One point paracompactification (metacompatification)

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Abstract. The notion of locally paracompact (metacompact) are studied and utilized to obtain the one point paracompactification (metacompatification). Several related results are also introduced.

Keywords: paracompact, metacompact, α-paracompact, α-metacompact.

1. Introduction

Paracompact spaces were first introduced by Dieudonné [3] in 1944 as a natural generalization of compact spaces still retaining enough structure to enjoy many properties of compact spaces. The notion of paracompactness gained stature with the proof, by A. H. Stone [7], that every metric space is paracompact and the subsequent use of the result is the solution of the general metrization problem by Bing [2], Nagata [5] and Smirnov [6].

In this paper, we define the notion of local paracompactness and study their main properties, we prove that any locally paracompact Hausdorff spaces can be embedded in a certain paracompact Hausdorff space called the one point paracompactification.

Finally, we define the notion of locally metacompactness, and by applying almost similar methods we obtain several results concerning locally metacompact spaces. For notions not defined here we refer the reader to Engelking [4]. All spaces are assumed to be T_2.

2. Locally paracompact spaces

In this section locally paracompact spaces are defined, different results concerning them and their relations with some topological spaces are studied.

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Definition 2.1. *A space X is called locally paracompact if each point of X has an open neighbourhood U such that $\overline{U}$ is paracompact.*

Clearly, every paracompact space is locally paracompact. However, the converse need no be true as the following example tells.

Example. Let Ω_o be the set of all ordinal numbers less than the first uncountable ordinal Ω equipped with the order topology whose subbase consists of sets of the form $\{x : x < \alpha\}$ or $\{x : \alpha < x\}$ for some α in Ω_o. Then Ω_o is a locally compact T_2-space. Hence, it is a locally paracompact T_2-space. However, Ω_o is not paracompact since the open cover $\{x : x < \alpha\}$ has no locally finite open refinement.

Definition 2.2. *A subset A of a topological space (X, τ) is called α-paracompact if every cover of A by members of τ has a locally finite refinement by members of τ which covers A.*

Remark. It is easy to see that if a subset A of a topological space (X, τ) is α-paracompact, then A is a paracompact subspace.

Theorem 2.1. *Let A be a closed paracompact subset of a space X and F be a closed subset of the interior of A. Then F is α-paracompact.*

Theorem 2.2. *A closed subset of an α-paracompact set is α-paracompact.*

Theorem 2.3. *An α-paracompact subset A of a T_2-space X is closed.*

Theorem 2.4. *Let (X, τ) be a locally paracompact T_2-space. Then each point of X has an open neighbourhood U such that $\overline{U}$ is α-paracompact.*

Proof. Since X is locally paracompact T_2-space, it is not difficult to see that X is a T_3-space. Since X is locally paracompact, each $x \in X$ has an open neighbourhood V such that $\overline{V}$ is paracompact. Since X is a T_3-space. there is an open set U such that $x \in U \subseteq \overline{U} \subseteq V \subseteq \overline{V}$. Thus, it follows from Theorem 2.1 that $\overline{U}$ is α-paracompact. □

Theorem 2.5. *Every closed subset of a locally paracompact space is locally paracompact.*

Proof. Let X be a locally paracompact space and F be closed in X. If $x \in F$, then there exists an open neighbourhood U of x such that $\overline{U}$ is paracompact. Now $U \cap F$ is open in F and $x \in U \cap F$. Also, $\overline{U \cap F}^F = \overline{U \cap F} \cap F$ is closed in X, and thus closed in $\overline{U}$, but $\overline{U}$ is paracompact, so, $\overline{U \cap F}^F$ is paracompact (as a closed subset of a paracompact space is paracompact). □

Theorem 2.6. *Let $X = \cup\{F_\alpha : \alpha \in \Lambda\}$, where $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed cover of X. Then X is locally paracompact if and only if F_α is locally paracompact.*

We recall the following known result.

Lemma 2.7. *Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is paracompact. Then X is paracompact.*

Theorem 2.8. *Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is a locally paracompact space. Then X is a locally paracompact space.*

Proof. Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is a locally paracompact space. Since Y is locally paracompact, for each $x \in X$, there is an open neighbourhood K of $f(x)$ such that $\overline{K}$ is paracompact. Since f is continuous, there is an an open neighbourhood G of x such that $f(G) \subseteq K$ and $f(\overline{G}) \subseteq \overline{f(G)}$. Thus, $f(\overline{G}) \subseteq \overline{K}$, and therefore, $\overline{G} \subseteq f^{-1}(\overline{K})$. Since $\overline{K}$ is closed and $f : X \overset{onto}{\longrightarrow} Y$ is perfect, so, $f : f^{-1}(\overline{K}) \overset{onto}{\longrightarrow} \overline{K}$ is perfect. Since $\overline{K}$ is paracompact, it follows from Lemma 2.7 that $f^{-1}(\overline{K})$ is paracompact, but $\overline{G}$ is closed in $f^{-1}(\overline{K})$, so, $\overline{G}$ is paracompact (as a closed subset of a paracompact space is paracompact). Hence, X is locally paracompact. □

Corollary 2.8.1. *The product of a compact space X and a locally paracompact space Y is a locally paracompact space.*

Proof. Let X be compact space and Y be a locally paracompact space. Since X is compact, the projection function $P_Y : X \times Y \to Y$ is a perfect onto function. Since Y is locally paracompact, it follows from Theorem 2.8 that $X \times Y$ is locally paracompact. □

Recall that a space X is called C-scattered [8] if every nonempty closed subspace A of X has a point $x \in A$ with a compact neighbourhood in A.

Theorem 2.9. *If X is a paracompact C-scattered space, then X is locally compact.*

Proof. Let X be a paracompact C-scattered space. Let $X^{(1)} = \{y \in X : y$ does not have a compact neighbourhood in $X\}$. Suppose that, for some ordinal number $\alpha > 1$, $X^{(\alpha)}$ has been defined. If $\beta = \alpha + 1$, define $X^{(\beta)} = (X^{(\alpha)})^{(1)}$ and $X^{(\beta)} = \bigcap_{\alpha<\beta} X^{(\alpha)}$ otherwise. If X is C-scattered, it is easy to check that $X^{(\alpha)} = \phi$ for some ordinal α. For $\alpha = 1$, if $X^{(\alpha)} = \phi$, then X is locally compact. Suppose that the result holds for all $\beta < \alpha$, i.e if $X^{(\beta)} = \phi$ for $\beta < \alpha$, then X is locally compact. Now we want to prove that if $X^{(\alpha)} = \phi$, then X is locally compact.

Case 1. There is α such that $\alpha = \beta + 1$ for some $\beta < \alpha$. In this case $X^{(\alpha)} = \phi$ implies $X^{(\beta)}$ is locally compact. For each $y \in X^{(\beta)}$, there exists an open U_y, containing y such that $\overline{U_y} \cap X^{(\beta)}$ is compact. For each $x \in X - X^{(\beta)}$ there is an open set U_x containing x such that $x \in \overline{U_x} \subseteq X - X^{(\beta)}$ Hence for each $x \in X - X^{(\beta)}$, $(\overline{U_x}^{(\beta)}) = \phi$. By the induction assumption, for each $x \in X - X^{(\beta)}$, $\overline{U_x}$ is locally compact. Now $\underline{U} = \{U_x : x \in X - X^{(\beta)}\} \cup \{U_x : x \in X^{(\beta)}\}$ is

an open cover of X. Since X is paracompact, $\underline{U}$ has a locally finite closed refinement, say $\underline{V}$. Let $\underline{V}' = \{\overline{V} : V \in \underline{V}, V \subseteq U_x$, for some $x \in X - X^{(\beta)}\}$. Now $\{X^{(\beta)}\} \cup \underline{V}'$ is a closed locally finite cover of X, so X is locally compact.
Case 2. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)} = \phi$. Consider the open cover $\underline{U} = \{X - X^{(\beta)} : \beta < \alpha\}$ of X. By the induction assumption, it is easy to see that $X - X^{(\beta)}$ is a locally compact subset of X, thus X is locally compact. □

Definition 2.3. *A space X is called bilocally compact if each point of X has an open neighbourhood U such that $\overline{U}$ is locally compact.*

Theorem 2.10. *If X is a locally paracompact C-scattered space, then X is a bilocally compact space.*

Proof. Let X be a locally paracompact C-scattered space. Then for each $x \in X$, there is an open neighbourhood U such that $\overline{U}$ is paracompact. Since X is C-scattered, it follows from Theorem 2.9 that $\overline{U}$ is locally compact. Hence X is bilocally compact. □

Theorem 2.11. *If X is a paracompact bilocally compact space, then $X \times Y$ is paracompact for any paracompact space Y.*

Proof. Since X is a paracompact bilocally compact space, for each $x \in X$ there is an open neighbourhood U_x of x such that $\overline{U_x}$ is locally compact and paracompact. Since X is paracompact,$\{U_x : x \in X\}$ has a locally finite closed refinement $\{F_\alpha : \alpha \in \Lambda\}$, where each F_α is locally compact and paracompact. Now $\{F_\alpha \times Y : \alpha \in \Lambda\}$ is a locally finite closed cover of $X \times Y$, where $F_\alpha \times Y$ is paracompact. Hence, $X \times Y$ is paracompact. □

Definition 2.4. *A subset A of a space X is called Lindelöf if every cover of A by open subsets of X has a countable subcover of A.*

Definition 2.5. *A space X is called locally Lindelöf if each point of X has a Lindelöf neighbourhood U_x.*

Theorem 2.12. *Let X be a locally paracompact separable space. Then X is locally Lindelöf.*

Proof. Since X be a locally paracompact separable space. For each $x \in X$ there is an open neighbourhood U such that $\overline{U}$ is paracompact. Since X is separable and U is open, U is separable.too, and thus, $\overline{U}$ is also separable. Therefore, $\overline{U}$ is both paracompact and separable, but it is known that a paracompact separable space is Lindelöf, so $\overline{U}$ is Lindelöf. Hence, X is locally Lindelöf. □

3. One point paracompactification

A paracompactification of a topological space (X, τ) is a paracompact space (X^*, τ^*), that contains (X, τ) as a dense subspace. A paracompactification is called the N-point paracompactification if $X^* - X$ consists of N points. If $N = 1$, X^* is called the one point paracompactification of X.

Theorem 3.1. *Let (X, τ) be a locally paracompact T_2-space, $p \notin X$ and $X^* = X \cup \{p\}$. Let τ^* be a family of subsets of X^* of the following types:*

1. *All open subsets of X.*
2. *All subsets of X of the form $X^* - K$, where K is an α-paracompact subset of X.*
3. *The sets X^* and ϕ.*

Then each of the following holds:

(i) *τ^* is a topology on X^*.*

(ii) *(X^*, τ^*) is a T_2- space.*

(iii) *(X^*, τ^*) is a paracompact space.*

(iv) *(X, τ) is dense in (X^*, τ^*).*

Proof. (i) Let $\bigcup_{\alpha\in\Lambda} E_\alpha$ be an arbitrary union of members of τ^*. Now if $E_\alpha = X^*$ for some α then $\bigcup_{\alpha\in\Lambda} E_\alpha = X^* \in \tau^*$. Suppose that $E_\alpha \neq X^*$ for each $\alpha \in \Lambda$ then $\bigcap_{\alpha\in\Lambda} E_\alpha = L_p \cup M_q$ where L_p is a union of elements of E_α's of type 1. and M_q is a union of elements of E_α's of type 2. then L_p is of type 1. $M_q = \cup X^* - K_q = X^* - \cap K_q$, where K_q is an α-paracompact subset of X for each q. Since an α-paracompact subset of T_2-space is closed and any intersection of closed subsets of an α-paracompact set is α-paracompact, we get that $\cap K_q$ is α-paracompact, therefore M_q is of type 2. Hence it is suffices to prove that a union of a set A of type 1 and a set B of type 2 is in τ^*. Now $A \cup B = (X - (X - A)) \cup (X^* - K)$ where K is an α-paracompact closed subset of X.
$A \cup B = (X^* - (X^* - A)) \cup (X^* - K) = X^* - ((X^* - A) \cap K)$. Since $(X^* - A) \cap K)$ is an α-paracompact closed subset of X, we get that $A \cup B$ is of type 2, so $A \cup B \in \tau^*$. Let $G, H \in \tau^*$, if both G, H are of type 1, then $G \cap H$ is of type 1 also, and if both G, H are of type 2, then $G \cap H$ is of type 2 also. Let G be of type 1 and H be of type 2, then $G \cap H = G \cap (X^* - M)$ where M is $\alpha-$paracompact, now $G \cap H = (X^* - (X^* - G)) \cap (X^* - M) = X^* - ((X^* - G) \cup M)$, hence $G \cap H \in \tau^*$.

(ii) To show that (X^*, τ^*) is a T_2-space. Let $x \neq y$ are elements in X. It is suffices to assume that $y = p$ and $x \neq p$. Since X is a locally paracompact

T_2-space, there is an open neighbourhood U containing x such that $\overline{U}$ is α-paracompact. Now $V = X^* - \bar{U}$ is an open set containing y and $V \cap U = \phi$. Hence X^* is a T_2-space.

(iii) To show that (X^*, τ^*) is paracompact let $\vec{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any open cover of X^*. Let $p \in U_{\alpha_o}$ for some $\alpha_o \in \Lambda$. Then $U_{\alpha_o} = X^* - R$ where R is an α-paracompact closed subset of X. Since $\vec{U}$ is an open cover of R, there is a open locally finite refinement $\vec{M}$ of $\vec{U}$ which covers R. Therefore, $\vec{H} = \vec{M} \cup \{U_{\alpha_o}\}$ is an open locally finite refinement $\vec{U}$ that covers X. Hence, X is paracompact.

(iv) (X, τ) is dense in (X^*, τ^*), since for each open set U in X^*, we have $U \cap X \neq \phi$. □

Observe that (X^*, τ^*) in the above theorem represents the one point paracompactification of (X, τ).

4. Locally metacompact spaces

In this section we introduce the notion of locally metacompact spaces and obtain several results using almost similar techniques that used in the case of locally paracompact spaces.

Definition 4.1. *A space X is called locally metacompact if each point of X has an open neighbourhood U such that $\overline{U}$ is metacompact.*

Remark. Obviously every metacompact space is locally metacompact, however, the space Ω_o of Example **??** is an example of a locally metacompact space which is not metacompact.

Remark. Since every paracompact space is metacompact, it is clear that every locally paracompact space is locally metacompact.

Definition 4.2. *A subset A of a topological space (X, τ) is called α-metacompact if every cover of A by members of τ has a point finite refinement by members of τ which covers A.*

Remark. Clearly, if a subset A of a topological space (X, τ) is α-paracompact, then A is α-metacompact.

Remark. It is easy to see that if a subset A of a topological space (X, τ) is α-metacompact, then A is a metacompact subspace.

Theorem 4.1. *Let A be a closed metacompact subset of a space X and F be a closed subset of the interior of A. Then F is α-metacompact.*

Theorem 4.2. *A closed subset of an α-metacompact set is α-metacompact.*

Theorem 4.3. *Let X be a locally metacompact T_2-space. Then each point of X has a open neighbourhood U such that $\overline{U}$ is α-metacompact.*

Proof. Similar to the proof of Theorem 2.4. □

Theorem 4.4. *Every closed subspace of a locally metacompact space is locally metacompact.*

Proof. Let X be a locally metacompact space and F be closed in X. If $x \in F$, then there exists an open neighbourhood U of x such that $\overline{U}$ is metacompact. Now $U \cap F$ is open in $F$and $x \in U \cap F$. Also, $\overline{U \cap F}^F = \overline{U \cap F} \cap F$ is closed in X, and thus closed in $\overline{U}$, but $\overline{U}$ is metacompact, so, $\overline{U \cap F}^F$ is metacompact (as a closed subset of a metacompact space is metacompact). □

Theorem 4.5. *Let $X = \cup\{F_\alpha : \alpha \in \Lambda\}$, where $\{F_\alpha : \alpha \in \Lambda\}$ is a locally finite closed cover of X. Then X is locally metacompact if and only if each F_α is locally metacompact.*

The following lemma is a (probably) known result, the straightforward proof is omitted.

Lemma 4.6. *Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is metacompact. Then X is metacompact.*

Theorem 4.7. *Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is a locally metacompact space. Then X is a locally metacompact space.*

Proof. Let $f : X \overset{onto}{\longrightarrow} Y$ be a perfect function, where Y is locally metacompact space. Since Y is locally metacompact, for each $x \in X$, there is an open neighbourhood K of $f(x)$ such that $\overline{K}$ is metacompact. Since f is continuous, there is an an open neighbourhood G of x such that $f(G) \subseteq K$ and $f(\overline{G}) \subseteq \overline{f(G)}$. Thus, $f(\overline{G}) \subseteq \overline{K}$, and therefore, $\overline{G} \subseteq f^{-1}(\overline{K})$. Since $\overline{K}$ is closed and $f : X \overset{onto}{\longrightarrow} Y$ is perfect, so, $f : f^{-1}(\overline{K}) \overset{onto}{\longrightarrow} \overline{K}$ is perfect. Since $\overline{K}$ is metacompact, it follows from Lemma 4.6 that $f^{-1}(\overline{K})$ is metacompact, but $\overline{G}$ is closed in $f^{-1}(\overline{K})$, so, $\overline{G}$ is metacompact (as a closed subset of a metacompact space is metacompact). Hence, X is locally metacompact. □

Corollary 4.7.1. *The product of a compact space X and a locally metacompact space Y is a locally metacompact space.*

Proof. Let X be compact space and Y be a locally metacompact space. Since X is compact, the projection function $P_Y : X \times Y \to Y$ is a perfect onto function. Since Y is locally metacompact, it follows from Theorem 4.7 that $X \times Y$ is locally metacompact. □

Lemma 4.8. *A metacompact separable space X is Lindelöf.*

Proof. Let X be a metacompact separable space and let $\mathcal{U}$ be an open cover of X. Since X is metacompact, $\mathcal{U}$ has a point finite open refinement $\mathcal{V}$, say. We will show that $\mathcal{V}$ is countable, to do so, assume not, that is, assume that

$\mathcal{V}$ is uncountable. Since X is separable, X has a countable dense set A, say. Thus, $A \cap V \neq \emptyset$, for every $V \in \mathcal{V}$, but A is countable and $\mathcal{V}$ is uncountable, so there exists $x \in A$ such that x belongs to uncoutably many members of $\mathcal{V}$. This contradicts that X is metacompact. Thus, $\mathcal{V}$ is countable, but $\mathcal{V}$ is a refinement of $\mathcal{U}$, so $\mathcal{U}$ has a countable subcover. Hence, X is Lindelöf. □

Since every locally paracompact space is locally metacompact (Remark 4), the following theorem generalizes Theorem 2.12.

Theorem 4.9. *Let X be a locally metacompact separable space. Then X is locally Lindelöf.*

Proof. Since X is locally metacompact, each point x of X has a open neighbourhood U_x such that $\overline{U_x}$ is metacompact. Since X is separable and U_x is open, U_x is separable too, and thus $\overline{U_x}$ is also separable. By Lemma 4.8, $\overline{U_x}$ is Lindelöf. Hence, X is locally Lindelöf. □

Theorem 4.10. *Let (X, τ) be a locally metacompact T_2- space, $p \notin X$ and $X^* = X \cup \{p\}$. Let τ^* be a family of all subsets of X^* of the following types:*

1. *All open subsets of X.*
2. *All sets of the form $X^* - K$, where K is an α-metacompact closed subset of X.*
3. *The sets X^* and ϕ.*

   Then each of the following holds:

(i) *τ^* is a topology on X^*.*

(ii) *(X^*, τ^*) is a metacompact T_2 space.*

(iii) *(X, τ) is dense in (X^*, τ^*).*

The one point metacompactification of a topological space X is a metacompact space (X^*, τ^*) that contains (X, τ) as a dense subspace and $X^* - X$ is a singleton set.

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