Generalities

Definition 3.1. *Let $\mathfrak{A}$ be a Lie $\mathbb{R}$-algebra. A p-cochain C of $\mathfrak{A}$ is an alternated p-linear application of $\mathfrak{A}^p$ in $\mathfrak{A}$ such that, if $p = 0$, C is being itself identified in an element of $\mathfrak{A}$.*

The cobord operator ∂ is defined by

$$\partial C\left(X^0,\ldots,X^p\right)=\sum_{0\leq i\leq p}(-1)^i\left[X^i,C\left(X^0,\ldots,\widehat{X^i},\ldots,X^p\right)\right]$$
$$+\sum_{0\leq i<j\leq p}(-1)^{i+j}\,C\left(\left[X^i,X^j\right],X^0,\ldots,\widehat{X^i},\ldots,\widehat{X^j},\ldots,X^p\right),$$

where $X^i\in\mathfrak{A}$, $i=0,1,\ldots,p$; and the symbol $\widehat{}$ designates the omission.

It implies that $\partial^2=0$. We will denote by $\mathcal{H}^p(\mathfrak{A})$ the corresponding cohomology spaces. It is called Chevalley-Eilenberg's cohomology. If $p=0$, $\partial C=-ad_X$, where ad_X is the adjoint linear application $Y\longmapsto[X,Y]$. If $p=1$, then $\partial C[X,Y]=[CX,Y]+[X,CY]-C[X,Y]$ for all $X,Y\in\mathfrak{A}$. Thus, $\partial C=0$ means that C is a derivation of Lie algebra $\mathfrak{A}$ and $\mathcal{H}^1(\mathfrak{A})=\{0\}$ is equivalent to saying that any derivation of $\mathfrak{A}$ is inner.

3.2 Lie algebra defined by a vector valued 1-form

Let M be a connected, paracompact, differentiable manifold with $m+n$ dimension; L a vector valued 1-form of rank $m\geq 1$; θ_X Lie derivation with respect to the vector field X on M. The relation $\theta_X L=0$ is also written like this: $[X,L]=0$, in other words, for all $Y\in\chi(M)$, $[X,LY]=L[X,Y]$.

We denote by $\mathfrak{A}_L$ the set of vector fields X such that $\theta_X L=0$. According to Jacobi's identity, $\mathfrak{A}_L$ is a real Lie algebra. The study of such algebra is interesting (cf. [11]).

We suppose in the following sequence that Nijenhuis torsion of L is zero. We saw in the section 2.1 that the nullity of Nijenhuis torsion of L defines a foliation with m dimension. A. Lichnerowicz considered in [15] $\mathcal{L}$ Lie algebra of tangent vector fields to a foliation, $\mathcal{L}_{\mathcal{F}}$ Lie algebra of the infinitesimal automorphisms of foliation, that is, $X\in\mathcal{L}_{\mathcal{F}}$ if and only if $\mathcal{L}\supset[X,\mathcal{L}]$. He proved in particular that whatever the considered foliation is, $\mathcal{H}^1(\mathcal{L})\approx\mathcal{L}_{\mathcal{F}}/\mathcal{L}$, and $\mathcal{H}^1(\mathcal{L}_{\mathcal{F}})=\{0\}$. $\mathfrak{A}_L$ is a subalgebra of $\mathcal{L}_{\mathcal{F}}$ of the L-foliated variety. If L is the vector valued 1-form of the fibrous transverse to a foliation ($rang\,(L)=n$ et $L^2=0$), J. Lehmann-Lejeune [14] proved that the derivations of $\mathfrak{A}_L$ are adjoint linear applications of the normalizer of $\mathfrak{A}_L$ in $\chi(M)$. In this section, we will give some properties of $\mathfrak{A}_L$ that will be used later.

Proposition 3.1. *If $L\mathfrak{A}_L$ denotes the space image of $\mathfrak{A}_L$ by L, $L\mathfrak{A}_L$ is an ideal of $\mathfrak{A}_L$. In particular if $L^2=0$ then $L\mathfrak{A}_L$ is a commutative ideal.*

Proof. From the definition of $\mathfrak{A}_L$, we have immediately $[X,L\mathfrak{A}_L]\subset L\mathfrak{A}_L$ for all $X\in\mathfrak{A}_L$. The nullity of Nijenhuis torsion of L, for all X, $Y\in\chi(M)$ can be written as follows: $[LX,LY]=L[LX,Y]+L[X,LY]-L^2[X,Y]$. If $X\in\mathfrak{A}_L$, the relation becomes for all $Y\in\chi(M)$: $[LX,LY]=L[LX,Y]$, that is, according to the definition of $\mathfrak{A}_L$, we have $L\mathfrak{A}_L\subset\mathfrak{A}_L$. Similarly, if $Y\in\mathfrak{A}_L$, we have $[LX,LY]=L^2[X,Y]$, which proves that $[L\mathfrak{A}_L,L\mathfrak{A}_L]\subset L\mathfrak{A}_L$ and that, if $L^2=0$, we get $[LX,LY]=0$. □

Proposition 3.2. *$\mathfrak{A}_L$ makes the kernel of L to remain stable.*

Proof. From the definition of $\mathfrak{A}_L$, we have $[X, LY] = L[X,Y]$, for all $X \in \mathfrak{A}_L$ and $Y \in \chi(M)$. If Y is an element of the kernel of L then we get $L[X,Y] = 0$. □

Proposition 3.3. *L is an endomorphism of modules*

- *from the centralizer of $\mathfrak{A}_L$ in $\chi(M)$,*
- *the normalizer of $\mathfrak{A}_L$ in $\chi(M)$,*
- *the algebra of derivations of $\mathfrak{A}_L$.*

Proof. If Y is a vector field commuting with $\mathfrak{A}_L$, as indicated in the definition of $\mathfrak{A}_L$, we have successively $[X,Y] = 0 = L[Y,X] = [LY,X]$ for all $X \in \mathfrak{A}_L$; this proves that LY commutes with $\mathfrak{A}_L$.

If Y is an element of the normalizer of $\mathfrak{A}_L$, for all $X \in \mathfrak{A}_L$, we have $[X,Y] \in \mathfrak{A}_L$; that is, LY is also an element of the normalizer of $\mathfrak{A}_L$.

Let D be a derivation of $\mathfrak{A}_L$. For all $X, Y \in \mathfrak{A}_L$ we get $D[X,Y] = [DX,Y] + [X,DY]$; according to the definition of $\mathfrak{A}_L$ elements, we can write $LD[X,Y] = L[DX,Y] + L[X,DY] = [LDX,Y] + [X,LDY]$; therefore, $L \circ D$ is also a derivation of $\mathfrak{A}_L$. □

3.3 Lie algebras associated with complete recovery of vector fields

Let M be a connected, paracompact, differentiable manifold with n dimension; $\chi(M)$ the module of vector fields on M; $\overline{\chi(M)}$ the complete recovery of $\chi(M)$ on the tangent bundle TM; C the canonical field generated by the group of homotheties with a parameter on TM. Uppon a domain U of a chart of M, we can write, if $X = X^i \frac{\partial}{\partial x^i}$ then $\overline{X} = X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial y^i}$ and $C = y^i \frac{\partial}{\partial x^i}$, with $i, j = 1, \ldots, n$.

Proposition 3.4. *The centralizer of $\overline{\chi(M)}$ in $\chi(TM)$ is generated by C.*

Proof. The result is easily obtained by writing in local coordinates the relation

$$\left[Z, \overline{X}\right] = 0, \text{ avec } Z \in \chi(TM) \text{ et } \forall\, \overline{X} \in \overline{\chi(M)}. \quad \square$$

Proposition 3.5. *The normalizer of $\overline{\chi(M)}$ in $\chi(TM)$ is $\overline{\chi(M)} + \mathbb{R}C$.*

Proof. We consider $Z \in \chi(TM)$ such that for all $\overline{X} \in \overline{\chi(M)}$, we have $[Z, \overline{X}] \in \overline{\chi(M)}$. Since the elements of $\overline{\chi(M)}$ are homogeneous of degree 0, according to the identity of Jacobi, we get $[[C,Z], \overline{X}] = 0$ for all $\overline{X} \in \overline{\chi(M)}$. In conformity with Proposition 3.4, we obtain $[C,Z] = KC$ with $K \in \mathbb{R}$. Furthermore, Z is an element of normalizer of $\overline{\chi(M)}$, this is only possible if $Z \in \overline{\chi(M)} + \mathbb{R}C$. □

Theorem 3.1. *All derivations of $\overline{\chi(M)} + \mathbb{R}C$ are inner or having the D form such that D becomes null on $\overline{\chi(M)}$ and $D(C) = KC$, with $K \in \mathbb{R}$. The cohomology space of $\mathcal{H}^1\left(\overline{\chi(M)} + \mathbb{R}C\right)$ is therefore of dimension 1.*

Proof of Theorem 3.1. According to a result of [21], the derivative ideal of $\chi(M)$ equals to $\chi(M)$, $\overline{\chi(M)}$ and $\chi(M)$ being isomorphic, we get

$$\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}.$$

It implies that $\left[\overline{\chi(M)} + \mathbb{R}C, \overline{\chi(M)} + \mathbb{R}C\right] = \overline{\chi(M)}$. In other words, $\overline{\chi(M)}$ is a characteristic ideal of $\overline{\chi(M)} + \mathbb{R}C$. All derivation D de $\overline{\chi(M)} + \mathbb{R}C$ is therefore a derivation of $\overline{\chi(M)}$. Since all derivation of $\overline{\chi(M)}$ is inner, all derivation D not identically zero on $\overline{\chi(M)}$ has the form $D = ad_{\overline{Z}}$ with $\overline{Z} \in \overline{\chi(M)}$. If D is zero on $\overline{\chi(M)}$ then $D(C) = KC$ with $K \in \mathbb{R}$; because $\mathbb{R}C$ is the center of $\overline{\chi(M)} + \mathbb{R}C$ according to Proposition 3.4. Hence the result.

Remark 3.1. It is easy to find that the derivation D, which becomes zero on $\overline{\chi(M)}$ and such that $D(C) = KC$, is not an adjoint linear application of the vector fields on TM.

3.4 Case of a tangent structure

Let J be the vector valued 1-form defining on the manifold M the natural tangent structure. So, the rank of J is n which is the dimension of M, the square of J is zero so is its Nijenhuis torsion. The image space of J is the vertical space of TM. Lie algebra defined by J, according to [3] is $\mathfrak{A}_J = \overline{\chi(M)} + J\overline{\chi(M)}$.

Proposition 3.6. *The centralizer of $\mathfrak{A}_J$ in $\chi(TM)$ is zero.*

Proof. If an element Z of $\chi(TM)$ commutes with $\mathfrak{A}_J$, we have $\left[Z, \overline{\chi(M)}\right] = 0$ and $\left[Z, J\overline{\chi(M)}\right] = 0$. According to Proposition 3.4, $\left[Z, \overline{\chi(M)}\right] = 0$ if and only if $Z = KC$ for all $K \in \mathbb{R}$. Since $[C, J] = -J$, we get $\left[KC, J\overline{\chi(M)}\right] = 0$ which is equivalent to $K = 0$. □

Proposition 3.7. *The normalizer of $\mathfrak{A}_J$ in $\chi(TM)$ is $\mathfrak{A}_J + \mathbb{R}C$.*

Proof. The normalizer of $\mathfrak{A}_J$ can be calculated on the coordinates system and we find the result. □

Proposition 3.8. *$J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J$.*

Proof. We know that $J\mathfrak{A}_J$ is an ideal of $\mathfrak{A}_J$ as stated in Proposition 3.1. What is left to be proved is that, if D is meant to be any derivation of $\mathfrak{A}_J$, D preserves

$J\mathfrak{A}_J$. In other words, if $X \in \mathfrak{A}_J$, we must have $DJX \in J\mathfrak{A}_J$. In conformity with the property of J, namely $ImJ = KerJ$, we can say

$$JDJX = 0. \tag{4}$$

We assume that $X, Y \in \mathfrak{A}_J$. From definition of $\mathfrak{A}_J$ we have

$$J[X, Y] = [JX, Y] = [X, JY]. \tag{5}$$

According to Proposition 3.3, JD is a derivation of $\mathfrak{A}_J$; by applying JD to the relation (5) and taking into account that $J\mathfrak{A}_J$ is commutative, we get

$$JDJ[X, Y] = [JDJX, Y] = [X, JDJY]. \tag{6}$$

Thus we can write $2JDJ[X, Y] = [JDJX, Y] + [X, JDJY]$. In conformity with property (5) of $\mathfrak{A}_J$, the previous expression is written like this: $2JDJ[X, Y] = [DJX, JY] + [JX, DJY]$. Since D is a derivation of $\mathfrak{A}_J$, we have $2JDJ[X, Y] = D[JX, JY] = 0$. The last equality is due to the commutativity of $J\mathfrak{A}_J$. Consequently, as stated in relation (6), we find $[JDJX, Y] = 0$ for all X and Y of $\mathfrak{A}_J$. This means that $JDJX$ is an element belonging to the center of $\mathfrak{A}_J$. However, according to the Proposition 3.6, the center of $\mathfrak{A}_J$ is reduced to 0. Hence $JDJX = 0$. for all $X \in \mathfrak{A}_J$. In conformity with relation (4), it's the result we've been searching for. □

Proposition 3.9. *$J\mathfrak{A}_J$ is characteristic ideal of $J\mathfrak{A}_J + \mathbb{R}C$; $J\mathfrak{A}_J$, $J\mathfrak{A}_J + \mathbb{R}C$ and $\mathfrak{A}_J$ are characteristic ideals of $\mathfrak{A}_J + \mathbb{R}C$.*

Proof. It is immediate to note that $J\mathfrak{A}_J$ is a caracteristic ideal of $J\mathfrak{A}_J + \mathbb{R}C$. Indeed, as we see in the following properties: $J\mathfrak{A}_J = J\overline{\chi(M)}$ and $[C, J] = -J$, the derivative ideal of $J\mathfrak{A}_J + \mathbb{R}C$ is given by $[J\mathfrak{A}_J + \mathbb{R}C, J\mathfrak{A}_J + \mathbb{R}C] = J\mathfrak{A}_J$. similarly, $\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$; because $\mathfrak{A}_J = \overline{\chi(M)} + J\overline{\chi(M)}$ and $\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}$; we get $[\mathfrak{A}_J + \mathbb{R}C, \mathfrak{A}_J + \mathbb{R}C] = \mathfrak{A}_J$. $J\mathfrak{A}_J$ being a characteristic ideal of $\mathfrak{A}_J$ according to Proposition 3.8, and $\mathfrak{A}_J$ is a charactéristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ in coonformity with the result above; so $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$. We then just need to demonstrate that $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$. Let D be any derivation of $\mathfrak{A}_J + \mathbb{R}C$. Since $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, we just need to prove that the DC vector field is vertical. For all $\overline{X} \in \overline{\chi(M)}$, we have $\left[DC, \overline{X}\right] = D\left[C, \overline{X}\right] - \left[C, D\overline{X}\right]$, as we see $\left[C, \overline{X}\right] = 0$ and $\left[C, D\overline{X}\right]$ vertical, because $D\overline{X} \in \mathfrak{A}_J$, the expression $\left[DC, \overline{X}\right]$ is vertical. This implies that DC is a vertical field of $\mathfrak{A}_J + \mathbb{R}C$. □

Lemma 3.1. *Let φ be an $\mathbb{R}$-linear application $\mathbb{R}$ of $\mathfrak{A}_J$ in $J\mathfrak{A}_J$ such that $\varphi^2 = 0$ and $\varphi\left[\overline{X}, \overline{Y}\right] = \left[\varphi\overline{X}, \overline{Y}\right]$ for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$. So, $\varphi = KJ$ with $K \in \mathbb{R}$.*

The proof is obtained by a simple calculation in local coordinates.

Proposition 3.10. *Let D be a derivation of $\mathfrak{A}_J + \mathbb{R}C$ such that $D : \mathfrak{A}_J + \mathbb{R}C \longrightarrow J\mathfrak{A}_J + \mathbb{R}C$ so there is $\overline{X} \in \mathfrak{A}_J$ such that $D = ad_{J\overline{X} + KC}$ with $K \in \mathbb{R}$.*

Proof. $J\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, according to Proposition 3.9, for all $\overline{X} \in \overline{\chi(M)}$ we have $DJ\overline{X} \in J\mathfrak{A}_J$ because $J\mathfrak{A}_J = J\overline{\chi(M)}$. From relation $[C, J] = -J$, we also get $\left[C, J\overline{X}\right] = -J\overline{X}$. By applying the derivation D to the latter relation, we have $-DJ\overline{X} = \left[DC, J\overline{X}\right] + \left[C, DJ\overline{X}\right] = \left[DC, J\overline{X}\right] - DJ\overline{X}$, because $DJ\overline{X} \in J\mathfrak{A}_J$ is homogeneous of degree -1. This implies, for all $\overline{X} \in \overline{\chi(M)}$, $\left[DC, J\overline{X}\right] = 0$. We note that DC commutes with $J\mathfrak{A}_J$ and, $DC \in J\mathfrak{A}_J + \mathbb{R}C$ since $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ according to Proposition 3.9. This causes $DC \in J\mathfrak{A}_J$. So, there is $\overline{X}_C \in \mathfrak{A}_J$ such that

$$DC = J\overline{X}_C. \tag{7}$$

By hypothesis, for all $\overline{X} \in \overline{\chi(M)}$ we have $D\overline{X} \in J\mathfrak{A}_J + \mathbb{R}C$; Yet, $\overline{X} \in \mathfrak{A}_J$ and $\mathfrak{A}_J$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$, in fact, we have $D\overline{X} \in J\mathfrak{A}_J$.

From relation $\left[DC, \overline{X}\right] = D\left[C, \overline{X}\right] - \left[C, D\overline{X}\right]$, we get $\left[DC, \overline{X}\right] = D\overline{X}$. That is, according to the relation (7) we find

$$\left[J\overline{X}_C, \overline{X}\right] = D\overline{X}. \tag{8}$$

Now, we need to specify the value of D on $J\overline{\chi(M)}$.

We have $D\left[J\overline{X}, \overline{Y}\right] = DJ\left[\overline{X}, \overline{Y}\right] = \left[DJ\overline{X}, \overline{Y}\right]$ for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$, since by definition $J\left[\overline{X}, \overline{Y}\right] = \left[J\overline{X}, \overline{Y}\right]$ and by hypothesis $\left[J\overline{X}, D\overline{Y}\right] = 0$. Thus, DJ is an $\mathbb{R}$-linear application verifying the conditions of the Lemme, we get for all $\overline{X} \in \overline{\chi(M)}$,

$$DJ\overline{X} = -KJ\overline{X}, \tag{9}$$

with $K \in \mathbb{R}$. Therefore, according to relations (7), (8) and (9), we find

$$D\left(\overline{X} + J\overline{Y} + K'C\right) = ad_{J\overline{X}_C + KC}\left(\overline{X} + J\overline{Y} + K'C\right),$$

for all $\overline{X}, \overline{Y} \in \overline{\chi(M)}$ and $K' \in \mathbb{R}$.

Hence, $D = ad_{J\overline{X}_C + KC}$. □

Proposition 3.11. *Let $\mathfrak{A}$ be a Lie algebra, I a characteristic ideal of $\mathfrak{A}$, D a derivation on $\mathfrak{A}$, π the linear application: $\mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/I$.*

By assuming $\overset{\circ}{D}\pi(X) = \pi(DX)$ for all $X \in \mathfrak{A}$, $\overset{\circ}{D}$ defines a derivation on $\mathfrak{A}/I$. In particular, if $D = ad_X$ then $\overset{\circ}{D} = ad_{\pi(X)}$.

Proof. The proof is classic. □

Proposition 3.12. *Any derivation of the normalizer of $\mathfrak{A}_J$ in $\chi(TM)$ is inner.*

Proof. According to Proposition 3.9, $J\mathfrak{A}_J + \mathbb{R}C$ is a characteristic ideal of $\mathfrak{A}_J + \mathbb{R}C$ which is the normalizer of $\mathfrak{A}_J$; we can therefore consider the application π: $\mathfrak{A}_J + \mathbb{R}C \longrightarrow \frac{\mathfrak{A}_J + \mathbb{R}C}{J\mathfrak{A}_J + \mathbb{R}C}$ and π verifies the properties of the Proposition 3.11.

Since $\frac{\mathfrak{A}_J + \mathbb{R}C}{J\mathfrak{A}_J + \mathbb{R}C} \approx \overline{\chi(M)}$ and all derivation of $\overline{\chi(M)}$ is inner, then all derivation of quotient algebra has the form $ad_{\pi(X)}$ with $X \in \mathfrak{A}_J + \mathbb{R}C$. Let D be a derivation of $\mathfrak{A}_J+\mathbb{R}C$ such that $\overset{\circ}{D}=ad_{\pi(X)}$, if we pose $D'=D-ad_X$, by definition $\overset{\circ}{D'}=\overset{\circ}{D}-ad_{\pi(X)}=0$. In other words, $\pi\left(D'Y\right)=0$ for all $Y\in\mathfrak{A}_J+\mathbb{R}C$. So, we have $D'Y\in J\mathfrak{A}_J+\mathbb{R}C$ for all $Y\in\mathfrak{A}_J+\mathbb{R}C$. According to Proposition 3.10, there is $\overline{X'}\in\mathfrak{A}_J$ and $K\in\mathbb{R}$ such that $D'=ad_{J\overline{X'}+KC}$. Thus the expression of D is written as follows: $D=ad_{\overline{X}}+ad_{J\overline{X'}+KC}=ad_{\overline{X}+J\overline{X'}+KC}$. □

Proposition 3.13. *The derivation algebras of $\mathfrak{A}_J$ is isomorphic to that of its normalizer in $\chi(TM)$.*

Proof. $\mathfrak{A}_J$ being a characteristic ideal of $\mathfrak{A}_J+\mathbb{R}C$ according to Proposition 3.9, any derivation D' of $\mathfrak{A}_J+\mathbb{R}C$ is a derivation of $\mathfrak{A}_J$ by restriction of D' on $\mathfrak{A}_J$. Let D be any derivation of $\mathfrak{A}_J$, we will prove the extension of D on $\mathfrak{A}_J+\mathbb{R}C$. If D' is a derivation on $\mathfrak{A}_J+\mathbb{R}C$ such that the restriction of D' on $\mathfrak{A}_J$ equals to D: $D'_{|\mathfrak{A}_J}=D$; then $D'-D$ is a null derivation of $\mathfrak{A}_J$. Let D'' be a derivation of $\mathfrak{A}_J+\mathbb{R}C$ that becomes zero on $\mathfrak{A}_J$. So, for all $\overline{X}\in\overline{\chi(M)}$ we have $D''\left[C,\overline{X}\right]=\left[D''C,\overline{X}\right]+\left[C,D''\overline{X}\right]$; considering $\left[C,\overline{X}\right]=0$ and $D''\overline{X}=0$, we get $\left[D''C,\overline{X}\right]=0$ for all $\overline{X}\in\overline{\chi(M)}$. That is, $D"C$ commutes with $\overline{\chi(M)}$. In addition, we have

$$D"[C,JX]=[D"C,J\overline{X}]+[C,D"J\overline{X}].$$

Since $[C,J\overline{X}]=-J\overline{X}$ and $D"J\overline{X}=0$, we get $[D"C,J\overline{X}]=0$, for all $\overline{X}\in\chi(M)$. Yet, $\mathfrak{A}_J=\overline{\chi(M)}+J\overline{\chi(M)}$, we note that $D''C$ commutes with $\mathfrak{A}_J$ and, according to Proposition 3.6, we find $D''C=0$. By hypothesis, D'' becomes null on $\mathfrak{A}_J$, therefore D'' becomes zero on $\mathfrak{A}_J+\mathbb{R}C$. We have just proved the uniqueness of an extension of a derivation D of $\mathfrak{A}_J$ on $\mathfrak{A}_J+\mathbb{R}C$ and the extension of D must be zero on $\mathbb{R}C$. Reciprocally, it is immediate to note that by posing $D'=D$ on $\mathfrak{A}_J$ and $D'(C)=0$, D' is a derivation of $\mathfrak{A}_J+\mathbb{R}C$, because the centralizer of $\mathfrak{A}_J$ in $\mathfrak{A}_J+\mathbb{R}C$ is zero. □

Theorem 3.2 ([14])**.** *The derivations of $\mathfrak{A}_J$ correspond to the adjoint representation of $\mathfrak{A}_J+\mathbb{R}C$. The first cohomology space of $\mathfrak{A}_J$ is thus with 1 dimension: $dim\left(\mathcal{H}^1\left(\mathfrak{A}_J\right)\right)=1$.*

Proof of Theorem 3.2. This is an immediate consequence of Propositions 3.12 and 3.13.

Remark 3.2. According to a result seen by [3], Lie algebra $\mathfrak{A}_J$ is isomorphic to Lie algebra $\chi_D(M)$ of dual vector fields on M. In this section, we note that the derivation algebras $D\left(\chi_D(M)\right)$ of $\chi_D(M)$ is such that $\mathcal{H}^1\left(D\left(\chi_D(M)\right)\right)=0$ whereas $dim\left(\mathcal{H}^1\left(\chi_D(M)\right)\right)=1$. $\mathfrak{A}_J$ admits a commutative characteristic ideal $J\mathfrak{A}_J$. It is also $J\mathfrak{A}_J$ a characteristic ideal of its vertical normalizer which is $J\mathfrak{A}_J+\mathbb{R}C$.

3.5 Case of a connection with null curvature

In this section we develop another point of view of the case given in [16]. Let M be a connected, paracompact, differentiable manifold with n dimension; J the vector valued 1-form defining on M the natural tangent structure. A connection Γ in the sense of Grifone [8] can be defined by $J\Gamma = J$ and $\Gamma J = -J$, it is an almost produced structure ($\Gamma^2 = I$). By posing $h = \frac{1}{2}(I + \Gamma)$ and $v = \frac{1}{2}(I - \Gamma)$, h means the horizontal projector, v the vertical projector and $TTM = Im\,(v) + Im\,(h)$. The curvature of Γ is defined by $R = \frac{1}{2}[h, h]$. From the definitions of Γ, h and v, we can see that Lie algebra of $\mathfrak{A}_\Gamma$ is $\mathfrak{A}_\Gamma = \mathfrak{A}_h = \mathfrak{A}_v$.

In the following section, we will use the notation $\mathfrak{A}_h$.

Proposition 3.14. *The elements of $\mathfrak{A}_h$ are projectable vector fields.*

Proof. Let's consider $X \in \mathfrak{A}_h$. From definition of $\mathfrak{A}_h$, we have $[X, hY] = h[X, Y]$ for all $Y \in \chi(TM)$. Since h is semi-basic ($hJ = 0$), in particular, we have $h[X, JY] = 0$ for all $Y \in \chi(TM)$. In other words, $J[X, JY] = 0$. This means that X is projectable (see [11]). □

In the following, we assume that the curvature R is zero, that is, Nijenhuis torsion of h is zero.

Proposition 3.15. *The horizontal vector fields of $\mathfrak{A}_h$ form a characteristic ideal of $\mathfrak{A}_h$ isomorphic with $\chi(M)$. In particular, all derivation of $h\mathfrak{A}_h$ is inner.*

Proof. We consider the application $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_h$ where $\overline{\chi(M)}$ is the complete recovery of $\chi(M)$. Since the kernel of horizontal projector h is a vertical space, h is injective on $\overline{\chi(M)}$. Let's assume $\overline{X} \in \overline{\chi(M)}$, $h\overline{X}$ is a projectable vector fields, so we have $\left[h\overline{X}, h\right] JY = 0$ for all $Y \in \chi(TM)$. The nullity of the curvature ensures $\left[h\overline{X}, h\right] hY = 0$ for all $Y \in \chi(TM)$. Now, any vector field on TM is the direct sum of its horizontal part and its vertical part; we get $\left[h\overline{X}, h\right] = 0$ for all $\overline{X} \in \overline{\chi(M)}$. In other words, $h\left(\overline{\chi(M)}\right) = h\mathfrak{A}_h$. Hence the bijectivity of $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_h$. On the other hand, from the nullity of curvature and relation $h^2 = h$, we can write

$$h\left[\overline{X}, \overline{Y}\right] = h\left[h\overline{X}, \overline{Y}\right] + h\left[\overline{X}, h\overline{Y}\right] - \left[h\overline{X}, h\overline{Y}\right],$$

for all $\overline{X}$, $\overline{Y} \in \overline{\chi(M)}$. We have just seen that $h\overline{X}, h\overline{Y}$ are elements of $\mathfrak{A}_h$. So, we have $h\left[h\overline{X}, \overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$ and $h\left[\overline{X}, h\overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$. Thus, the nullity of the curvature is written as follows: $h\left[\overline{X}, \overline{Y}\right] = \left[h\overline{X}, h\overline{Y}\right]$. This proves that the application $h : \overline{\chi(M)} \longrightarrow h\mathfrak{A}_{ha}$ is an isomorphism of Lie algebras. In addition, $h\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_h$ according to Proposition 3.1. From the relation $\left[\overline{\chi(M)}, \overline{\chi(M)}\right] = \overline{\chi(M)}$ and by isomorphism, we have $[h\mathfrak{A}_h, h\mathfrak{A}_h] = h\mathfrak{A}_h$; that is, $h\mathfrak{A}_h$ is a characteristic ideal of $\mathfrak{A}_h$. □

Proposition 3.16. $\mathfrak{A}_h = h\mathfrak{A}_h \times v\mathfrak{A}_h$.

Proof. We notice that $v\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_h$, because we have $\mathfrak{A}_h = \mathfrak{A}_v$ and $[h,h] = [v,v] = 0$; according to Proposition 3.1, $v\mathfrak{A}_v$ which is $v\mathfrak{A}_h$ is an ideal of $\mathfrak{A}_v$ which is $\mathfrak{A}_h$ too. Let's consider $X \in \mathfrak{A}_h$. From the definition of $\mathfrak{A}_h$, we have $[X, hY] = h\,[X,Y]$ for all $Y \in \chi\,(TM)$. By applying h to the above relation, we get

$$h\,[X, hY] = h^2\,[X,Y]\,. \tag{10}$$

The nullity of the curvature is written like this: $[hX, hY] - h\,[hX,Y] + h^2\,[X,Y] - h\,[X,hY] = 0$; by considering relation (10), we find $[hX,hY] - h\,[hX,Y] = 0$ for all $Y \in \chi\,(TM)$. In other words, $hX \in \mathfrak{A}_h$. By means of an analogous reasoning for v, we obtain $vX \in \mathfrak{A}_h$. We have just proved that $\mathfrak{A}_h = h\mathfrak{A}_h \oplus v\mathfrak{A}_h$. What is to be proved next is that, for all $X,\ Y \in \mathfrak{A}_h$, we must have $[hX, vY] = 0$. by taking into account the definition of $\mathfrak{A}_h$, considering that $h^2 = h$ and $I - h = v$, we can write successively $[hX, vY] = [hX, Y] - [hX, hY] = h\,[X,Y] - h\,[X,Y] = 0$ for all $X,\ Y \in \mathfrak{A}_h$. □

Proposition 3.17. *$v\mathfrak{A}_h$ is the centralizer of $h\mathfrak{A}_h$ in $\chi\,(TM)$.*

Proof. Let Y be an element of the centralizer of $h\mathfrak{A}_h$. This implies

$$[hX, Y] = 0 \tag{11}$$

for all $X \in \mathfrak{A}_h$. By applying h to relation (11) and taking into account $hX \in \mathfrak{A}_h$, we get $[hX, hY] = 0$; which means that hY commutes with $h\mathfrak{A}_h$. According to Proposition 3.15, $h\mathfrak{A}_h$ is the set of horizontal and projectable fields; hY is therefore null. By applying v to relation (11), we also have

$$[hX, vY] = 0 \tag{12}$$

for all $X \in \mathfrak{A}_h$. We will demonstrate that $vY \in \mathfrak{A}_h$, that is,

$$[vY, h] = 0. \tag{13}$$

Note that equation (12) is always true on vertical fields. Therefore, in order to have the relation (13), we just need to verify onto the horizontal fields. To put it differently, we must have

$$v\,[vY, hZ] = 0 \tag{14}$$

for all $Z \in \chi\,(TM)$. However, according to Proposition 3.15, $h\mathfrak{A}_h$ generates locally the horizontal space; there is therefore $h_1, \ldots, h_n \in h\mathfrak{A}_h\,(U)$ such that $hZ \underset{U}{=} \sum_i f_i h_i$ where f_i are functions on TU. The relation (14) becomes $\sum_i f_i v\,[vY, h_i] + \sum_i (vY.f_i)\,v\,(h_i) = 0$. This is verified, in conformity with teh relation (12) for the first term, and $v \circ h = 0$ for the second term. Therefore, we have $vY \in \mathfrak{A}_h$. The reciprocal is obtained by Proposition 3.16. □

Theorem 3.3. *Let M be a connected, paracompact, differentiable manifold with n dimension; Γ a flat connection on M; $\chi(M)$ the module of vector fields on M. Lie algebra $\mathfrak{A}_\Gamma$ is isomorphic to $\chi(M) \times \chi(\mathbb{R}^n)$. The first cohomology space of Chevalley-Eilenberg $\mathcal{H}^1(\mathfrak{A}_\Gamma)$ of $\mathfrak{A}_\Gamma$ is therefore zero.*

Proof of Theorem 3.3. The Proposition 3.15 gives the isomorphism between $\chi(M)$ and $h\mathfrak{A}_h$. The ideal $v\mathfrak{A}_h$ is on vertical space and it is the centralizer of $h\mathfrak{A}_h$ according to Proposition 3.17, since $h\mathfrak{A}_h$ is the set of horizontal and projectable fields as stated in Proposition 3.15, $v\mathfrak{A}_h$ is therefore constituted by fields that come only from the fiber which is $\mathbb{R}^n$, according to Fröbenius' theorem since the horizontal space is completely integrable. In other words, $v\mathfrak{A}_h$ is isomorphic to $\chi(\mathbb{R}^n)$. Finally, Proposition 3.16 proves that $\mathfrak{A}_h = h\mathfrak{A}_h \times v\mathfrak{A}_h$. As for the cohomology, we apply a result of F. Takens' work on $\chi(M)$ and $\chi(\mathbb{R}^n)$.

Remark 3.3. If v is a projector on M (see section 2.6), an adaptation of the method used in this section on an adapted chart to a foliation makes it possible to have Lie algebra $\mathfrak{A}_v$ such that $\mathfrak{A}_v \approx \chi_h \times \chi_v$, where χ_h denotes the set of basic vector fields, that is, the vector fields on $Im(h)$, which only depend on $x^a,\ a = 1, \dots, n$, χ_v are the set of vector fields on $Im(v)$, which only depend on $y^i,\ i = 1, \dots, m$. All derivation of $\mathfrak{A}_v$ is therefore interior.

Example 3.1 (**Example of a vector valued 1-form admitting non-adjoint derivations of vector fields**). Let $M = \mathbb{R}^3$ be a manifold. We designate by (x, y, z) the coordinates system of $\mathbb{R}^3$ and by $\left(\dot{x}, \dot{y}, \dot{z}\right)$ that of the fiber of the tangent bundle $T\mathbb{R}^3$. The vector valued 1-form $L = dx \otimes C$ is defined on $T\mathbb{R}^3$, where C means the canonical field such that $C = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}$. It is easy to note that $L^2 = 0$ and $[L, L] = 0$; Nijenhuis torsion of L is therefore zero. So, Lie algebra $\mathfrak{A}_L$ is given by the following relation: $T \in \mathfrak{A}_L \Longleftrightarrow [T, dx \otimes C] = 0$. By assuming the coordinates of T by $\left(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}\right) \in T\mathbb{R}^3$, we find $X(x),\ Y(x, y, z),\ Z(x, y, z)$ and

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial x}(x) & \alpha_2^1(x,y,z) & \alpha_3^1(x,y,z) \\ \alpha_1^2(x,y,z) & \frac{\partial X}{\partial x}(x) & \alpha_3^2(x,y,z) \\ \alpha_1^3(x,y,z) & \alpha_2^3(x,y,z) & \frac{\partial X}{\partial x}(x) \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$X(x)$) is only just a function of x and $Y, Z,\ \alpha_j^i$ with $i \neq j$ are arbitrary functions of x, y, z. We see that by denoting the canonical divergence of $\mathbb{R}^3$ by φ, $D = (dx + \varphi) \otimes C$ is a non-adjoint derivation of a vector fields of $\mathfrak{A}_L$ on $T\mathbb{R}^3$.

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