Near BE-semigroups

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Abstract. In this paper, we introduce a new algebraic structure the so-called near BE-semigroup which is a generalization of a BE-semigroup. We provide some examples to show the existence of the structure. Furthermore, some properties of this structure are investigated.

Keywords: near BE-semigroups, unit divisors, deductive systems.

1. Introduction

Iseki and Tanaka defined BCK-algebra in [2] while Iseki defined BCI-algebra in [7]. It is well known that every BCK-algebra is a BCI-algebra, i.e. a BCI-algebra is a generalization of a BCK-algebra. Hu and Li defined another class of abstract algebras which is known as a BCH-algebra [6, 15]. They showed that

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every BCI-algebra is a BCH-algebra. For general information on BCK-algebras we refer to [3]. Neggers and Kim [12] gave the idea of a d-algebra which is a generalization of BCK-algebras, and they gave the idea of a B-algebra in [1]as well. They showed that it is equivalent to the idea of a group. Moreover, in [5] they discussed homomorphisms of B-algebras and explored some useful properties. Furthermore, the author in [10] introduced a new idea, known as a BH-algebra, which is another generalization of BCH, BCI and BCK-algebras. B-algebra was deeply studied by Walendziak in [16] in which he obtained other equivalent set of axioms for a B-algebra. The authors in [4] gave the idea of a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to a Boolean group. Moreover, in [11] the idea of a BM-algebra has been given which is a special case of B-algebras and they showed that a BM-algebra is equivalent to a 1-commutative B-algebra. In [8], the authors gave the idea of a BE-algebra which is a generalization of a BCK-algebra. In [13, 14], the authors gave the idea of ideals and upper sets in BE-algebras and obtained several properties of such ideals and upper sets. By using the idea of upper sets, they gave an equivalent condition for filters in BE-algebras. By combining BE-algebra and semigroup, Ahn and Kim gave the idea of a BE-semigroup and explored some elegant and interesting properties [9]. In the near past, a new generalization of BE-algebra, termed as PSRU-algebra, has been defined in [17] while in the paper [18] the author has worked on the generalizations of BCK-algebras and explored the implicative property of the generalized algebras.

2. Preliminaries

In this section, we define *near BE-semigroup* which is a generalization of a BE-semigroup. We give some examples and then some of their properties are discussed. Firstly, we recall the definition of BE-algebra [8].

Definition 2.1. *An algebraic structure ($\boldsymbol{W}$; $\diamond$, 1) is said to be a BE-algebra if the following conditions are satisfied:*

(*i*) $(\forall x \in \boldsymbol{W}), x \diamond x = 1,$

(*ii*) $(\forall x \in \boldsymbol{W}), x \diamond 1 = 1,$

(*iii*) $(\forall x \in \boldsymbol{W}), 1 \diamond x = x,$

(*iv*) $(\forall x, y, z \in \boldsymbol{W}), x \diamond (y \diamond z) = y \diamond (x \diamond z).$

The property (iv) *is known as exchange law. Also note that if ($\boldsymbol{W}$; $\diamond$, 1) is a BE-algebra, then $u \diamond (v \diamond u) = 1$ for any u, $v \in \boldsymbol{W}$. Let we define a relation "$\leq$" in the BE-algebra ($\boldsymbol{W}$; $\diamond$,1) as follows:*

Let a, $b \in \boldsymbol{W}$, then $a \leq b$ if and only if $a \diamond b = 1$.

Let us give some examples. For the following examples we refer to [8].

Example 2.1. (i) Let $\boldsymbol{W} = \{1, 2, 3, 4, 5, 6\}$ and define "$\diamond$" in $\boldsymbol{W}$ in the table given below:

$\diamond$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	2	4	4	5
3	1	1	1	4	4	4
4	1	2	3	1	2	3
5	1	1	2	1	1	2
6	1	1	1	1	1	1

Then $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra.

(ii) Let $\boldsymbol{W} = \{1, 2, 3, 4, 5\}$. Let "$\diamond$" be a binary operation in $\boldsymbol{W}$ defined in the following table:

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	1	1	1
3	1	2	1	1	1
4	1	1	1	1	1
5	1	2	3	4	1

Then $(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra.

A BE-algebra $(\boldsymbol{W}; \diamond, 1)$ is said to be self-distributive if $\forall\ u, v, w \in \boldsymbol{W}$, $u \diamond (v \diamond w) = (u \diamond v) \diamond (u \diamond w)$.

Let us state some properties of a self-distributive BE-algebra $(\boldsymbol{W}; \diamond, 1)$ which will be used later. For the following result we refer to [8].

Proposition 2.1. *Let $(\boldsymbol{W}; \diamond, 1)$ be a self-distributive BE-algebra and further let $u, v \in \boldsymbol{W}$ be such that $u \leq v$. Then $u \diamond w \leq v \diamond w$ for all $w \in \boldsymbol{W}$.*

Proof. It is straightforward. □

In the following definition, we introduce the new structure of *near BE-semigroup.*

Definition 2.2. *An algebraic system $(\boldsymbol{W}; \otimes, \diamond, 1)$ is said to be a near BE-semigroup if the following conditions are satisfied:*
(*i*) *$(\boldsymbol{W}; \otimes)$ is a semigroup,*
(*ii*) *$(\boldsymbol{W}; \diamond, 1)$ is a BE-algebra,*
(*iii*) *The operation "$\otimes$" is left distributive over the operation "$\diamond$" .*

Note that it will be more accurate to call what we have just defined a near left BE-semigroup. There is a similar definition of a near right BE-semigroup. Throughout in this paper, we shall consider a near left BE-semigroup and simply call it a near BE-semigroup. Also note that we shall use NBE-semigroup for a near BE-semigroup.

Let us give some examples in order to show the existence of the above structure.

Example 2.2. Let $\boldsymbol{W} = \{1, s, t, q\}$ and define two operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\diamond$	1	s	t	q
1	1	1	1	1
s	1	1	1	1
t	1	1	1	s
q	1	1	1	s

$\otimes$	1	s	t	q
1	1	s	t	q
s	1	1	t	q
t	1	s	1	q
q	1	1	1	1

Then, it can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup.

Example 2.3. Let $\boldsymbol{W} = \{1, s, t, r\}$ and define two operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\diamond$	1	s	t	r
1	1	1	1	1
s	1	1	1	1
t	1	1	1	1
r	1	s	1	r

$\otimes$	1	s	t	r
1	1	s	t	r
s	1	1	t	r
t	1	s	1	r
r	1	1	1	1

Then, it can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is an NBE-semigroup.

In the following, we provide some properties of NBE-semigroups. These properties are true for BE-semigroups as well.

Proposition 2.2. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be an NBE-semigroup. Then, the following statements hold:*

(i) $a \otimes 1 = 1, \forall a \in \boldsymbol{W}$,

(ii) $a \leq b \Rightarrow c \otimes a \leq c \otimes b, \forall a, b, c \in \boldsymbol{W}$.

Proof. (i) Let $a \in \boldsymbol{W}$, then

$a \otimes 1 = a \otimes (1 \diamond 1)$

$= (a \otimes 1) \diamond (a \otimes 1)$ ($\because$ by left distributive law)

$= 1$ ($\because$ by first property of a BE-algebra)

(ii) Let $a, b, c \in \boldsymbol{W}$ be such that $a \leq b$. Then

$(c \otimes a) \diamond (c \otimes b) = c \otimes (a \diamond b)$ ($\because$ by left distributive law)

$= c \otimes 1$ ($\because$ $a \diamond b = 1$)

$= 1$ ($\because$ by part (i))

Thus, $c \otimes a \leq c \otimes b$. □

3. Unit divisors

In this section, we define the unit divisors and characterize them by some of properties.

Definition 3.1. *In an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1), an element $p \neq 1$ is called a left unit divisor if their exists $q \neq 1 \in \boldsymbol{W} \ni p \otimes q = 1$. Similarly we can define a right unit divisor. An element $p \neq 1$ of $\boldsymbol{W}$ is said to be a unit divisor if it is both left and right unit divisor.*

In the following, we state and prove some properties for NBE-semigroups. These properties are true in case of BE-semigroups as well.

Theorem 3.1. *Let us suppose that the left cancellation law holds for "$\otimes$" in an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1), that is, $s \otimes g = s \otimes h \Rightarrow g = h \, \forall \, s \neq 1$, g, $h \in \boldsymbol{W}$, then $\boldsymbol{W}$ contains no left unit divisors.*

Proof. Let us suppose that the left cancellation law holds for "$\otimes$ " in $\boldsymbol{W}$. Let us suppose that s, $g \in \boldsymbol{W}$ such that $s \otimes g = 1$ and $s \neq 1$.

Then $s \otimes g = 1 = s \otimes 1$ ($\because$ by Proposition 2.2)

$\Rightarrow g = 1$ ($\because$ by left cancellative law)

Thus, $\boldsymbol{W}$ contains no left unit divisors. □

Theorem 3.2. *Let ($\boldsymbol{W}$; $\otimes$, $\diamond$,1) be an NBE-semigroup which contains no left unit divisors. Then, the left cancellation law holds for the operation "$\otimes$".*

Proof. Suppose that g, h, $k \in \boldsymbol{W}$ are such that $g \otimes h = g \otimes k$ and $g \neq 1$.

Therefore,

$g \otimes (h \diamond k) = (g \otimes h) \diamond (g \otimes k) = 1$ and

$g \otimes (k \diamond h) = (g \otimes k) \diamond (g \otimes h) = 1$.

As there is no left unit divisor in $\boldsymbol{W}$, so it follows that

$h \diamond k = 1 = k \diamond h \Rightarrow h = k$. □

4. Deductive systems

In this section, we define deductive systems and give some examples. Furthermore, we discuss some of their properties.

Definition 4.1. *A non-empty subset $\boldsymbol{G}$ of an NBE-semigroup ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) is said to be a deductive system (which is shortly denoted by DS) if the following conditions are satisfied:*

(*i*) $\boldsymbol{W} \otimes \boldsymbol{G} \subseteq \boldsymbol{G}$,

(*ii*) $\forall \, a \in \boldsymbol{G}$ *and* $\forall \, p \in \boldsymbol{W} \ni a \diamond p \in \boldsymbol{G} \Rightarrow p \in \boldsymbol{G}$.

Let us explain deductive systems by giving some examples.

Example 4.1. Let $\boldsymbol{W} = \{1, s, t, r\}$ and define the two binary operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ in the following tables:

$\otimes$	1	s	t	r
1	1	1	1	1
s	1	1	1	1
t	1	1	1	s
r	1	1	1	s

$\diamond$	1	s	t	r
1	1	s	t	r
s	1	1	t	r
t	1	1	1	r
r	1	1	1	1

It can be easily verified that ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) is an NBE-semigroup. Let us assume $\boldsymbol{G} = \{1,\ s\}$. Then $\boldsymbol{G}$ is a DS, but $\boldsymbol{E} = \{1,\ r\}$ is not a DS of $\boldsymbol{W}$, because $r \otimes r = s \notin \boldsymbol{E}$ or $r \diamond s = 1 \in \boldsymbol{E}$, $r \in \boldsymbol{E}$, but $s \notin \boldsymbol{E}$

Example 4.2. Let $\boldsymbol{W} = \{1,\ s,\ t,\ q\}$ and define the two binary operations "$\otimes$" and "$\diamond$" in $\boldsymbol{W}$ as follows:

$\otimes$	1	s	t	q
1	1	1	1	1
s	1	s	1	1
t	1	1	1	1
q	1	1	1	t

$\diamond$	1	s	t	q
1	1	s	t	q
s	1	1	t	q
t	1	1	1	q
q	1	1	1	1

It can be easily shown that ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) is an NBE-semigroup. Let $\boldsymbol{G} = \{1,\ t\}$, then $\boldsymbol{G}$ is a DS, but $\boldsymbol{E} = \{1,\ q\}$ is not an DS of $\boldsymbol{W}$, because $q \otimes q = t \notin \boldsymbol{E}$ or $q \diamond t = 1 \in \boldsymbol{E}$, $q \in \boldsymbol{E}$ but $t \notin \boldsymbol{E}$.

Let us state and prove some properties of deductive systems for NBE-semigroups. These properties hold in case of BE-semigroups as well. Firstly we define a special subset of a BE-algebra which will be used later.

Definition 4.2. *Let us suppose that (*$\boldsymbol{W}$*;* $\diamond$*, 1) is a BE-algebra, and let* $p,\ q \in \boldsymbol{W}$*. Then, we define:* $\boldsymbol{A}(p,q) = \{z \in \boldsymbol{W} : p \diamond (q \diamond z) = 1\}$.

It should be noted that $\boldsymbol{A}$ $(p,\ q)$ is clearly non-empty because we have 1, p, q $\in \boldsymbol{A}(p,\ q)$. Let us give an example in order to understand the above definition.

Example 4.3. In Example 2.1 (i), we have $\boldsymbol{A}(2,3) = \{1,2,3,4,5\}$ while in Example 2.1 (ii), we have $\boldsymbol{A}(4,5) = \{1,2,4,5\}$.

Proposition 4.1. *Let (*$\boldsymbol{W}$*;* $\otimes$*,* $\diamond$*, 1) be an NBE-semigroup and* $\boldsymbol{G}$ *be a DS. Then* $\boldsymbol{A}(p,q) \subseteq \boldsymbol{G}, \forall p,q \in \boldsymbol{G}$.

Proof. Suppose that $x \in \boldsymbol{A}(p,\ q)$, then $p \diamond (q \diamond x) = 1$ and $1 \in \boldsymbol{G}$. By the second property of a DS, it follows that $x \in \boldsymbol{G}$. This completes the proof. □

Theorem 4.1. *Let us assume that* $\{\boldsymbol{G}_i : i \in \boldsymbol{I}\}$ *is a family of DSs of an NBE-semigroup (*$\boldsymbol{W}$*;* $\otimes$*,* $\diamond$*, 1). Then* $\cap \boldsymbol{G}_i$ *is a DS of* $\boldsymbol{W}$*.*

Proof. It is straightforward. □

In the following definition, we introduce a particular subset of an NBE-semigroup.

Definition 4.3. *Let (*$\boldsymbol{W}$*;* $\otimes$*,* $\diamond$*, 1) be an NBE-semigroup and let* $\boldsymbol{G}$ *be a subset of* $\boldsymbol{W}$*. The intersection of all DSs of* $\boldsymbol{W}$ *containing* $\boldsymbol{G}$ *is said to be the DS generated by* $\boldsymbol{G}$*, and it is represented by the symbol* $<\boldsymbol{G}>$*.*

It is very clear from the above definition that if $\boldsymbol{G}$ and $\boldsymbol{E}$ are subsets of an NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ such that $\boldsymbol{G} \subseteq \boldsymbol{E}$, then $<\boldsymbol{G}> \subseteq <\boldsymbol{E}>$. Furthermore, if $\boldsymbol{G}$ is an DS of $\boldsymbol{W}$, then one can have $<\boldsymbol{G}>= \boldsymbol{G}$.

Before the next result, note that an NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ is said to be self-distributive if $(\boldsymbol{W}; \diamond, 1)$ is a self-distributive BE-algebra.

Theorem 4.2. *Let $(\boldsymbol{W}; \otimes, \diamond, 1)$ be a self-distributive NBE-semigroup and let $\boldsymbol{G}$ be a non-empty subset of $\boldsymbol{W}$ such that $\boldsymbol{W} \otimes \boldsymbol{G} \subseteq \boldsymbol{G}$. Then, we have*
$<\boldsymbol{G}>= \{a \in \boldsymbol{W} : q_n \diamond (\ldots \diamond (q_1 \diamond a)\ldots) = 1 \text{ for some } q_1,\ldots,q_n \in \boldsymbol{G}\}$.

Proof. Let $\boldsymbol{B} = \{a \in \boldsymbol{W} : q_n \diamond (\ldots \diamond (q_1 \diamond a)\ldots) = 1 \text{ for some } q_1,\ldots,q_n \in \boldsymbol{G}\}$. Assume that $b \in \boldsymbol{B}$, then we have some $q_1,\ldots,q_n \in \boldsymbol{G} \ni q_n \diamond (\ldots \diamond (q_1 \diamond b)\ldots)$ $=1$. Now, let $g \in \boldsymbol{W}$, then $1 = g \otimes 1 = g \otimes (q_n \diamond (\ldots \diamond (q_1 \diamond b)\ldots)) = (g \otimes q_n) \diamond (\ldots \diamond ((g \otimes q_1) \diamond (g \otimes b)) \ldots)$.

As $g \otimes q_i \in \boldsymbol{G}$ for $i = 1,\ldots,n$, it follows that $g \otimes b \in \boldsymbol{B}$. Take $g, a \in \boldsymbol{W}$ such that $a \diamond g \in \boldsymbol{B}$ and $a \in \boldsymbol{B}$. Then, by definition of $\boldsymbol{B}$, we have some $q_1,\ldots,q_n$, $r_1,\ldots, r_m \in \boldsymbol{G}$ such that $q_n \diamond (\ldots \diamond (q_1 \diamond (a \diamond g))\ldots) = 1$ and $r_m \diamond (\ldots \diamond (r_1 \diamond a)\ldots) = 1$.

From the fourth property of BE-algebra and from the above first equation it follows that $a \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) = 1$. In other words, we have $a \leq q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)$. Furthermore, from the above second equation and from Proposition 2.2, we have $1 \leq r_m \diamond (\ldots \diamond (r_1 \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots)$.

Therefore, $r_m \diamond (\ldots \diamond (r_1 \diamond (q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots) = 1$. This implies that $g \in \boldsymbol{B}$. Thus $\boldsymbol{B}$ is a DS of $\boldsymbol{W}$. Clearly $\boldsymbol{G} \subseteq \boldsymbol{B}$. Let $\boldsymbol{M}$ be another DS which contains $\boldsymbol{G}$. In order to show $\boldsymbol{B} \subseteq \boldsymbol{M}$, let us suppose that $a \in \boldsymbol{B}$. Then, we have some $q_1,\ldots,q_n \in \boldsymbol{G}$ such that $q_n \diamond (\ldots \diamond (q_1 \diamond a)...) = 1$. Thus, by definition of deductive systems, we have $a \in \boldsymbol{M}$ and thus $\boldsymbol{B} \subseteq \boldsymbol{M}$. This completes the proof. □

Furthermore, we have the following important fact.

Remark 4.1. The union of two DSs $\boldsymbol{G}$ and $\boldsymbol{E}$ of a self-distributive NBE-semigroup $(\boldsymbol{W}; \otimes, \diamond, 1)$ is not necessary to be a DS.

Example 4.4. Let $\boldsymbol{W} = \{1, s, t, u, v\}$ and define two binary operations in $\boldsymbol{W}$ in the following tables:

$\otimes$	1	s	t	u	v
1	1	1	1	1	1
s	1	1	1	1	1
t	1	1	1	1	1
u	1	1	1	1	s
v	1	1	1	1	s

$\diamond$	1	s	t	u	v
1	1	s	t	u	v
s	1	1	t	t	v
t	1	s	1	s	v
u	1	1	1	1	v
v	1	1	t	t	1

It can be easily verified that $(\boldsymbol{W}; \otimes, \diamond, 1)$ is a self-distributive NBE-semigroup. Let $\boldsymbol{G} = \{1, s\}$ and $\boldsymbol{E} = \{1, t\}$. Then $\boldsymbol{G}$ and $\boldsymbol{E}$ are DSs of $\boldsymbol{W}$. Although, we have $\boldsymbol{G} \cup \boldsymbol{E} = \{1, s, t\}$ is not a DS of $\boldsymbol{W}$, because $t \diamond u = s \in \boldsymbol{G} \cup \boldsymbol{E}$ but $u \notin \boldsymbol{G} \cup \boldsymbol{E}$.

In the following, we prove another property of self-distributive NBE-semigroups.

Theorem 4.3. *Let ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1) be a self-distributive NBE-semigroup and further assume that $\boldsymbol{G}$ and $\boldsymbol{E}$ are DSs of ($\boldsymbol{W}$; $\otimes$, $\diamond$, 1). Then, we have*

$<\boldsymbol{G}\cup\boldsymbol{E}> = \{a \in \boldsymbol{W} : p \diamond (q \diamond a) = 1 \text{ for some } p \in \boldsymbol{G},\ q \in \boldsymbol{E}\}$.

Proof. Let $\boldsymbol{T} = \{a \in \boldsymbol{W} : p \diamond (q \diamond a) = 1 \text{ for some } p \in \boldsymbol{G},\ q \in \boldsymbol{E}\}$. Clearly, $\boldsymbol{T} \subseteq <\boldsymbol{G}\cup\boldsymbol{E}>$. Let us suppose that $g \in <\boldsymbol{G}\cup\boldsymbol{E}>$. Then, by Theorem 4.2, we have some $q_1, \ldots, q_n \in \boldsymbol{G}\cup\boldsymbol{E} \ni q_n \diamond (\ldots \diamond (q_1 \diamond g)\ldots) = 1$. Let $q_i \in \boldsymbol{G}\ \forall\, i = 1, \ldots, n$, then $g \in \boldsymbol{G}$. Therefore $g \in \boldsymbol{T}$ as $g \diamond (1 \diamond g) = 1$. Similarly, if $q_i \in \boldsymbol{E}\ \forall\, i = 1, \ldots, n$, then $g \in \boldsymbol{E}$. Therefore $g \in \boldsymbol{T}$ as $1 \diamond (g \diamond g) = 1$. Let some of $q_1, \ldots, q_n \in \boldsymbol{G}$ and others belongs to $\boldsymbol{E}$, then we may suppose $q_1, \ldots, q_t \in \boldsymbol{G}$ and $q_{t+1}, \ldots, q_n \in \boldsymbol{E}$ for $1 \leq t \leq n$ without loss of generality. Let us assume that $u = q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)$. Then, we have $q_n \diamond (\ldots \diamond (q_{t+1} \diamond u)\ldots) = q_n \diamond (\ldots \diamond (q_{t+1} \diamond (q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)))\ldots) = 1$. Hence $u \in \boldsymbol{E}$. Now, let $v = u \diamond g = (q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) \diamond g$, then we have $q_t \diamond (\ldots \diamond (q_1 \diamond v)\ldots) = q_t \diamond (\ldots \diamond (q_1 \diamond ((q_t \diamond (\ldots \diamond (q_1 \diamond g)\ldots)) \diamond g)) \ldots)$

$= (q_t \diamond (\ldots \diamond (q_1 \diamond g) \ldots)) \diamond (q_t \diamond (\ldots \diamond (q_1 \diamond g) \ldots) = 1$.

It follows that $v \in \boldsymbol{G}$. As $u \diamond (v \diamond g) = v \diamond (u \diamond g) = v \diamond v = 1$, then $g \in \boldsymbol{T}$. This completes the proof. □

5. Conclusion

In this paper, we have studied near BE-semigroups. We have defined it in the usual way by excluding one of the distributive laws. It is a generalization of a BE-semigroup. We have seen that the said structure exists by constructing some non-trivial examples. Moreover, we have generalized some properties of BE-semigroups to near BE-semigroups. At the end, we have discussed different classes of near BE-semigroups and investaged some of their properties. We have defined near left BE-semigroup and named it near BE-semigroup but we have seen that we can define near right BE-semigroup as well by excluding left distributive law from a BE-semigroup. Furthermore, analogues of the results which are true for near left BE-semigroups can be proved.

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Accepted: October 30, 2020