Transmuted shanker distribution: properties and applications

Loai Alzoubi*
Department of Mathematics
Al Al-Bayt University
Mafraq-25113
Jordan
loai67@aabu.edu.jo

Mohammed M. Gharaibeh
Department of Mathematics
Al Al-Bayt University
Mafraq-25113
Jordan
mmgharaibeh@gmail.com

Raed Alzghool
Al-Balqa Applied University
Department of Mathematics
Salt-19117
Jordan
r_zghool@hotmail.com

Abstract. In this paper a new continuous distribution is proposed. This distribution will generalize Shanker distribution to the transmuted Shanker distribution using the quadratic rank transmutation map. Some properties of this distribution are studied. A numerical study is conducted to calculate the mean, standard deviation, skewness, kurtosis and the coefficient of variation of this distribution. An application to a data set is conducted. It shows that the new distribution performs better than Shanker distributions and some other distributions, like Shanker itself and Lindley in performance of the values of $-2lnL, W, A, AIC, KS$-statistic and p-value.
Keywords: Shanker distribution, transmuted Shanker distribution, moments, skewness, kurtosis, entropy, order statistics, quantile function, median, MLE.

1. Introduction

Classical families of distributions might not be adequate for modeling many practical data. Therefore, generalizing the existing distributions by adding one or more parameters allows the resulted distributions to be more appropriate to fit real life data. [23] employed the quadratic rank transmutation map to generate a general and flexible family of distributions. This family is called transmuted family of distributions. We use the quadratic rank transmutation where $(f(x))$ and $(F(x))$ are the *pdf* and CDF of the base distribution map to

∗. Corresponding author

introduce a new distribution called transmuted Shanker distribution (TSHD). This proposed distribution is a generalization of Shanker distribution.

The CDF of the quadratic rank transmutation map is defined as:

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x), \qquad |\lambda| \leq 1. \tag{1}$$

The pdf is defined as:

$$g(x) = f(x)((1+\lambda) - 2\lambda F(x)), \qquad |\lambda| \leq 1. \tag{2}$$

Many authors used these maps to generalize some distributions. [3] proposed the transmuted Janardan distribution. [7] has worked out the transmuted Mukherjee-Islam distribution: a generalization of Mukherjee-Islam distribution. Transmuted Burr type XII distribution: a generalization of the Burr type XII distribution [15]. A generalization of the new Weibull-Pareto distribution [2]. [1] worked out the transmuted two-parameter Lindley distribution. [11] generalized two distributions through her master thesis, namely: the transmuted Gamma-Gompertz and transmuted Generalized Type-II Half-Logistics Distributions. [10] worked out the transmuted reciprocal and the transmuted two-parameter weighted exponential distributions. [18] used the quadratic transmutation map to generalize the power function and Type-I half logistic distributions. [6] generalized Sujatha and Amarendra distributions using quadratic transmutation map. [5] used the same map to generalize Rani and generalized Akash distributions. [19] used the quadratic map to propose transmuted Akash distribution and the cubic map to produce the cubic Mukherjee-Islam distribution. [13] proposed the transmuted Aradhana distribution. [14] generalized Ishita distribution to transmuted Ishita with applications.

The remaining of this paper is organized as follows: Section 2, defines pdf and cdf of the proposed distribution. In Section 3, we defined the reliability, hazard rate functions. While in Section 5, the maximum likelihood estimates of the TSHD parameters are derived. The quantile function and the median of the distribution are derived in Section 6. Rényi entropy is defined in Section 7. An application to real data set is presented in Section 8.

2. Transmuted Shanker distribution

[20] has suggested a one-parameter distribution for modeling real lifetime data sets, it is called Shanker distribution. This distribution is a mixture of exponential (θ) and gamma ($2,\theta$). [20] defined his distribution as:

Definition 2.1. *A random variable X is said to have Shanker distribution if its probability density function (pdf) and cumulative distribution function (CDF)*

are defined as [20]:

$$f(x) = \begin{cases} \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x}, & x > 0, \quad \theta > 0, \\ 0, & o.w. \end{cases} \tag{3}$$

$$F(x) = \begin{cases} 1 - \frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}, & x > 0, \quad \theta > 0, \\ 0, & o.w. \end{cases} \tag{4}$$

The transmuted Shanker distribution *pdf* and CDF are; respectively, defined as:

$$\begin{aligned} g(x) &= \Big(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\Big)\Big(1+\lambda-2\lambda\Big(1-\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big),\\ &= \Big(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\Big)\Big(1-\lambda+2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big)\\ &= \frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x}\\ &+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x}\; x,\theta>0, |\lambda|\le 1, \end{aligned} \tag{5}$$

$$\begin{aligned} G(x) &= (1+\lambda)\Big(1-\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big) - \lambda\Big(1-\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big)^2\\ &= 1-(1-\lambda)\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x} + \lambda\Big(\frac{\theta^2+1+\theta x}{\theta^2+1}\Big)^2 e^{-2\theta x}\; x,\theta>0, |\lambda|\le 1. \end{aligned} \tag{6}$$

3. Reliability and hazard rate functions

The reliability function (survival function) is defined as:

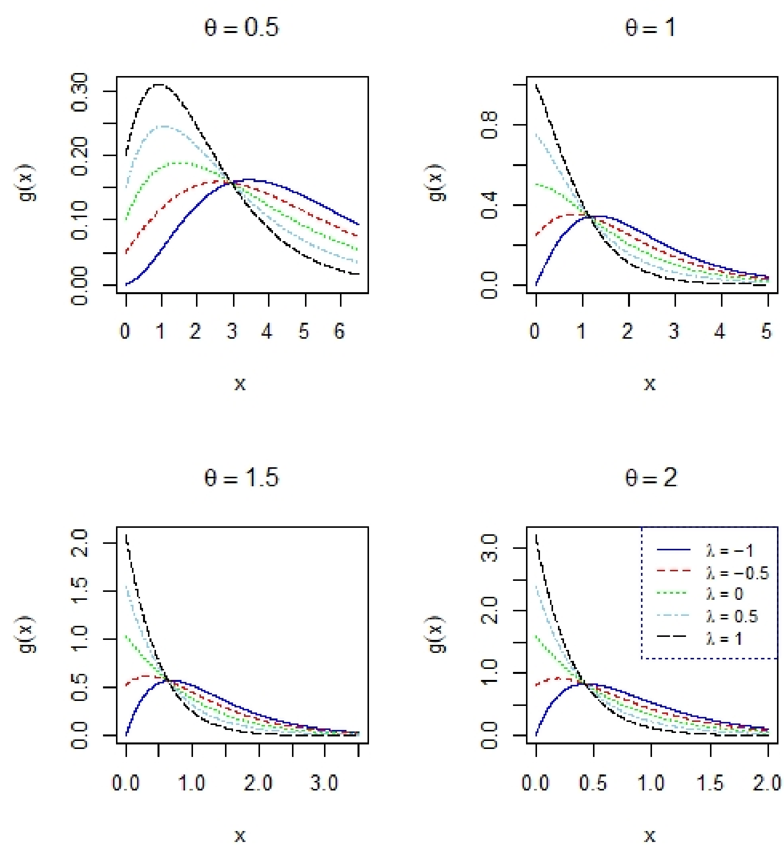
$$R(t) = p(T \geq t) = 1 - G(t).$$

Therefore, the reliability function for TSHD is given by:

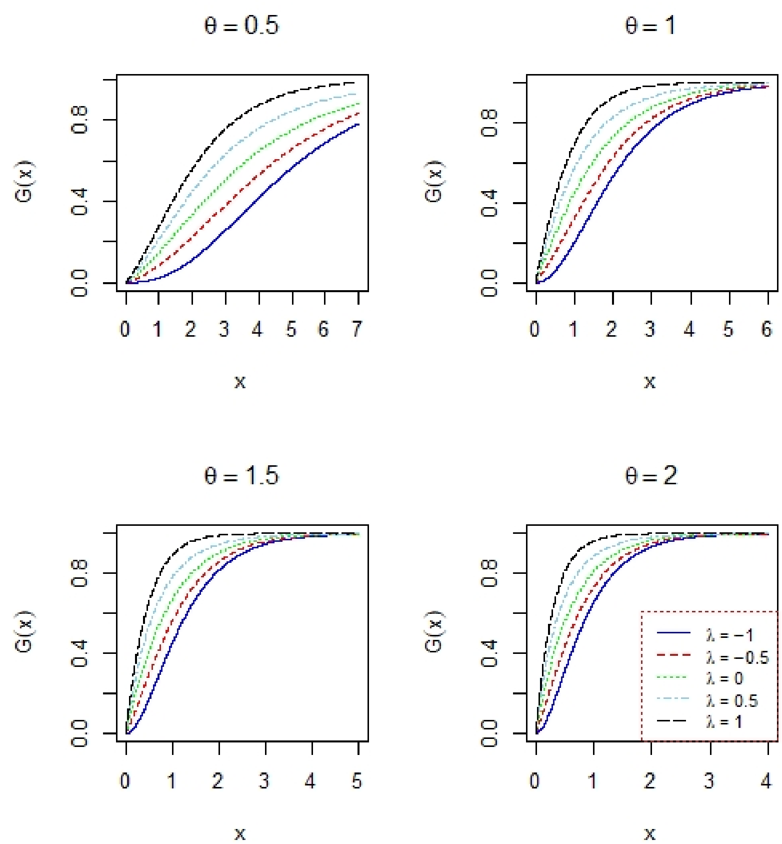
$$\begin{aligned} R(t) &= 1 - \Big[1-(1-\lambda)\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x} + \lambda\Big(\frac{\theta^2+1+\theta x}{\theta^2+1}\Big)^2 e^{-2\theta x}\Big]\\ &= (1-\lambda)\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x} - \lambda\Big(\frac{\theta^2+1+\theta x}{\theta^2+1}\Big)^2 e^{-2\theta x}\\ &= \frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big[1-\lambda-\lambda\Big(\frac{\theta^2+1+\theta x}{\theta^2+1}\Big)e^{-\theta x}\Big]. \end{aligned} \tag{7}$$

The hazard rate function is defined to be $h(t) = \frac{g(t)}{1-G(t)}$

$$
\begin{aligned}
h(t) &= \frac{\left(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\right)\left(1-\lambda+2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\right)}{\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\left[1-\lambda-\lambda\left(\frac{\theta^2+1+\theta x}{\theta^2+1}\right)e^{-\theta x}\right]} \\
&= \frac{\theta^2(\theta+x)\left(1-\lambda+2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\right)}{(\theta^2+1+\theta x)\left[1-\lambda-\lambda\left(\frac{\theta^2+1+\theta x}{\theta^2+1}\right)e^{-\theta x}\right]} \\
&= \frac{\theta^2(\theta+x)\Big((1-\lambda)(\theta^2+1)+2\lambda(\theta^2+1+\theta x)e^{-\theta x}\Big)}{(\theta^2+1+\theta x)\Big[(1-\lambda)(\theta^2+1)-\lambda(\theta^2+1+\theta x)e^{-\theta x}\Big]}. \qquad (8)
\end{aligned}
$$

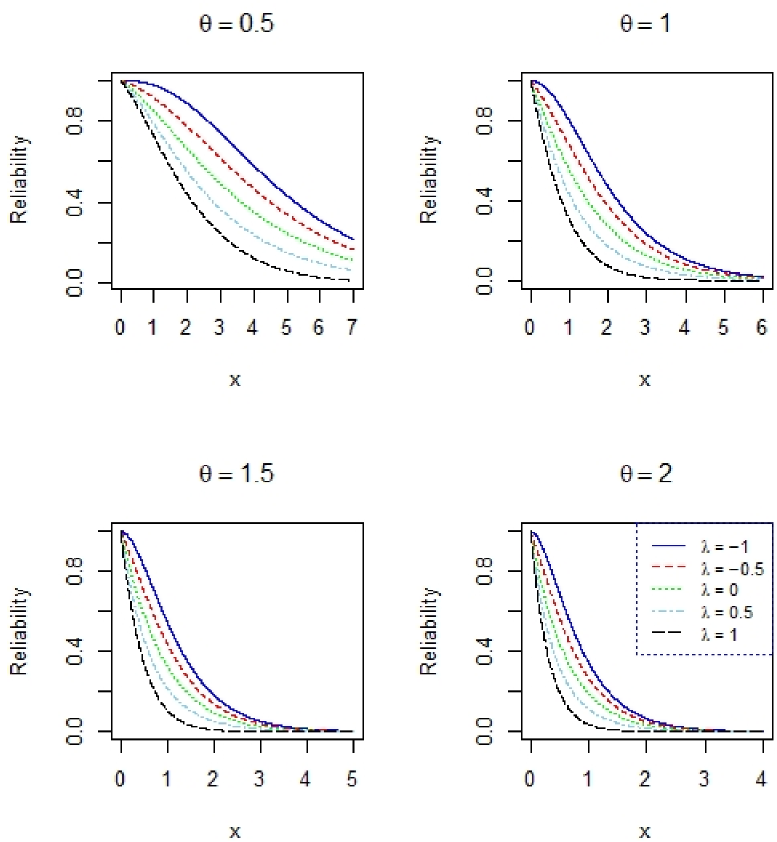


a) The pdf of the TSHD for different values of λ and θ.

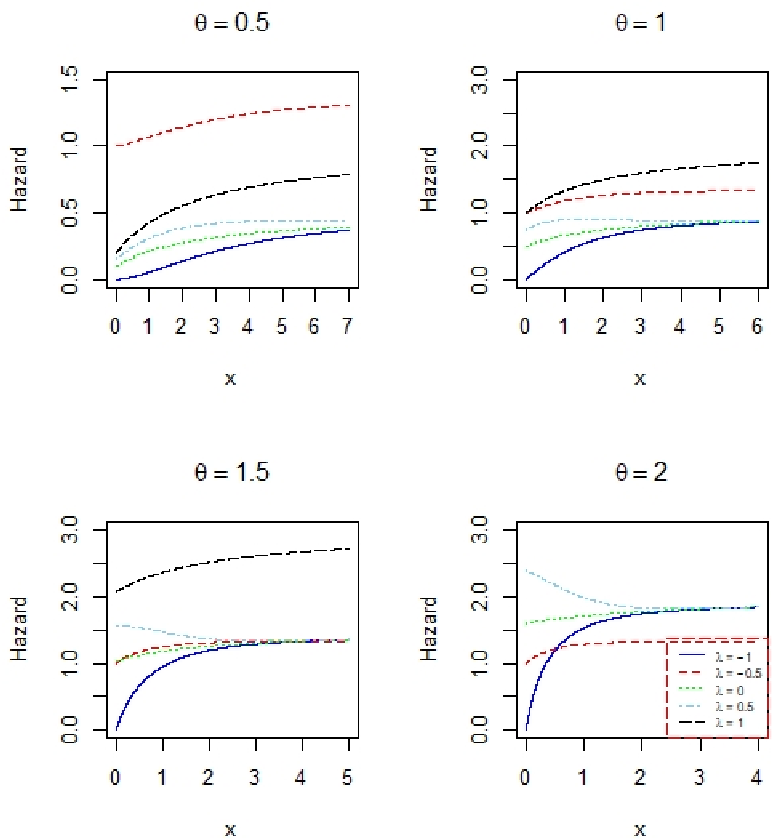


b) The cdf of the TSHD for different values of λ and θ.

Figures a and b show the plots of the *pdf* and CDF of TSHD for values of $\theta = 0.5, 1, 1.5$ and 2 and values of $\lambda = -1, -0.5, 0, 0.5$ and 1.



c) The reliability of the TSHD for different values of λ and θ.



d) The hazard rate function of the TSHD for different values of λ and θ.

Figures c and d show the plots of the reliability and hazard rate functions of TSHD for values of $\theta = 0.5, 1, 1.5$ and 2 and values of $\lambda = -1, -0.5, 0, 0.5$ and 1.

4. Properties of the TSHD

4.1 Moments

Theorem 4.1. *Let X be a TSHD random variable, then the r^{th} moment is defined to be*

$$E(X^r) = \frac{(1-\lambda)\Gamma(r+1)}{\theta^{r-3}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(r+2)}{\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+1)}{2^{r-1}\theta^{r-2}(\theta^2+1)} + \frac{\lambda(2\theta^2+1)\Gamma(r+2)}{2^r\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+3)}{2^{r+1}\theta^{r-1}(\theta^2+1)} \tag{9}$$

Proof.

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r g(x)dx, \\
&= \int_0^\infty x^r \left[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x}\right. \\
&\left. + \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x}\right]dx \\
&= \frac{(1-\lambda)\theta^3}{\theta^2+1}\int_0^\infty x^r e^{-\theta x}dx \\
&+ \frac{(1-\lambda)\theta^2}{\theta^2+1}\int_0^\infty x^{r+1}e^{-\theta x}dx + \frac{2\lambda\theta^3}{\theta^2+1}\int_0^\infty e^{-2\theta x}dx \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}\int_0^\infty x^{r+1}e^{-2\theta x}dx + \frac{2\lambda\theta^3}{\theta^2+1}\int_0^\infty x^{r+2}e^{-2\theta x}dx \\
&= \frac{(1-\lambda)\theta^3}{\theta^2+1}\frac{\Gamma(r+1)}{\theta^r} + \frac{(1-\lambda)\theta^2}{\theta^2+1}\frac{\Gamma(r+2)}{\theta^{r+1}} + \frac{2\lambda\theta^3}{\theta^2+1}\frac{\Gamma(r+1)}{(2\theta)^r} \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}\frac{\Gamma(r+2)}{(2\theta)^{r+1}} + \frac{2\lambda\theta^3}{\theta^2+1}\frac{\Gamma(r+3)}{(2\theta)^{r+2}} \\
&= \frac{(1-\lambda)\Gamma(r+1)}{\theta^{r-3}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(r+2)}{\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+1)}{2^{r-1}\theta^{r-2}(\theta^2+1)} \\
&+ \frac{\lambda(2\theta^2+1)\Gamma(r+2)}{2^r\theta^{r-1}(\theta^2+1)} + \frac{\lambda\Gamma(r+3)}{2^{r+1}\theta^{r-1}(\theta^2+1)}.
\end{aligned}$$

□

Therefore, the first four moments can be calculated by putting $r = 1, 2, 3$ and 4 in Equation (9). Thus

$$\begin{aligned}
E(X) &= \frac{(1-\lambda)\Gamma(2)}{\theta^{-2}(\theta^2+1)} + \frac{(1-\lambda)\Gamma(3)}{(\theta^2+1)} + \frac{\lambda\Gamma(2)}{\theta^{-1}(\theta^2+1)} \\
&+ \frac{\lambda(2\theta^2+1)\Gamma(3)}{2(\theta^2+1)} + \frac{\lambda\Gamma(4)}{2^2(\theta^2+1)} \\
&= \frac{1}{(\theta^2+1)}\left[(1-\lambda)\theta^2 + 2(1-\lambda) + \lambda\theta + \lambda(2\theta^2+1) + \frac{3}{2}\lambda\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\theta^2+1)}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right],\\
E(X^2) &= \frac{(1-\lambda)\Gamma(3)}{\theta^{-1}(\theta^2+1)}+\frac{(1-\lambda)\Gamma(4)}{\theta(\theta^2+1)}+\frac{\lambda\Gamma(3)}{2(\theta^2+1)}\\
&+\frac{\lambda(2\theta^2+1)\Gamma(4)}{2^2\theta(\theta^2+1)}+\frac{\lambda\Gamma(5)}{2^3\theta(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[2\theta(1-\lambda)+\frac{6(1-\lambda)}{\theta}+\lambda+\frac{3\lambda(2\theta^2+1)}{2\theta}+\frac{3\lambda}{\theta}\right]\\
&= \frac{1}{(\theta^2+1)}\left[2\theta(1-\lambda)+\lambda+\frac{6\lambda\theta^2-3\lambda+12}{2\theta}\right]\\
&= \frac{1}{(\theta^2+1)}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right],\\
E(X^3) &= \frac{(1-\lambda)\Gamma(4)}{(\theta^2+1)}+\frac{(1-\lambda)\Gamma(5)}{\theta^2(\theta^2+1)}+\frac{\lambda\Gamma(4)}{2^2\theta(\theta^2+1)}\\
&+\frac{\lambda(2\theta^2+1)\Gamma(5)}{2^3\theta^2(\theta^2+1)}+\frac{\lambda\Gamma(6)}{2^4\theta^2(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[6(1-\lambda)+\frac{3\lambda}{2\theta}+\frac{24(1-\lambda)}{\theta^2}+\frac{3\lambda(2\theta^2+1)}{\theta^2}+\frac{15\lambda}{2\theta^2}\right],\\
E(X^4) &= \frac{(1-\lambda)\Gamma(5)}{\theta(\theta^2+1)}+\frac{120(1-\lambda)}{\theta^3(\theta^2+1)}+\frac{24\lambda}{8\theta^2(\theta^2+1)}\\
&+\frac{120\lambda(2\theta^2+1)}{16\theta^3(\theta^2+1)}+\frac{720\lambda}{32\theta^3(\theta^2+1)}\\
&= \frac{1}{(\theta^2+1)}\left[\frac{24(1-\lambda)}{\theta}+\frac{24\lambda}{\theta^2}+\frac{15\lambda\theta^2-90\lambda+120}{\theta^3}\right]\\
&= \frac{1}{\theta^3(\theta^2+1)}\left[3\theta^2(8-3\lambda)+24\lambda\theta+30(4-3\lambda)\right].
\end{aligned}
$$

Therefore, the variance of TSHD random variable is defined to be:

$$
\begin{aligned}
\sigma^2 = Var(X) = E(X^2)-(E(X))^2\\
&= \frac{1}{(\theta^2+1)}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right]\\
&-\left[\frac{1}{(\theta^2+1)}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right]\right]^2
\end{aligned}
$$

$$= \frac{-1}{4\theta(\theta^2+1)^2}\Bigg[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3$$

$$+ 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\Bigg]. \tag{10}$$

The coefficient of variation (cv) is defined to be the ratio of standard deviation of the random variable to its expected value, that is $cv = \frac{\sqrt{var(X)}}{E(X)} = \frac{\sigma}{\mu}$. Therefore,

$$cv = \frac{\left[\frac{-1}{\theta}\Big[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3 + 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\Big]\right]^{\frac{1}{2}}}{\left[2(1+\lambda)\theta^2 + 2\lambda\theta + \lambda + 4\right]}.$$

Skewness is a measure of symmetry, or more precisely, the lack of symmetry. A distribution, or data set, is symmetric if it looks the same to the left and right of the center point. It is defined as

$$Sk(X) = \frac{E(X^3) - 3E(X^2)E(X) + 2(E(X))^3}{\sigma^3}.$$

Thus

$$sk(X) = \frac{\left[\begin{array}{l} \frac{1}{(\theta^2+1)}\left[6(1-\lambda) + \frac{3\lambda}{2\theta} + \frac{24(1-\lambda)}{\theta^2} + \frac{3\lambda(2\theta^2+1)}{\theta^2} + \frac{15\lambda}{2\theta^2}\right] \\ \frac{-3}{(\theta^2+1)^2}\left[(2+\lambda)\theta + \lambda + \frac{12-3\lambda}{2\theta}\right]\left[(1+\lambda)\theta^2 + \lambda\theta + \frac{\lambda}{2} + 2\right] \\ + \frac{2}{(\theta^2+1)^3}\left[(1+\lambda)\theta^2 + \lambda\theta + \frac{\lambda}{2} + 2\right]^3 \end{array}\right]}{\left[\begin{array}{l} \frac{-1}{4\theta(\theta^2+1)^2}\Big[4(\lambda+1)^2\theta^5 + 4(2\lambda^2+\lambda-2)\theta^4 + 8(\lambda^2+2\lambda+2)\theta^3 \\ + 2(2\lambda^2+9\lambda-16)\theta^2 + (\lambda^2+4\lambda+16)\theta + 8(2\lambda-3)\Big] \end{array}\right]^{3/2}}.$$

Kurtosis is a measure of whether the data are heavy-tailed or not. That is, data sets with high kurtosis tend to have heavy tails, or outliers. Data sets with low kurtosis tend to have light tails, or lack of outliers. Kurtosis is defined as

$$Kur(X) = \frac{E(X^4) - 4E(X^3)(E(X)) + 6(E(X))^2(Var(X)) + 3(E(X))^4}{(Var(X))^2},$$

$$Kur(X) = \frac{\left[\begin{aligned}&\frac{1}{\theta^3(\theta^2+1)}\left[3\theta^2(8-3\lambda)+24\lambda\theta+30(4-3\lambda)\right]\\ &-\frac{4}{(\theta^2+1)^2}\left[6(1-\lambda)+\frac{3\lambda}{2\theta}+\frac{24(1-\lambda)}{\theta^2}+\frac{3\lambda(2\theta^2+1)}{\theta^2}+\frac{15\lambda}{2\theta^2}\right]\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right]\\ &-\frac{3}{2\theta(\theta^2+1)^3}\left[(2+\lambda)\theta+\lambda+\frac{12-3\lambda}{2\theta}\right]\Big[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4\\ &+8(\lambda^2+2\lambda+2)\theta^3+2(2\lambda^2+9\lambda-16)\theta^2+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\Big]\\ &\times\Big[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4+8(\lambda^2+2\lambda+2)\theta^3+2(2\lambda^2+9\lambda-16)\theta^2\\ &+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\Big]+\frac{3}{(\theta^2+1)^4}\left[(1+\lambda)\theta^2+\lambda\theta+\frac{\lambda}{2}+2\right]^4\end{aligned}\right]}{\left[\begin{aligned}&\frac{-1}{4\theta(\theta^2+1)^2}\Big[4(\lambda+1)^2\theta^5+4(2\lambda^2+\lambda-2)\theta^4+8(\lambda^2+2\lambda+2)\theta^3\\ &+2(2\lambda^2+9\lambda-16)\theta^2+(\lambda^2+4\lambda+16)\theta+8(2\lambda-3)\Big]\end{aligned}\right]^2}$$

Table 1: The mean, standard deviation, skewness, kurtosis and the coefficient of variation of the TSHD for different values of λ when $\theta =$ 0.5, 1, 1.5 and 2

λ	θ	μ	σ_x	Sk	Kur	CV (%)	θ	μ	σ_x	Sk	Kur	CV (%)
-1.0	0.5	5.080	2.945	1.221	5.384	57.974	1.5	1.292	0.901	1.425	6.125	69.760
-0.5	0.5	4.340	2.967	1.243	5.380	68.369	1.5	1.082	0.888	1.513	6.401	82.062
0.0	0.5	3.600	2.800	1.452	6.121	77.778	1.5	0.872	0.822	1.779	7.593	94.301
0.5	0.5	2.860	2.404	1.768	7.887	84.065	1.5	0.662	0.689	2.201	10.351	104.186
1.0	0.5	2.120	1.620	1.338	5.559	76.433	1.5	0.452	0.432	1.807	7.711	95.617
-1.0	1.0	2.188	1.424	1.313	5.682	65.090	2.0	0.895	0.645	1.502	6.482	72.106
-0.5	1.0	1.844	1.417	1.376	5.826	76.834	2.0	0.748	0.632	1.600	6.831	84.609
0.0	1.0	1.500	1.323	1.620	6.796	88.192	2.0	0.600	0.583	1.876	8.159	97.183
0.5	1.0	1.156	1.121	1.997	9.057	96.959	2.0	0.453	0.486	2.321	11.231	107.471
1.0	1.0	0.812	0.726	1.608	6.650	89.377	2.0	0.305	0.299	1.906	8.329	98.073

Table 1 shows that the values of the mean are decreasing as the values of the distribution parameters are increasing. It also shows that the standard deviation values are increasing in general. TSHD is positively skewed as all values of the skewness are positive. The diffusional kurtosis is a quantitative measure of the degree to which the diffusion displacement probability distribution deviates from a Gaussian form (kurtosis = 3). As all kurtosis values are greater than 3, this tells us that TSHD is a heavily tailed distribution.

4.2 Moment generating function

Theorem 4.2. *The moment generating function (MGF) of the TSHD random variable is given by*

$$M_X(t) = \frac{\theta^2}{\theta^2+1}\Bigg[\frac{(1-\lambda)\theta}{(t-\theta)} + \frac{(1-\lambda)}{(t-\theta)^2} + \frac{2\lambda\theta}{(t-2\theta)} + \frac{2\lambda(2\theta^2+1)}{(t-2\theta)^2} + \frac{4\lambda\theta}{(t-2\theta)^3}\Bigg]. \tag{11}$$

Proof.

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^\infty e^{tx} g(x)dx \\
&= \int_0^\infty e^{tx}\Bigg[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-\theta x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-2\theta x} \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-2\theta x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-2\theta x}\Bigg]dx \\
&= \int_0^\infty \Bigg[\frac{(1-\lambda)\theta^3}{\theta^2+1}e^{-(\theta-t)x} + \frac{(1-\lambda)\theta^2}{\theta^2+1}xe^{-(\theta-t)x} + \frac{2\lambda\theta^3}{\theta^2+1}e^{-(2\theta-t)x} \\
&+ \frac{2\lambda\theta^2(2\theta^2+1)}{\theta^2+1}xe^{-(2\theta-t)x} + \frac{2\lambda\theta^3}{\theta^2+1}x^2e^{-(2\theta-t)x}\Bigg]dx \\
&= \frac{\theta^2}{\theta^2+1}\Bigg[\frac{(1-\lambda)\theta}{(t-\theta)} + \frac{(1-\lambda)}{(t-\theta)^2} + \frac{2\lambda\theta}{(t-2\theta)} + \frac{2\lambda(2\theta^2+1)}{(t-2\theta)^2} + \frac{4\lambda\theta}{(t-2\theta)^3}\Bigg]. \square
\end{aligned}$$

5. Maximum likelihood estimates

Definition 5.1. *Let $X_1, X_2, ..., X_n$ be a random sample size n with a pdf $g(x)$. The likelihood function is defined as the joint density of the random sample, which is defined as*

$$\ell = L(\lambda, \theta | x_1, x_2, ..., x_n) = \prod_{i=1}^{n} g(x_i|\lambda,\theta).$$

Hence, the likelihood function is given by

$$L = \prod_{i=1}^{n}\Bigg(\Big(\frac{\theta^2(\theta+x_i)}{\theta^2+1}e^{-\theta x_i}\Big)\Big(1-\lambda+2\lambda\frac{\theta^2+1+\theta x_i}{\theta^2+1}e^{-\theta x_i}\Big)\Bigg)$$

$$= \prod_{i=1}^{n} \Big(\frac{\theta^2(\theta + x_i)}{\theta^2 + 1} e^{-\theta x_i}\Big) \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} e^{-\theta x_i}\Big),$$

$$L = \prod_{i=1}^{n} \Big(\frac{\theta^2}{\theta^2 + 1}\Big) \prod_{i=1}^{n} (\theta + x_i)$$

$$\times e^{-\sum_{i=1}^{n} \theta x_i} \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1} e^{-\theta x_i}\Big). \tag{12}$$

Therefore, the log-likelihood function is given by:

$$\ell = lnL = ln\Bigg(\prod_{i=1}^{n} \Big(\frac{\theta^2}{\theta^2 + 1}\Big) \times \prod_{i=1}^{n} (\theta + x_i) \times e^{-\sum_{i=1}^{n} \theta x_i}$$

$$\times \prod_{i=1}^{n} \Big(1 - \lambda + 2\lambda \Big(\frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1}\Big) e^{-\theta x_i}\Big)\Bigg),$$

$$\ell = 2nln\theta - nln(\theta^2 + 1) + \sum_{i=1}^{n} ln(\theta + x_i) - \theta \sum_{i=1}^{n} x_i$$

$$+ \sum_{i=1}^{n} ln\Bigg[1 - \lambda + 2\lambda \Bigg(\frac{\theta^2 + 1 + \theta x_i}{\theta^2 + 1}\Bigg) e^{-\theta x_i}\Bigg]. \tag{13}$$

Deriving with respect to the parameters, we get

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n} \frac{(-1 + 2(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}{(1 - \lambda)(\theta^2 + 1) + 2\lambda(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}\Bigg],$$

$$\frac{\partial \ell}{\partial \theta} = \frac{2n}{\theta(\theta^2 + 1)} + \sum_{i=0}^{n} \Bigg[\frac{1}{\theta + x_i}\Bigg] - \sum_{i=1}^{n} x_i$$

$$- \sum_{i=1}^{n} \frac{2\lambda x_i(\theta^3 + \theta^2 x_i + 3\theta + x_i)}{(\theta^2 + 1)\Big[(1 - \lambda)(\theta^2 + 1) e^{\theta x_i} + 2\lambda(\theta^2 + 1 + \theta x_i)\Big]}.$$

Estimates of distribution parameters can be found by equating the derivatives of the log-likelihood function in (13) to zero. Thus,

$$\sum_{i=0}^{n} \frac{(-1 + 2(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}{(1 - \lambda)(\theta^2 + 1) + 2\lambda(\theta^2 + 1 + \theta x_i) e^{-\theta x_i}}\Bigg] = 0,$$

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2\lambda x_i(\theta^3 + \theta^2 x_i + 3\theta + x_i)}{(\theta^2 + 1)\Big[(1 - \lambda)(\theta^2 + 1) e^{\theta x_i} + 2\lambda(\theta^2 + 1 + \theta x_i)\Big]} - \sum_{i=0}^{n} \Bigg[\frac{1}{\theta + x_i}\Bigg]$$

$$= \frac{2n}{\theta(\theta^2 + 1)}.$$

There is no exact solutions to this non-linear system of equations, but we can solve it numerically.

6. Quantile function

Consider a random variable X with a continuous cumulative distribution function $G(x)$. If $G(x)$ is a strictly increasing function, then the quantile X_q is the value satisfies $q = G^{-1}(x_q)$. Therefore,

$$\begin{aligned} (1+\lambda)F(x_q) - \lambda\big(F(x_q)\big)^2 &= q, \\ \lambda\big(F(x_q)\big)^2 - (1+\lambda)F(x_q) - q &= 0. \end{aligned}$$

Solving this equation with respect to, x

$$\begin{aligned} F(x_q) &= \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}, \\ 1 - \frac{\theta^2+1+\theta x_q}{\theta^2+1} e^{-\theta x_q} &= \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}, \\ \frac{\theta^2+1+\theta x_q}{\theta^2+1} e^{-\theta x_q} &= 1 - \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}, \\ (\theta^2+1+\theta x_q) e^{-\theta x_q} &= \left[1 - \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right](\theta^2+1). \end{aligned}$$

This nonlinear equation has no exact solution, but it can be solved numerically.

7. Rényi entropy

Theorem 7.1. *The Rényi entropy of $\beta \geq 0$ for TSHD is defined is*

$$E_R = \Big(\frac{1}{1-\beta}\Big) log \sum_{i=0}^{\beta}\sum_{k=0}^{\beta}\sum_{r=0}^{i} \binom{\beta}{i}\binom{\beta}{k}\binom{i}{r}\Big(\frac{(2\lambda)^i \theta^{3\beta-2k-1}(1-\lambda)^{\beta-i}\Gamma(k+r+1)}{(\theta^2+1)^{\beta+r}(\beta+i)^{k+r+1}}\Big).$$

Proof.

$$\begin{aligned} E_R &= \Big(\frac{1}{1-\beta}\Big)\log\int_0^\infty (g(x))^\beta dx \\ &= \Big(\frac{1}{1-\beta}\Big)\log\int_0^\infty \left[\Big(\frac{\theta^2(\theta+x)}{\theta^2+1}e^{-\theta x}\Big)\Big(2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x} + 1 - \lambda\Big)\right]^\beta dx \\ &= \Big(\frac{1}{1-\beta}\Big)\log\Big(\frac{\theta^2}{\theta^2+1}\Big)^\beta \int_0^\infty \sum_{i=0}^{\beta}\Bigg[(\theta+x)^\beta e^{-\theta\beta x}\binom{\beta}{i} \\ &\quad \cdot \Big(2\lambda\frac{\theta^2+1+\theta x}{\theta^2+1}e^{-\theta x}\Big)^i (1-\lambda)^{\beta-i}\Bigg] dx \end{aligned}$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \binom{\beta}{i} \Big(\frac{(2\lambda)^i (1-\lambda)^{\beta-i} \theta^{2\beta}}{(\theta^2+1)^{\beta+i}}\Big) \int_0^\infty (\theta+x)^\beta (\theta^2+1+\theta x)^i e^{-(\beta+i)\theta x} dx$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \sum_{k=0}^{\beta} \sum_{r=0}^{i} \binom{\beta}{i} \binom{\beta}{k} \binom{i}{r} \Big(\frac{(2\lambda)^i (1-\lambda)^{\beta-i} \theta^{2\beta}}{(\theta^2+1)^{\beta+i}}\Big) \theta^{\beta-k} (\theta)^r (\theta^2+1)^{i-r}$$

$$\times \int_0^\infty x^{k+r} e^{-(\beta+i)\theta x} dx$$

$$= \Big(\frac{1}{1-\beta}\Big) \log \sum_{i=0}^{\beta} \sum_{k=0}^{\beta} \sum_{r=0}^{i} \binom{\beta}{i} \binom{\beta}{k} \binom{i}{r} \Big(\frac{(2\lambda)^i \theta^{3\beta-2k-1} (1-\lambda)^{\beta-i} \Gamma(k+r+1)}{(\theta^2+1)^{\beta+r} (\beta+i)^{k+r+1}}\Big).$$

□

8. Applications

In this section, an application to real life data is presented to illustrate the usefulness of the proposed distribution in fitting such data. The data set represents the times between successive failures of air conditioning equipment in a Boeing 720 airplane and is given as: 74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326. (Data reference: [17])

The proposed distribution is used to fit this data in addition to the following distributions:

- Lindley distribution [16]: $f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x > 0, \theta > 0.$
- Ishita distribution [22]: $f(x) = \frac{\theta^3}{\theta^3+2}(\theta + x^2)e^{-\theta x}; x > 0, \theta > 0.$
- Akash distribution [21]: $f(x) = \frac{\theta^3}{\theta^2+2}(1 + x^2)e^{-\theta x} \quad ; x > 0, \theta > 0.$
- Shanker distribution [20]: $f(x) = \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-\theta x} \quad ; x > 0, \theta > 0.$

For all these fitted distributions, we compute the Cramer-von Mises Criterion (W), Anderson-Darling Criterion (A), Akaike Information Criterion (AIC), -2 log likelihood ($-2lnL$), Kolmogorov Smirnov (KS) statistic and its p-value and the results are reported in Table 2. It is clear from Table 2 that the proposed distribution has the smallest values of the $-2lnL$, W, A, AIC, KS-statistic and largest p-value among all fitted distributions. Consequently, the proposed distribution is more adequate to fit this real life data than other fitted distributions (Lindley, Ishita, Akash, Shanker).

Table 3 provides the maximum likelihood estimates (MLEs) of the parameters of the fitted distributions and their confidence intervals.

9. Conclusion

This paper proposed the two-parameter transmuted Shanker distribution. This distribution is a generalization of Shanker distribution. Its properties were stud-

Table 2: $-2lnL$, W, A, AIC, KS statistic and its p-value for fitted distributions.

Distribution	$-2lnL$	***W***	***A***	***AIC***	***KS-Statistic***	***p-value***
Lindley	181.3412	0.1769775	1.023697	183.3412	0.3858	0.02302
Ishita	194.3217	0.1738195	1.00636	196.3217	0.4557	0.003939
Akash	194.2997	0.1742572	1.00889	196.2997	0.4556	0.003946
Shanker	181.5784	0.1735927	1.005124	183.5784	0.388	0.02187
Transmuted Shanker	179.3349	0.1567872	0.920379	183.3349	0.3267	0.0814

Table 3: The MLEs of the parameters of the fitted distributions and their confidence intervals

Distribution	**Parameter**	**MLE**	**Standard error**	**95% Confidence Interval**	
				Lower Limit	**Upper Limit**
Lindley	θ	0.01638021	0.002979601	0.01054	0.02222
Ishita	θ	0.02473989	0.003681941	0.017523	0.031956
Akash	θ	0.02473481	0.003680836	0.01752	0.031949
Shanker	θ	0.01651167	0.003003094	0.010626	0.022398
Transmuted Shanker	θ	0.01491089	0.002985547	0.009059	0.020763
	λ	0.5779029	0.21318088	0.160068375	0.995737425

ied and investigated. Application to real data was presented and showed that the proposed TSHD outperforms Shanker distribution and some other considered distributions. For future work, it might be of interest to consider this distribution to adaptive methods suggested by [9, 8, 4].

References

[1] M. Al-khazaleh, A. Al-Omari, A. Al-Khazaleh, *Transmuted two-parameter lindley distribution*, Journal of Statistics Applications and Probability, 5 (2016), 1-11.

[2] A. Al-Omari, A. Al-Khazaleh, L. Alzoubi, *A generalization of the new Weibulll-Pareto distribution*, Revista Investigacion Operacional, 41 (2020), 138-146.

[3] A. Al-Omari, A.M. Al-khazaleh, L.M. Alzoubi, *Transmuted Janardan distribution: a generalization of the Janardan distribution*, Journal of Statistics Applications & Probability, 5 (2017), 1-11.

[4] L. Al-Zoubi, *Adaptive inference for multi-stage unbalanced exponential survey data based on a simulation from an intercept-only mode*, Electronic Journal of Applied Statistical Analysis (EJASA), 8 (2015), 136-153.

[5] N. Almawajdeh, *Generalization of rani and generalized akash distributions*, Master thesis, (2019), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[6] B. Almousa, *Generalization of Sujatha and Amarendra distributions using quadratic transmutation map*, Master thesis, (2019), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[7] L. Alzoubi, *Transmuted Mukherjee-Islam distribution: a generalization of Mukherjee-Islam distribution*, Journal of Mathematics Research, 9 (2017), 135-144.

[8] L.M. Al-Zoubi, *Adaptive inference and design for multistage surveys*, PhD thesis, School of Mathematics and Applied Statistics, PhD Thesis, 2011.

[9] L.M. Al-Zou′bi, R.G. Clark, D.G. Steel, *Adaptive inference for multi-stage survey data*, Communications in Statistics Simulation and Computation, 39 (2010), 1334-1350.

[10] H.A. Alsikeek, *Quadratic transmutation map for reciprocal distribution and two-parameter weighted exponential distribution*, Master Thesis (2018), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[11] R.M. AzZwideen, *Transmutation maps for gamma-gompertz and generalized type-II logistics distributions*, Master thesis (2017), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[12] Mohammed M. Gharaibeh, *Transmuted Aradhana distribution: properties and application*, Jordan Journal of Mathematics and Statistics (JJMS), 13 (2020), 287-304.

[13] Mohammed M. Gharaibeh, *Transmuted Aradhana distribution: properties and application*, Jordan Journal of Mathematics and Statistics (JJMS), 13 (2020), 287-304.

[14] Mohammed M. Gharaibeh, Amer I. Al-Omari, *Transmuted Ishita distribution and its applications*, Journal of Statistical Applications & Probability, 8 (2019), 67-81.

[15] A.M. Khazaleh, *Transmuted burr type XII distribution: a generalization of the Burr type XII distribution*, International Mathematical Forum, 11 (2016), 547-556.

[16] F. Merovci, I. Elbatal, *Transmuted Lindley-geometric distribution and its applications*, Journal of Statistics Applications & Probability, 3 (2014), 77-91.

[17] F. Proschan, *Theoretical explanation of observed decreasing failure rate*, Technometrics, 42 (2000), 7-11.

[18] F.S. Rabaiah, *Generalizations of power function and type-I half logistic distributions using quadratic transmutation map*, Master Thesis, (2018), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[19] F. Saadeh, *Generalization to some distributions using transmutation maps*, Master Thesis (2019), Department of Mathematics, Al Al-Bayt University, Mafraq, Jordan.

[20] R. Shanker, *Shanker distribution and its applications*, International Journal of Statistics and Applications, 5 (2015), 338–348.

[21] R. Shanker,*Akash distribution and its applications*, International Journal of Probability and Statistics, 4 (2015), 65-75.

[22] R. Shanker, K.K. Shukla, *Ishita distribution and its applications*, Biometrics & Biostatistics International Journal, 5 (2017), 1-9.

[23] W.T. Shaw, I.R. Buckley, *The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank rransmutation map*, Technical Report, 2007.

Accepted: January 14, 2021