On property $(a\omega)$ and hypercyclic/supercyclic operators

Abdellah Akrym*
Chouaib Doukkali University
Faculty of Sciences
El Jadida
Morocco
akrym.maths@gmail.com

Abdeslam EL BAKKALI
Chouaib Doukkali University
Faculty of Sciences
El Jadida
Morocco
abdeslamelbakkalii@gmail.com

Abdelkhalek FAOUZI
Chouaib Doukkali University
Faculty of Sciences
El Jadida
Morocco
faouzi.a@ucd.ac.ma

Abstract. In this paper, we show that property $(a\omega)$ holds for the adjoint of hypercyclic/ supercyclic operator. Also, we characterize hypercyclic/ supercyclic operators satisfying the property $(a\omega)$. We establish that for a hypercyclic/ supercyclic operators, the property $(a\omega)$ holds if and only if property (ω) holds, if and only if a-Weyl's theorem holds.

Keywords: a-Weyl's theorem, a-Browder's theorem, property(ω), property$(a\omega)$, hypercyclic/ supercyclic operators, single valued extension property.

1. Introduction

An operator acting on Banach space satisfies property $(a\omega)$ if the complement of its Weyl spectrum in its spectrum is its set of eigenvalues of finite multiplicity which are isolated in the approximate spectrum. This class is a variant of Weyl's theorem (Weyl's theorem has been introduced in [8]) that has been introduced in [6] and studied in many papers e.g. [5, 6]. In [21], Rakočević was introduced two variants of Weyl's theorem, the so called a-Weyl's theorem and the property (ω) studied also in [2, 3, 12, 21]. Note that property (ω) and $(a\omega)$ are also related to Browder's, a-Browder's, Weyl's and a-Weyl's theorems [1, 2, 5, 6].

∗. Corresponding author

In this work, we show that property $(a\omega)$ holds for the adjoint of hypercyclic/supercyclic operator. Also, we prove that under hypercyclicity or supercyclicity hypothesis, property $(a\omega)$, property $(a\omega)$ and a-Weyl's theorem are equivalent.

Hypercyclic/supercyclic operators satisfying property (ω) or Weyl's theorem was studied in [12, 11]. In [2], it was proved that property (ω) holds for T if and only if a-Browder's theorem hold for T and $p_{00}^a(T) = \pi_{00}(T)$. In [12], it was proved that for hypercyclic or supercyclic operator, property (ω) holds for T if and only if $\pi_{00}(T) = \pi_{00}(T^*)$. In [4], the authors characterize hypercyclic and supercyclic operators satisfying the property (gw). It is natural to ask about the relation between hypercyclic/supercyclic operators and property $(a\omega)$. In [5], characterization of property $(a\omega)$ in terms of Browder's theorem was given. This paper is a continuation of these results, more precisely, we show that for hypercyclic or supercyclic operator property $(a\omega)$ holds if and only if $\pi_{00}^a(T) = \pi_{00}(T^*)$.

2. Preliminaries

Throughout this paper, $\mathcal{X}$ denotes a generally infinite-dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ the algebra of bounded linear operators on $\mathcal{X}$. We denote the null space and range of $T \in \mathcal{B}(\mathcal{X})$ by $N(T)$ and $R(T)$, respectively. Let $\alpha(T) := \dim N(T)$, $\beta(T) := \text{codim } R(T)$ be the nullity and the deficiency of T respectively. T^*, $\sigma(T)$, $\sigma_p(T)$, and $\sigma_a(T)$ denote respectively the adjoint, the spectrum, the point spectrum, and the approximate point spectrum of T.

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called a semi-Fredholm operator if $T(\mathcal{X})$ is closed and $\alpha(T) < \infty$ or $\beta(T) < \infty$. In the sequel, $\rho_{sF}(T)$ will denote the semi-Fredholm resolvent set of T. If both $\alpha(T)$ and $\beta(T)$ are finite, T is called Fredholm operator. The index of a Fredholm operator T is given by

$$ind(T) := \alpha(T) - \beta(T).$$

An operator $T \in \mathcal{B}(\mathcal{X})$ is called Weyl if it is Fredholm of index zero. The ascent $p(T)$ and the descent $q(T)$ are given by

$$p(T) = \inf \left\{ n \in \mathbb{N} \ : \ N(T^n) = N(T^{n+1} \right\},$$
$$q(T) = \inf \left\{ n \in \mathbb{N} \ : \ R(T^n) = R(T^{n+1} \right\},$$

where the infimum over the empty set is taken ∞. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, [1, 19, 17].

An operator T is called Browder if it is Fredholm of finite ascent and descent; equivalently ([13, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \in \mathbb{C} \setminus \{0\}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum

$\sigma_w(T)$, and the Browder spectrum $\sigma_b(T)$ of $T \in B(X)$ are defined by ([1], [13])

$$\begin{aligned} \sigma_e(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\}, \\ \sigma_w(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}, \\ \sigma_b(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}, \end{aligned}$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_f(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc K (respectively iso K) for the accumulation (respectively isolated) points of a subset $K \subseteq \mathbb{C}$. $T \in \mathcal{B}(\mathcal{X})$. We denote by (see [1] page 164 and 176)

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

the set of isolated eigenvalues of finite multiplicity, and by

$$\pi_{00}^a(T) := \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

we denote the set of eigenvalues of finite multiplicity which are isolated in $\sigma_a(T)$. We write

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) \ : \ \lambda I - T \text{ is Browder operator }\},$$

for the set of Riesz points of T, and

$$p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{\lambda \in \sigma_a(T) \ : \ \lambda I - T \in \mathcal{B}_+(\mathcal{X})\},$$

where $\mathcal{B}_+(\mathcal{X})$ denote the class of all upper semi-Browder operators. It should be noted that

$$p_{00}(T) \subseteq p_{00}^a(T) \subseteq \pi_{00}^a(T), \ p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T) \text{ and } p_{00}(T) = p_{00}(T^*).$$

We say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{X})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and that Browder's theorem holds for $T \in \mathcal{B}(\mathcal{X})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

To describe some of the above mentioned results, we consider the following sets [1, 2]. The class of all upper semi-Fredholm operators

$$\Phi_+(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed }\},$$

the class of all lower semi-Fredholm operators

$$\Phi_-(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ \beta(T) < \infty\},$$

the class of all semi-Fredholm operators and the class of all Fredholm operators are

$$\Phi_{\pm}(\mathcal{X}) := \Phi_{+}(\mathcal{X}) \cup \Phi_{-}(\mathcal{X}) \text{ and } \Phi(\mathcal{X}) := \Phi_{+}(\mathcal{X}) \cap \Phi_{-}(\mathcal{X}),$$

respectively. The class of all upper semi-Browder operators and the class of all lower semi-Browder are defined by

$$\mathcal{B}_{+}(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ p(T) < \infty\} \text{ and } \mathcal{B}_{-}(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) \ : \ q(T) < \infty\},$$

respectively. The class of all Browder operators is

$$\mathcal{B}_0(\mathcal{X}) := \mathcal{B}_{+}(\mathcal{X}) \cap \mathcal{B}_{-}(\mathcal{X}).$$

The class of all upper semi-Weyl operators and the class of all lower semi-Weyl operators will be denoted by

$$\mathcal{W}_{+}(\mathcal{X}) := \{T \in \Phi_{+}(\mathcal{X}) : ind(T) \leq 0\}, \ \mathcal{W}_{-}(\mathcal{X}) := \{T \in \Phi_{-}(\mathcal{X}) : \ ind(T) \geq 0\},$$

respectively. The class of all Weyl operators is defined by

$$\mathcal{W}(\mathcal{X}) := \mathcal{W}_{+}(\mathcal{X}) \cap \mathcal{W}_{-}(\mathcal{X}).$$

These classes of operators motivate the following spectra. The upper semi-Browder spectrum $\sigma_{ub}(T)$ and the lower semi-Browder spectrum $\sigma_{lb}(T)$ of T are defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_{+}(\mathcal{X})\} \text{ and } \sigma_{lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_{-}(\mathcal{X})\}.$$

The upper semi-Weyl spectrum $\sigma_{uw}(T)$ and the lower semi-Weyl spectrum $\sigma_{lw}(T)$ of T are defined by

$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_{+}(\mathcal{X})\} \text{ and } \sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}_{-}(\mathcal{X})\}.$$

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be bounded below if T is injective and has closed range (and then $\sigma_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below }\}$).

It is well known [1, 19], that

$$\sigma_{uw}(T) \subseteq \sigma_a(T) \text{ and } \sigma_{lw}(T) \subseteq \sigma_a(T^*).$$

Recall (see [14]) that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be satisfying Browder's theorem if $\sigma_w(T) = \sigma_b(T)$, or equivalently

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

and T is said to be satisfy a-Browder's theorem if $\sigma_{uw}(T) = \sigma_{ub}(T)$, or equivalently

$$\sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T).$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and T is said to be satisfy a-Weyl's theorem if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T).$$

We say that property (ω) holds for T if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T).$$

It is well known [10, 14, 21] that for $T \in \mathcal{B}(\mathcal{X})$ we have

$$\text{a-Weyl's theorem} \implies \text{Weyl's theorem} \implies \text{Browder's theorem},$$

$$\text{a-Weyl's theorem} \implies \text{a-Browder's theorem} \implies \text{Browder's theorem},$$

$$\text{property } (\omega) \implies \text{Weyl's theorem}.$$

A variant of Weyl's theorem has been introduced by Berkani and Zariouh in [6]. The operator $T \in \mathcal{B}(\mathcal{X})$ is said to satisfy property $(a\omega)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}^a(T).$$

The following theorem [5, Theorem 2.2] establishes the precise relationship between property $(a\omega)$ and Browder's theorem.

Theorem 2.1 ([5]). *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) *Browder's theorem holds for T and $\pi_{00}^a(T) = p_{00}(T)$.*

We say that $T \in \mathcal{B}(\mathcal{X})$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0, the only analytic solution $f : U \longrightarrow \mathcal{X}$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function, we say that T has the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ [1, 9, 19].

In [1], it was shown that

$$\lambda_0 \in \text{ iso } \sigma_a(T) \Longrightarrow T \text{ has SVEP at } \lambda_0. \tag{1}$$

In particular if the point spectrum $\sigma_p(T)$ is empty then T satisfied the SVEP.

$$p(\lambda_0 I - T) < \infty \Longrightarrow T \text{ has SVEP at } \lambda_0. \tag{2}$$

Dually,

$$q(\lambda_0 I - T) < \infty \Longrightarrow T^* \text{ has SVEP at } \lambda_0. \tag{3}$$

Moreover, if we assume that $\lambda_0 I - T \in \Phi_{\pm}(\mathcal{X})$, then the implications (1), (2) and (3) are equivalences [2].

Let $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{X}$ is separable. The orbit of a vector $x \in \mathcal{X}$ under T is defined by

$$Orb(T, x) := \{T^n x \ : \ n \geq 0\}.$$

The operator T is said to be hypercyclic if there is some vector $x \in \mathcal{X}$ such that $Orb(T, x)$ is dense in $\mathcal{X}$. Such vector x is called a hypercyclic vector for T. The set of all hypercyclic vectors for T will be denoted by $\mathcal{HP}(\mathcal{X})$. The operator T is said to be supercyclic [18] if there is some vector $x \in \mathcal{X}$ such that

$$\mathbb{C}Orb(T, x) := \{\alpha T^n x \ : \ \alpha \in \mathbb{C}, \ n \geq 0\},$$

is dense in $\mathcal{X}$. Such vector x is called a supercyclic vector for T , and the set of all supercyclic vectors under T will be denoted by $\mathcal{SP}(\mathcal{X})$.

3. Main results

The following Lemma was proved by second the author and Tajmouati in [12, Proof of Proposition 2.1]. We give here the proof for completeness.

Lemma 3.1. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then Weyl's theorem holds for T^*. Moreover,*

$$p_{00}(T) = p_{00}^a(T). \tag{4}$$

Proof. We show first that T^* satisfies Weyl's theorem. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, then, by [15], $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\alpha\}$, for some $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\alpha \notin \sigma_b(T^*)$, hence T^* has the SVEP. Consequently a-Browder's theorem (in particular Browder's theorem) holds for T and T^*. Suppose that $\sigma_p(T^*) = \emptyset$ then $\pi_{00}(T^*) = \emptyset$. Since $p_{00}(T^*) \subset \pi_{00}(T^*)$, thus $p_{00}(T^*) = \pi_{00}(T^*) = \emptyset$, since Browder's theorem holds for T^*, it follows, by [2, Theorem 2.16], T^* satisfies Weyl's theorem. Now, if $\sigma_p(T^*) = \{\alpha\}$, for some $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\alpha \notin \sigma_b(T^*)$, then $\pi_{00}(T^*) = \{\alpha\}$. Since $p_{00}(T^*) \subset \pi_{00}(T^*)$, thus $\sigma(T^*) \setminus \sigma_b(T^*) = p_{00}(T^*) = \{\alpha\}$, consequently $p_{00}(T^*) = \pi_{00}(T^*) = \{\alpha\}$. Hence, by [2, Theorem 2.16], Weyl's theorem holds for T^*.

In the other hand, as a-Browder's theorem holds for T, thus $\sigma_{ub}(T) = \sigma_{uw}(T)$ and $\sigma_b(T) = \sigma_w(T)$. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$ and $\sigma_{uw}(T) = \sigma_w(T)$, see [3, Theorem 1.5]. Therefore,

$$p_{00}(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = p_{00}^a(T). \quad \square$$

To motivate the following theorem we start by an example which shows that the adjoint of supercyclic operator satisfies property $(a\omega)$.

Example 1. Consider the operator shift T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \dots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots\right), \quad \text{for all } (x_n) \in l^2(\mathbb{N}).$$

Then T is surjective quasi-nilpotent, $\overline{\cup_{k\geq 1} N(T^k)} = \mathcal{X}$, $\sigma(T) = \{0\}$ and $\pi_{00}^a(T) = \pi_{00}(T) = \{0\}$, thus, by [20, corollary 1], the operator T is supercyclic. We know that the adjoint of T is defined by

$$T^*(x_1, x_2, x_3, \dots) = \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \dots\right).$$

Since T^* are quasi-nilpotent, thus $\sigma(T) = \sigma(T^*) = \sigma_w(T) = \sigma_w(T^*) = \{0\}$. But T^* is injective $\pi_{00}(T^*) = \pi_{00}^a(T^*) = \emptyset$. Since, by Lemma 3.1, T^* satisfies Weyl's theorem. Hence, by [5, Theorem 2.4], T^* satisfies property ($a\omega$).

In the following, we will give sufficient conditions for the adjoint of hypercyclic/supercyclic operator to satisfies property ($a\omega$).

Theorem 3.1. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. If*

$$\textit{iso } \sigma_a(T^*) \subset \{\lambda \in \mathbb{C} \ : \ R(\lambda I - T^*) \textit{ is closed } \},$$

then property ($a\omega$) holds for T^*.

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then, by Lemma 3.1, T^* has the SVEP and Weyl's theorem holds for T^*.

Now, since Weyl's theorem holds for T^*, thus, in order to show that property ($a\omega$) holds for T^* it suffices to prove that $q(\lambda I - T^*) < \infty$ for all $\lambda \in \pi_{00}^a(T^*)$ [5, Theorem 2.4]. Let $\lambda \in \pi_{00}^a(T^*)$, then λ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(\lambda I - T^*) < \infty$ then $R(\lambda I - T^*)$ is closed. But T^* has the SVEP at λ, thus $0 < p(\lambda I - T^*) < \infty$. Hence, by [1, Theorem 3.4], $q(\lambda I - T^*) < \infty$. □

In the next Theorem, we will shows that a hypercyclic/supercyclic operator T satisfies property ($a\omega$) if and only if a-Weyl's theorem holds for T.

Theorem 3.2. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then T satisfies property ($a\omega$) if and only if a-Weyl's theorem holds for T.*

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, then, by Lemma 3.1, T^* has the SVEP.

($\Longrightarrow$) Suppose that T satisfies property ($a\omega$), thus, by [5, Theorem 2.3], Weyl's theorem holds for T. Hence, by [1, Theorem 3.108], a-Weyl's theorem holds for T.

($\Longleftarrow$) Suppose that a-Weyl's theorem holds for T. Then $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$ and Weyl's theorem holds for T, thus $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$ and $\sigma_{uw}(T) = \sigma_w(T)$, see [3, Theorem 1.5]. Hence $\pi_{00}^a(T) = \sigma(T) \setminus \sigma_w(T)$. Therefore property ($a\omega$) holds for T. □

Example 2. Consider the operator shift T defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right) \quad \text{for all } (x_n) \in l^2(\mathbb{N}).$$

Then the operator T is supercyclic (Example 1). Since T^* is injective quasinilpotent, thus $\sigma(T^*) = \{0\}$ and $\pi_{00}(T^*) = \emptyset$. Hence $\pi^a_{00}(T) \neq \pi_{00}(T^*)$. On the other hand,

$$\{0\} = \pi^a_{00}(T) \neq \sigma(T) \setminus \sigma_w(T) = \emptyset.$$

Therefore, T does not satisfy property $(a\omega)$.

In the following, we give necessary and sufficient conditions for hypercyclic/supercyclic operators to have property $(a\omega)$.

Theorem 3.3. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) $\pi^a_{00}(T) = \pi_{00}(T^*)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that property $(a\omega)$ holds for T, then by Theorem 2.1, $p_{00}(T) = \pi^a_{00}(T)$. Since Weyl's theorem holds for T^*, thus

$$\pi_{00}(T^*) = p_{00}(T^*) = p_{00}(T).$$

Therefore $\pi^a_{00}(T) = \pi_{00}(T^*)$.

(1) $\Longleftarrow$ (2) Suppose that $\pi^a_{00}(T) = \pi_{00}(T^*)$. Since T^* satisfies Weyl's theorem, thus $p_{00}(T^*) = \pi_{00}(T^*)$. Then

$$\pi_{00}(T^*) = p_{00}(T^*) = p_{00}(T).$$

Which imply that $p_{00}(T) = \pi^a_{00}(T)$. Hence

$$\sigma(T) \setminus \sigma_b(T) = \sigma(T) \setminus \sigma_w(T) = \pi^a_{00}(T).$$

Therefore, the property $(a\omega)$ holds for T. $\square$

It was shown in [15] that if $T \in \mathcal{HP}(\mathcal{X})$ then $\sigma_p(T^*) = \emptyset$. Using this result, we get a necessary and sufficient condition for an operator $T \in \mathcal{HP}(\mathcal{X})$ to satisfies property $(a\omega)$.

Corollary 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$ such that $T \in \mathcal{HP}(\mathcal{X})$. Then property $(a\omega)$ holds for T if and only if*

$$\pi^a_{00}(T) = \emptyset.$$

Proof. Assume that $p^a_{00}(T) = \emptyset$. Since $T \in \mathcal{HP}(\mathcal{X})$. Thus $\sigma_p(T^*) = \emptyset$, hence $\pi^a_{00}(T) = \pi_{00}(T^*) = \emptyset$. Therefore, by Theorem 3.3, property $(a\omega)$ holds for T. For the converse, assume that property $(a\omega)$ holds for T. Since $T \in \mathcal{HP}(\mathcal{X})$, thus $\pi_{00}(T^*) = \emptyset$. Hence $\pi^a_{00}(T) = \emptyset$. $\square$

Corollary 3.2. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$ such that $\pi^a_{00}(T) = \pi_{00}(T^*)$. Then a-Weyl's theorem hols for T.*

Proof. Since $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$ with $\pi^a_{00}(T) = \pi_{00}(T^*)$. Thus, by Theorem 3.3, property ($a\omega$) holds for T. Hence, by Theorem 3.2, a-Weyl's theorem hols for T. □

Given $T \in \mathcal{B}(\mathcal{X})$. Let $\rho_K(T)$ denote the Kato resolvent set of T;

$$\rho_K(T) := \left\{\lambda \in \mathbb{C} \;:\; R(\lambda I - T) \text{ is closed }, N(\lambda I - T) \subset \cap_{k\geq 1} R(\lambda I - T)^k\right\}.$$

It is well-known that $\rho_K(T)$ is open and $\rho_K(T) = \rho_K(T^*)$. Let

$$\rho_{su}(T) := \{\lambda \in \mathbb{C} \;:\; (\lambda I - T)\mathcal{X} = \mathcal{X}\}.$$

Let $\sigma_{su}(T) = \mathbb{C} \setminus \rho_{su}(T)$ be the surjectivity spectrum of T. It is well-known that $\rho_{su}(T)$ is open and $\rho_{su}(T) \subseteq \rho_K(T)$.

We denote by $\mathcal{H}_c(\sigma(T))$ the set of all complex-valued functions which are analytic and non constant in a neighborhood of the spectrum $\sigma(T)$, where the operator $f(T)$ is defined by the classical functional calculus.

Corollary 3.3. *Let $\mathcal{X}$ be a separable Banach space and suppose that $T \in \mathcal{B}(\mathcal{X})$ is such that $\lambda I - T$ is surjective and $\overline{\cup_{k\geq 1} N(\lambda I - T)^k} = \mathcal{X}$ for some $\lambda \in \rho_{su}(T)$. Then $f(T)$ satisfies property ($a\omega$) for every $f \in \mathcal{H}_c(\sigma(T))$.*

Proof. First, we will show that $\text{iso}\,\sigma_a(f(T)) \subseteq f(\text{ iso}\,\sigma_a(T))$. Let $\lambda_0 \in \text{iso}\,\sigma_a(f(T))$. By the spectral mapping theorem for the approximate point spectrum, $\lambda_0 \in \text{ iso}\, f(\sigma_a(T))$. Then there exists $\mu_0 \in \sigma_a(T)$ such that $f(\mu_0) = \lambda_0$. Denote by G the component of the domain of f which contains μ_0. If $\mu_0 \in \text{ acc}\,\sigma_a(T)$, then there exists a sequence $(\mu_n)_{n\geq 1} \subset G \cap \sigma_a(T)$ of distinct scalars such that $\mu_n \longrightarrow \mu_0$ as $n \longrightarrow \infty$. Let $K = \{\mu_0, \mu_1, \mu_2, \ldots\}$, then K is a compact subset of G, thus f may assume the value $\lambda_0 = f(\mu_0)$ only a finite number of points of K, hence for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \longrightarrow f(\mu_0) = \lambda_0$ as $n \longrightarrow \infty$, $\lambda_0 \in \text{ acc}\, f(\sigma_a(T))$, and this is a contradiction. Hence $\mu_0 \in \text{ iso}\,\sigma_a(T)$. Therefore

$$\text{iso}\,\sigma_a(f(T)) \subseteq f(\text{ iso}\,\sigma_a(T)). \tag{5}$$

Since $\overline{\cup_{k\geq 1} N(\lambda I - T)^k} = \mathcal{X}$ for some $\lambda \in \rho_{su}(T)$. Thus, by [7, Theorem 3], $f(T)$ is supercyclic for $f \in \mathcal{H}_c(\sigma(T))$, and since T^* has de SVEP, hence $\sigma(T) = \sigma_a(T)$. Moreover, we have $f(T)^* = f(T^*)$ for every $f \in \mathcal{H}_c(\sigma(T))$. On the other hand, by [16, Proposition 3], $\sigma(T)$ is connected. But

$$\sigma(f(T)^*) = \sigma(f(T)) = f(\sigma(T)).$$

Thus, $\sigma(f(T))$ and $\sigma(f(T)^*)$ are connected. Then,

$$\pi_{00}(f(T)) = \pi_{00}(f(T)^*) = \emptyset,$$

by (5), we have $\pi^a_{00}(f(T)) = \emptyset$. Hence, $\pi^a_{00}(f(T)) = \pi_{00}(f(T)^*)$. Therefore, by Theorem 3.3, $f(T)$ satisfies property ($a\omega$) for every $f \in \mathcal{H}_c(\sigma(T))$. □

In the following result we shows that for hypercyclic/supercyclic operator property (ω) and property $(a\omega)$ are equivalent.

Theorem 3.4. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Property $(a\omega)$ holds for the operator T if and only if T satisfies property (ω).*

Proof. Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then T^* has the SVEP, thus a-Browder's theorem (in particular Browder's theorem) holds for T, hence as above,

$$p_{00}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = p^a_{00}(T). \tag{6}$$

$(\Longrightarrow)$ Assume that T satisfies property $(a\omega)$, then, by Theorem 3.2, a-Weyl's theorem (in particular Weyl's theorem) hols for T. Thus

$$\sigma_a(T) \setminus \sigma_{uw}(T) = \pi^a_{00}(T).$$

Hence, $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T)$. Therefore, T satisfies property (ω).

$(\Longleftarrow)$ Assume that property (ω) holds for T. Then Weyl's theorem holds for T. Thus

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T).$$

But $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$, thus $\pi_{00} = \pi^a_{00}(T)$, $(\sigma(T) = \sigma_a(T))$. Hence $\pi^a_{00}(T) = \sigma(T) \setminus \sigma(T)$. Therefore, property $(a\omega)$ holds for T. $\square$

In the following, some characterizations of hypercyclic/supercyclic operators satisfying property $(a\omega)$ will be given.

Corollary 3.4. *Let $T \in \mathcal{HP}(\mathcal{X}) \cup \mathcal{SP}(\mathcal{X})$. Then the following statements are equivalent.*

(1) *The property $(a\omega)$ holds for T.*

(2) *The property (ω) holds for T.*

(3) $\pi^a_{00}(T) = \pi_{00}(T^*)$.

(4) $\pi_{00}(T) = \pi_{00}(T^*)$

Proof. By Theorem 3.4, we have ((1) $\Longleftrightarrow$ (2)). Using Theorem 3.3, we get ((1) $\Longleftrightarrow$ (3)). For ((2) $\Longleftrightarrow$ (4)) we use [12, Proposition 2.1]. $\square$

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