A matrix expansion solution for a hyperbolic system of time-fractional PDEs with variable coefficients

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Abstract. In this research, we introduce a series solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of Caputo fractional derivative. An appropriate expansion of matrix functions is derived and used to create a series solution for the target problem and the residual power series method is also used to determine the coefficients of the series solution. To test our proposed method, we discuss four interesting and important applications. The first three applications are set up so that the exact solution is already known whereas the last application is set up without knowing the solution in advance to test the predictability of the solution or obtain a suitable approximate solution. Numerical results are analyzed to confirm the ability of the used method and to verify the solution obtained. The surface graphs of the solution are plotted to illustrate the behavior of the solution in various conditions. Mathematica 7 software is used to calculate the numerical and symbolic quantities.
Keywords: fractional partial differential equations, hyperbolic systems, power series.

1. Introduction

Many natural phenomena have been modeled through partial differential equations (PDEs), especially in physics, engineering, chemistry, and biology, as well as in humanities [5, 25]. A wide range of PDEs can be classified under the name of hyperbolic PDEs which has the following general form [25, 21, 15, 34, 29]:

$$u_t\,(x,t) = a\,(x,t)\,u_x\,(x,t) + b\,(x,t)\,u\,(x,t) + f(x,t), x \in S, t > 0, \tag{1}$$

subject to

$$u\,(x,0) = u_0\,(x)\,. \tag{2}$$

The equations of compressible fluid flow and the Euler equations are examples of PDEs that can be reduced to hyperbolic PDEs when the effects of viscosity and heat conduction are neglected [34]. In addition, many mathematical models are appearing as hyperbolic systems of PDEs that have the general form:

(3) $$U_t(x,t) = A(x,t)\,U_x(x,t) + B(x,t)\,U(x,t) + F(x,t),\ x \in S, t \geq 0,$$

subject to

(4) $$U(x,0) = U_0(x),$$

where $U(x,t)$, $F(x,t) \in M_{n\times 1}$, $n \in \mathbb{N}$ are vector functions of two variables x and t, $U_0(x) \in M_{n\times 1}$ is vector function of x, $A(x,t)$, $B(x,t) \in M_{n\times n}$ are matrix functions of two variables x and t, and $A(x_0,t_0)$ is diagonalizable with real eigenvalues for every $(x_0,t_0) \in S \times [0,\infty)$. The system in Eqs. (3) and (4) is said to be strictly hyperbolic if the eigenvalues of $A(x_0,t_0)$ are all distinct whereas it is said to be elliptic at a point (x_0,t_0) if none of the eigenvalues of $A(x_0,t_0)$ are real for every $(x_0,t_0) \in S \times [0,\infty)$.

In recent decades, many mathematical models have been reformulated using the concept of fractional calculus because they are found to reflect the phenomenon that has been modeled in a more precise and realistic way by replacing the ordinary derivative with a fractional derivative of the model. The concept of fractional calculus is derivatives and integrals of fractional order and dates back to the seventeenth century [24, 6] and has recently gained considerable interest because of its extensive use in widespread fields, for instance, biological, chemical, engineering and applied physics such as nonlinear oscillation, waves, and diffusion as we mentioned [24, 6, 33, 18, 23, 35]. In fact, from that date until now, there are many definitions of fractional differential and integral operators, the most popular definition is the Caputo fractional differential operator, $\mathfrak{D}_t^{\alpha}$, which is defined by the following formula [24, 6]:

(5) $$\mathfrak{D}_t^{\alpha} u(x,t) = \begin{cases} J_t^{m-\alpha} \frac{\partial^m}{\partial t^m} u(x,t), & m-1 < \alpha < m, t > t_0 \geq 0, \\ \dfrac{\partial^m}{\partial t^m} u(x,t), & \alpha = m, \end{cases}$$

where $m \in \mathbb{N}$ and J_t^{β} is the Riemann-Liouville fractional integral operator of order $\beta > 0$ with respect to $t \geq t_0 \geq 0$ which is defined by the following formula:

(6) $$J_t^{\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\tau)^{\beta-1} u(x,\tau)\, d\tau,\ t > \tau > t_0 \geq 0.$$

For more details about the properties of the two previous definitions, you can refer to the references [24, 6, 33, 18, 23, 35]. The most useful properties that we need in this research are:

1. $J_t^{\alpha}(t-t_0)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-t_0)^{\mu+\alpha}, \mu > -1, t \geq t_0 \geq 0.$
2. $J_t^{\alpha}\lambda = \frac{\lambda}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}, \lambda$ is a constant.

3. $\mathfrak{D}_t^{\alpha}(t-t_0)^{\mu}=\begin{cases}\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-t_0)^{\mu-\alpha}, & \mu\notin\{0,1,2,\ldots,m-1\}, \\ 0, & \mu\in\{0,1,2,\ldots,m-1\}\end{cases}.$

4. $\mathfrak{D}_t^{\alpha}\lambda=0, \lambda$ is a constant.

It is difficult to find exact solutions for fractional differential and integral equations, for this reason, it is usually used analytical and numerical methods to find approximate solutions to those equations. In recent decades, many methods have been used to find analytical and numerical solutions for fractional differential and integral equations such as the Adomian decomposition method [7], the homotopy perturbation method [28], the variational iteration method [28], the homotopy analysis method [8], and other methods [3, 9, 10].

In the last five years, the residual power series method (RPSM) has achieved an advanced rank among the methods used to find approximate solutions for many fractional differential and integral equations. It has been used in finding exact and approximate solutions for many equations such as homogeneous and nonhomogeneous time and space-fractional telegraph equation [12], linear and nonlinear neutral fractional pantograph Equations [14], space and time-fractional linear and nonlinear KdV-Burgers equation [11], multi-energy groups of neutron diffusion equations [31], fractional multi-pantograph system [13] and other equations. RPSM is characterized by its ease and speed in finding solutions for equations in the form of a power series. In fact, RPSM is a technique for finding the coefficients of the power series without having to find a recurrence relation which we normally obtain from equating the corresponding coefficients in the series. The RPSM is a good alternate proceeding for gaining analytic solutions for fractional PDEs.

Several articles are interested in providing analytical and numerical solutions to fractional PDEs of hyperbolic type, such as Caputo time fractional-order hyperbolic telegraph equation [32], time-fractional hyperbolic PDEs [17, 4, 27, 2, 1], time-fractional diffusion equation [22], fractional advection-dispersion flow equations [26] and other hyperbolic equations. However, a limited number of researches provided analytical as well as numerical solutions for hyperbolic systems of time-fractional PDEs. Kochubei [20] presented a numerical analytical solution for homogenous hyperbolic fractional systems, Hendy et al. [16] introduced a solution for two-dimensional fractional systems was provided by a separate contrast scheme. Therefore, more research is needed in providing analytical and numerical solutions for such systems due to their importance in many applications as mentioned above.

The present work aims to extend the application of RPSM to construct approximate solutions of a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of the Caputo's fractional derivative which are given in the form of the following model:

$$U_t^{(\alpha)}(x,t)=A(x,t)\,U_x^{(\beta)}(x,t)+B(x,t)\,U(x,t)+F(x,t), x\in S, t\geq 0, \tag{7}$$

subject to the following initial condition:

$$U(x,0) = U_0(x), \tag{8}$$

where $0 < \alpha, \beta \leq 1$, $U_t^{(\alpha)}(x,t) = \mathfrak{D}_t^{\alpha} U(x,t)$ refers to the Caputo's time-fractional derivative of order α of the multivariable vector function $U(x,t)$, $U_x^{(\beta)}(x,t) = \mathfrak{D}_x^{\beta} U(x,t)$ refers to the Caputo's space-fractional derivative of order β of the multivariable vector function $U(x,t)$ and the definitions of all terms and variables in the Eqs. (7) and (8) are the same as those in Eqs. (3) and (4).

This paper is arranged as follows: In Section 2, the analysis of matrix fractional power series is prepared. In Section 3, the construction of fractional power series solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients in the sense of the Caputo's fractional derivative is presented using RPSM. In Section 4 application models, graphical and numerical simulations are performed in order to illustrate the capability and the simplicity of the proposed method. Finally, conclusions are presented in Section 5.

2. Analysis of matrix fractional power series

In this section, we present some definitions and theories regarding matrix analysis and the matrix power series which are playing a central role in constructing the RPSM solution to a hyperbolic system of time-fractional partial differential equations with variable coefficients.

Definition 2.1 ([19]). *The Riemann-Liouville fractional integral operator of order $\alpha = 0$ of a matrix functions $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$ is defined as:*

$$J_t^{\alpha} U(x,t) = [J_t^{\alpha} u_{ij}(x,t)]_{r\times k},\ 1 \leq i \leq r,\ 1 \leq j \leq k. \tag{9}$$

Definition 2.2 ([19]). *The Caputo's time-fractional derivative operator of order α of a matrix function $U(x,t) = [u_{ij}(x,t)] \in M_{r\times k}$ is:*

$$\mathfrak{D}_t^{\alpha} U(x,t) = [\mathfrak{D}_t^{\alpha} u_{ij}(x,t)]_{r\times k},\ 1 \leq i \leq r,\ 1 \leq j \leq k. \tag{10}$$

Lemma 2.1. *If $m-1 < \alpha = m$ and $m \in \mathbb{N}$, then*

$$(i)\qquad \mathfrak{D}_t^{\alpha} J_t^{\alpha} U(x,t) = U(x,t),$$

$$(ii)\qquad J_t^{\alpha} \mathfrak{D}_t^{\alpha} U(x,t) = U(x,t) - \sum_{j=0}^{m-1} \frac{\partial^j U(x,0^+)}{\partial t^j} \frac{(t-t_0)^j}{j!},\ t > t_0.$$

Definition 2.3. *Let $A_k \in M_{m\times n}$. We say that a sequence $\{A_k\}$ converges to a matrix $A \in M_{m\times n}$ with respect to a matrix norm $\|\bullet\|$ on $M_{m\times n}$ if and only if $\lim_{k\to\infty} \|A_k - A\| = 0$. If $\{A_k\}$ converges to A with respect to $\|\bullet\|$, we write $\lim_{k\to\infty} A_k = A$.*

Definition 2.4. *For $0<\alpha\leq 1$, $x\in S$, and $t\geq t_0$ a matrix power series (MPS) of the form:*

$$\sum_{m=0}^{\infty} H_m(x)(t-t_0)^{\,m\alpha} = H_0(x)+H_1(x)(t-t_0)^{\alpha}+H_2(x)(t-t_0)^{2\alpha}+\dots, \tag{11}$$

is called a bivariate fractional MPS around t_0, where t is a variable and $H_m(x)\in M_{r\times k}$ are matrix functions called series coefficients.

Theorem 2.1. *Assume that $U(x,t)=[u_{ij}(x,t)]\in M_{r\times k},\ 1\leq i\leq r,\ 1\leq j\leq k$ such that $u_{ij}(x,t)\in C(S\times[t_0,t_0+T))$ and $\mathfrak{D}_t^{w\alpha}u_{ij}(x,t)\in C(S\times(t_0,t_0+T))$ for $1\leq i\leq r,\ 1\leq j\leq k,\ w=0,1,2,\dots,n+1$ where $\mathfrak{D}_t^{w\alpha}=\mathfrak{D}_t^{\alpha}\,.\,\mathfrak{D}_t^{\alpha}\,\dots\,\mathfrak{D}_t^{\alpha}$ (w-times) and $\alpha>0$. Then*

$$J_t^{(n+1)\alpha}\mathfrak{D}_t^{(n+1)\alpha}U(x,t)=\frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha},\ t_0\leq\xi\leq t<t_0+T. \tag{12}$$

Proof. Of the operator definition in Eqs. (6) and (9), we have

$$\begin{aligned}
J_t^{(n+1)\alpha}\mathfrak{D}_t^{(n+1)\alpha}U(x,t) &= \frac{1}{\Gamma((n+1)\alpha)}\int_{t_0}^{t}(t-y)^{(n+1)\alpha-1}\mathfrak{D}_y^{(n+1)\alpha}U(x,y)\,dy\\
&= \frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha)}\int_{t_0}^{t}(t-y)^{(n+1)\alpha-1}dy\\
&\quad \text{(Based on the second mean value theorem for integral)}\\
&= \frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha},\ t_0\leq\xi\leq t<t_0+T. \quad \square
\end{aligned}$$

Theorem 2.2. *Assume that $U(x,t)=[u_{ij}(x,t)]\in M_{r\times k},\ 1\leq i\leq r,\ 1\leq j\leq k$ such that $u_{ij}(x,t)\in C(S\times[t_0,t_0+T))$ and $\mathfrak{D}_t^{w\alpha}u_{ij}(x,t)\in C(S\times(t_0,t_0+T))$ for $1\leq i\leq r,\ 1\leq j\leq k,\ w=0,1,2,\dots,n+1$ where $\alpha\in(0,1]$. Then*

$$\begin{aligned}
U(x,t) &= \sum_{m=0}^{n}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}\\
&\quad +\frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha},\ t_0\leq\xi\leq t\leq t_0+T.
\end{aligned} \tag{13}$$

Proof. From Theorem 2.1, it suffices to demonstrate that:

$$J_t^{(n+1)\alpha}\mathfrak{D}_t^{(n+1)\alpha}U(x,t)=U(x,t)-\sum_{m=0}^{n}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}.$$

According to Lemma 2.1, it is easy to show that the formula is correct for $n = 0$ and $n = 1$. Thus, inductively, we prove the theorem as follows:

$$\begin{aligned} J_t^{(n+2)\alpha}\mathfrak{D}_t^{(n+2)\alpha}U(x,t) &= J_t^{\alpha}\left(\left(J_t^{(n+1)\alpha}\mathfrak{D}_t^{(n+1)\alpha}\right)\mathfrak{D}_t^{\alpha}U(x,t)\right) \\ &= J_t^{\alpha}\left(\mathfrak{D}_t^{\alpha}U(x,t) - \sum_{m=0}^{n}\frac{\mathfrak{D}_t^{(m+1)\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}\right) \\ &= J_t^{\alpha}\mathfrak{D}_t^{\alpha}U(x,t) - \sum_{m=0}^{n}\frac{\mathfrak{D}_t^{(m+1)\alpha}U(x,t_0)}{\Gamma((m+1)\alpha+1)}(t-t_0)^{(m+1)\alpha} \\ &= U(x,t) - U(x,t_0) - \sum_{m=1}^{n+1}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha} \\ &= U(x,t) - \sum_{m=0}^{n+1}\frac{\mathfrak{D}_t^{m\alpha}U(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha}. \end{aligned}$$

Thus, the proof of Theorem 2.2 is completed. □

Let's call the series (13); bivariate fractional matrix Taylor's formula (BFMTF) of the matrix function $U(x,t)$. As any series, the tail of the series (13), $R_n(x,t) = \frac{\mathfrak{D}_t^{(n+1)\alpha}U(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha}$, is called the nth remainder for the Taylor series of $U(x,t)$. The function $P(x,t) = U(x,t) - R_n(x,t)$ is an approximate function for $U(x,t)$ and the accuracy of the approximation increases as $R_n(x,t)$ decreases. Finding a bound for $R_n(x,t)$ gives an indication of the accuracy of the approximation $P(x,t) \approx U(x,t)$. The following corollary provides such a bound.

Corollary 2.1 (The Remainder Estimation)**.** *Assume that* $\mathfrak{D}_t^{(n+1)\alpha}U(x,t)$, $\alpha \in (0,1]$ *is defined on* $(S \times (t_0, t_0+d))$. *If* $\left\|\mathfrak{D}_t^{(n+1)\alpha}U(x,t)\right\| \le M(x)$ *on* $t_0 \le t \le d$ *and fixed* x *for some matrix norm* $\|\bullet\|$, *then the reminder* $R_n(x,t)$ *of the BFMTF of* $U(x,t)$ *satisfies:*

$$\left\|\mathfrak{D}_t^{(n+1)\alpha}U(x,t)\right\| \le \frac{M(x)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha}, t_0 \le t \le d. \tag{14}$$

Note that when $n \to \infty$, the Taylor's formula (13) becomes of the form:

$$U(x,t) = \sum_{m=0}^{\infty}\frac{U_t^{(m\alpha)}(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha},\ x\in S, t_0\le t<t_0+T, \tag{15}$$

which can be applied directly throughout this work.

Finally, it is worth to mention that if $\alpha = 1$, then the BFMTF (15) becomes as follows:

$$U(x,t) = \sum_{m=0}^{\infty}\frac{\partial^m U(x,t_0)}{m!\partial t^m}(t-t_0)^m,\ t_0\le t<t_0+T, \tag{16}$$

which is the bivariate classical matrix Taylor's formula of a matrix function.

3. Construction of the RPSM solution for a hyperbolic system

In this section, we construct an approximate solution to the hyperbolic system of time-fractional PDEs with variable coefficients given in Eqs. (7) and (8) by using RPSM. To achieve it, we rewrite the Eqs. (7) and (8) as follow:

$$U_t^{(\alpha)}(x,t) - A(x,t)\,U_x^{(\beta)}(x,t) - B(x,t)\,U(x,t) - F(x,t) = 0, \tag{17}$$

$0 < \alpha, \beta \leq 1,\ x \in S, t \geq 0$, subject to the initial condition:

$$U(x,0) = U_0(x). \tag{18}$$

Suppose that the solution of the IVP (17) and (18) has the following expansion:

$$U(x,t) = \sum_{m=0}^{\infty} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S, \quad 0 \leq t < R, \tag{19}$$

where $H_m(x) = U_t^{(m\alpha)}(x,0) \in M_{r\times 1},\ n = 0,1,2,\ldots,$ and $A(x,t), B(x,t)$ & $F(x,t)$ have the following BFMTF:

$$\begin{aligned} A(x,t) &= \sum_{m=0}^{\infty} \frac{A_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t, \\ B(x,t) &= \sum_{m=0}^{\infty} \frac{B_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t, \\ F(x,t) &= \sum_{m=0}^{\infty} \frac{F_t^{(m\alpha)}(x,0)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t. \end{aligned} \tag{20}$$

Of course, dealing with a finite series is easier than dealing with an infinite series. For this reason, the RPSM deals with a finite series while calculating coefficients of the series solution. Therefore, we define the kth-truncated series of $U(x,t)$ as follows:

$$U_k(x,t) = \sum_{m=0}^{k} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha},\ 0 < \alpha \leq 1, x \in S,\ 0 \leq t < R. \tag{21}$$

$U(x,t)$ satisfies the initial conditions (18), so from Eq. (19), we obtain $U(x,0) = H_0(x) = H(x)$. On the other hand, from Eq. (21) the initial guess approximation of $U(x,t)$ should be $U_0(x,t) = H_0(x)$. As a result, we can reformulate the expansion of Eq. (21) in the following form:

$$U_k(x,t) = H(x) + \sum_{m=1}^{k} \frac{H_m(x)}{\Gamma(1+m\alpha)} t^{m\alpha}, \tag{22}$$

$$0 < \alpha \leq 1, x \in S,\ 0 \leq t < R,\ k = 1,2,3,\ldots$$

To apply the RPSM for finding the value of coefficients $H_m(x), m{=}1,2,3,\ldots,k$ in the series expansion of Eq. (22), we define the residual matrix function (RMF) for Eq. (17) as:

$$\text{RMF}\,(x,t) = U_t^{(\alpha)}\,(x,t) - A\,(x,t)\,U_x^{(\beta)}\,(x,t) - B\,(x,t)\,U\,(x,t) - F\,(x,t)\,, \tag{23}$$

and the k-th residual matrix function (RMF_k) of the style form:

$$\begin{aligned}\text{RMF}_k\,(x,t) = {} & (U_k)_t^{(\alpha)}\,(x,t) - A\,(x,t)\,(U_k)_x^{(\beta)}\,(x,t) \\ & - B\,(x,t)\,U_k\,(x,t) - F\,(x,t)\,,\end{aligned} \tag{24}$$

where $0 < \alpha, \beta \leq 1, x \in S,\ 0 \leq t < t_0 + R$.

Substitute the expansions of $U\,(x,t)\,, A\,(x,t)\,, B\,(x,t)$ and $F\,(x,t)$ into Eqs. (23) and (24) to obtain the following form of the RMF and RMF_k functions:

$$\begin{aligned}\text{RMF}\,(x,u) = {} & \sum_{m=0}^{\infty} \mathcal{H}_{(m+1)}\,(x) t^{m\alpha} - \sum_{m=0}^{\infty} \mathcal{A}_m\,(x)\,t^{m\alpha} \\ & - \sum_{m=0}^{\infty} \mathcal{B}_m\,(x)\,t^{m\alpha} - \sum_{m=0}^{\infty} \mathcal{F}_m\,(x)\,t^{m\alpha},\end{aligned} \tag{25}$$

and

$$\text{RMF}_k\,(x,t) = \sum_{m=0}^{k} \big(\mathcal{H}_{(m+1)}\,(x) - \mathcal{A}_m\,(x) - \mathcal{B}_m\,(x) - \mathcal{F}_m(x)\big) t^{m\alpha}, \tag{26}$$

where

$$\begin{aligned}\mathcal{H}_{(m+1)}\,(x) &= \frac{H_{(m+1)}(x)}{\Gamma\,(1+m\alpha)}, \\ \mathcal{A}_m\,(x) &= \sum_{i=0}^{m} \frac{A_t^{(i\alpha)}\,(x,0)}{\Gamma\,(1+i\alpha)} \frac{H_{m-i}^{(\beta)}\,(x)}{\Gamma\,(1+(m-i)\alpha)}, \\ \mathcal{B}_m\,(x) &= \sum_{i=0}^{m} \frac{B_t^{(i\alpha)}\,(x,0)}{\Gamma\,(1+i\alpha)} \frac{H_{m-i}(x)}{\Gamma\,(1+(m-i)\alpha)}, \\ \mathcal{F}_m\,(x) &= \frac{F_t^{(m\alpha)}\,(x,0)}{\Gamma\,(1+m\alpha)}.\end{aligned} \tag{27}$$

The main idea of the RPSM can be shown in the following facts:

1. $\text{RMF}(x,t){=}\ 0$.
2. $\lim_{k\to\infty} \text{RMF}_k\,(x,t) = \text{RMF}\,(x,t)$ for each $x \in S$ and $0 \leq t < R$.
3. $\mathfrak{D}_t^{(j-1)\alpha}\text{RMF}_k\,(x,0) = 0,\ j = 1,2,3,\ldots$ (by (4)).

4. $\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}\,(x,0) = \mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,0),\ j = 1,2,3,\ldots.$

For the form of the coefficients formulas $H_m\,(x),\ m = 1,2,3,\ldots,k$ in Eq. (22), we apply the operator $\mathfrak{D}_t^{(j-1)\alpha}$ on the Eq. (26) to get the following formula:

$$\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,t) =$$

$$\sum_{m=j-1}^{k}\left(\mathcal{H}_{(m+1)}\,(x) - \mathcal{A}_m\,(x) - \mathcal{B}_m\,(x) - \mathcal{F}_m(x)\right)\frac{\Gamma\,(1+m\alpha)\,t^{(m-j+1)\alpha}}{\Gamma\,(1+(m-j+1)\alpha)}. \tag{28}$$

For $j = 1$, the fact $\mathfrak{D}_t^{(j-1)\alpha}\mathrm{RMF}_k\,(x,0) = 0$ and formula (28) lead to the algebraic equation $\mathcal{H}_1\,(x) - \mathcal{A}_0\,(x) - \mathcal{B}_0\,(x) - \mathcal{F}_0\,(x) = 0$. According to the relations in Eq. (27), we have the second coefficient in the series (21) which is given in the following formula:

$$H_1\,(x) = A\,(x,0)\,H^{(\beta)}\,(x) + B\,(x,0)\,H\,(x) + F(x,0). \tag{29}$$

Similarly, for $j = 2$, the third coefficient of the series (21) has the following form:

$$\begin{aligned} H_2\,(x) &= (\mathcal{A}_1\,(x) + \mathcal{B}_1\,(x) + \mathcal{F}_1\,(x))\ \ \Gamma\,(1+\alpha) \\ &= A\,(x,0)\,H_1^{(\beta)}\,(x) + A_t^{(\alpha)}\,(x,0)\,H^{(\beta)}\,(x) + B\,(x,0)\,H_1\,(x) \\ &\quad + B_t^{(\alpha)}\,(x,0)\,H\,(x) + F_t^{(\alpha)}\,(x,0)\,. \end{aligned} \tag{30}$$

This procedure can be repeated up to arbitrary order coefficients of the bivariate fractional MPS solution of Eqs. (17) and (18) that are obtained. So, it is clear that for $j = k$, the coefficient $H_k(x)$ of the series (21) will be given in the following recurrence relation:

$$\begin{aligned} &H_0\,(x) = H\,(x) \\ &H_k(x) = \ \Gamma(1+(k-1)\alpha)(\mathcal{A}_{k-1}(x) + \mathcal{B}_{k-1}(x) + \mathcal{F}_{k-1}(x)), k = 1,2,\ldots, \end{aligned} \tag{31}$$

where $\mathcal{A}_{m-1},\ \mathcal{B}_{m-1}$ and $\mathcal{F}_{m-1}$ are given in Eq. (27). Therefore, the exact solution of the IVP (17) and (18) will be given in the following series form:

$$\begin{aligned} U(x,t) =& H(x) + \sum_{m=1}^{\infty}(\mathcal{A}_{m-1}(x) + \mathcal{B}_{m-1}(x) \\ &+ \mathcal{F}_{m-1}(x))\frac{\Gamma(1+(m-1)\alpha)}{\Gamma\,(1+m\alpha)}t^{m\alpha}, \end{aligned} \tag{32}$$

and the kth-approximation of the solution of the IVP (17) and (18) will be as follows:

$$\begin{aligned} U_k\,(x,t) =& H\,(x) + \sum_{m=1}^{k}(\mathcal{A}_{m-1}\,(x) + \mathcal{B}_{m-1}\,(x) \\ &+ \mathcal{F}_{m-1}\,(x))\frac{\Gamma\,(1+(m-1)\,\alpha)}{\Gamma\,(1+m\alpha)}t^{m\alpha}, \end{aligned} \tag{33}$$

where $x \in S,\ 0 \le t < R$.

4. Applications and numerical simulations

To test our proposed method, we present in this section four interesting and important applications. The first three applications are set up so that the exact solution is already known whereas the last application is set up without knowing the solution in advance to test the predictability of the solution or obtain a suitable approximate solution. All symbolic and numerical calculations were performed using Mathematica 7 software.

Application 4.1. *Consider the following homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$U_t^{(\alpha)}(x,t) - A(x,t)\,U_x^{(1)}(x,t) - B(x,t)\,U(x,t) = 0,\ 0 < \alpha \le 1, x \in \mathbb{R}, t \ge 0, \tag{34}$$

subject to the following initial condition:

$$U(x,0) = \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{35}$$

where

$$A(x,t) = \begin{pmatrix} t^{\alpha}x & \frac{1}{2x} \\ x^2 & -\frac{t^{3\alpha}}{2x} \end{pmatrix},\quad B(x,t) = \begin{pmatrix} t^{4\alpha}x - t^{\alpha} & -t^{2\alpha} \\ \frac{t^{5\alpha}+t^{2\alpha}x^4}{x} + \frac{t^{\alpha}x\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} & -x^2 \end{pmatrix},$$

and the exact solution is $U(x,t) = \begin{pmatrix} x \\ 1 + t^{2\alpha}x^2 \end{pmatrix}$.

To obtain a matrix expansion solution for this application using RPSM, assume that the solution takes the form of the series in Eq. (19). According to the initial condition in Eq. (35), the first coefficient of the expansion Eq. (19) $H_0(x) = H(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}$. The coefficients of expansion are computed by recurrence relation in Eq. (31) and the formulas in Eq. (27). The functions of Eq. (27) for this application has the following expressions

$$\begin{aligned}
&\mathcal{A}_0(x) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix},\quad \mathcal{B}_0(x) = \begin{pmatrix} 0 \\ -x^2 \end{pmatrix},\\
&\mathcal{A}_1(x) = \begin{pmatrix} x \\ 0 \end{pmatrix},\quad \mathcal{B}_1(x) = \begin{pmatrix} -x \\ \frac{x^2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \end{pmatrix},\\
&\mathcal{A}_2(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \mathcal{B}_2(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\\
&\mathcal{A}_m(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad \mathcal{B}_m(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad m = 3,4,5,\ldots.
\end{aligned} \tag{36}$$

So, by Eq. (31), the coefficients of the expansion in Eq. (19) are given by the following vector functions:

$$H_0(x) = \begin{pmatrix} x \\ 1 \end{pmatrix},$$
$$H_1(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$H_2(x) = \begin{pmatrix} 0 \\ x^2 \Gamma(1+2\alpha) \end{pmatrix}, \tag{37}$$
$$H_m(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad m = 4, 5, 6, \dots .$$

Therefore, the matrix expansion solution of the IVP (34) and (35) would be as follows:

$$U(x,t) = \begin{pmatrix} x \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 \end{pmatrix} t^{2\alpha}, \tag{38}$$

which coincides with the exact solution.

Application 4.2. *Consider the following non-homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$\begin{aligned} &U_t^{(\alpha)}(x,t) + A(x,t)\, U_x^{(1)}(x,t) + B(x,t)\, U(x,t) \\ &= F(x,t), 0 < \alpha \le 1, x \in \mathbb{R},\ t \ge 0, \end{aligned} \tag{39}$$

subject to the following initial condition:

$$U(x,0) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \tag{40}$$

where

$$A(x,t) = \begin{pmatrix} x & t^\alpha \\ 2x + t^{2\alpha} & 0 \end{pmatrix}, \quad B(x,t) = \begin{pmatrix} t^\alpha & x \\ 0 & 2x + t^{2\alpha} \end{pmatrix},$$
$$F(x,t) = \begin{pmatrix} x^3 + e^x \Gamma(1+\alpha) + (2+e^x)\, x t^\alpha + (x + e^x)\, t^{2\alpha} \\ 2x^3 + \left(2x e^x + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right) t^\alpha + (2+x)\, x t^{2\alpha} + e^x t^{3\alpha} + t^{4\alpha} \end{pmatrix}.$$

According to the construction in Section 3, assume that the IVP (39) and (40) has a matrix expansion solution as in Eq. (19). The initial coefficient of the expansion in Eq. (19) is determined from the initial condition Eq. (40) which is equal to $H_0(x) = H(x) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$. By the coefficients of Eq. (39), $A(x,t)$,

$B(x,t)$ and $F(x,t)$, the functions in Eq. (27) can be determined as follows:

$$\begin{aligned}
&\mathcal{A}_0(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_0(x)=\begin{pmatrix}-x^3\\-2x^3\end{pmatrix},\quad \mathcal{F}_0(x)=\begin{pmatrix}x^3+\mathrm{e}^x\Gamma(1+\alpha)\\2x^3\end{pmatrix},\\
&\mathcal{A}_1(x)=\begin{pmatrix}-(2+\mathrm{e}^x)x\\-2\mathrm{e}^x x\end{pmatrix},\quad \mathcal{B}_1(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_1(x)=\begin{pmatrix}(2+\mathrm{e}^x)x\\2\mathrm{e}^x x+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\end{pmatrix},\\
&\mathcal{A}_2(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_2(x)=\begin{pmatrix}-\mathrm{e}^x-x\\-x(2+x)\end{pmatrix},\quad \mathcal{F}_2(x)=\begin{pmatrix}\mathrm{e}^x+x\\x(2+x)\end{pmatrix},\\
&\mathcal{A}_3(x)=\begin{pmatrix}0\\-\mathrm{e}^x\end{pmatrix},\quad \mathcal{B}_3(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_3(x)=\begin{pmatrix}0\\\mathrm{e}^x\end{pmatrix},\\
&\mathcal{A}_4(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_4(x)=\begin{pmatrix}0\\-1\end{pmatrix},\quad \mathcal{F}_4(x)=\begin{pmatrix}0\\1\end{pmatrix},\\
&\mathcal{A}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{B}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad \mathcal{F}_m(x)=\begin{pmatrix}0\\0\end{pmatrix},\quad m=5,6,7,\dots.
\end{aligned}\tag{41}$$

These functions are the basic components of the expansion coefficients in Eq. (19) and by compensating them in recurrence relation in Eq. (31), we get those coefficients which are given as follows:

$$\begin{aligned}
H_0(x)&=\begin{pmatrix}0\\x^2\end{pmatrix},\\
H_1(x)&=\begin{pmatrix}\mathrm{e}^x\Gamma(1+\alpha)\\0\end{pmatrix},\\
H_2(x)&=\begin{pmatrix}0\\\Gamma(1+2\alpha)\end{pmatrix},\\
H_m(x)&=\begin{pmatrix}0\\0\end{pmatrix},\quad m=3,4,5,\dots .
\end{aligned}\tag{42}$$

Thus, the matrix expansion solution of the IVP (39) and (40) would be as follows:

$$U(x,t)=\begin{pmatrix}0\\x^2\end{pmatrix}+\begin{pmatrix}\mathrm{e}^x\\0\end{pmatrix}t^\alpha+\begin{pmatrix}0\\1\end{pmatrix}t^{2\alpha},\tag{43}$$

which is coinciding with the exact solution $U(x,t)=\begin{pmatrix}\mathrm{e}^x t^\alpha\\x^2+t^{2\alpha}\end{pmatrix}$.

Application 4.3. *Consider the following non-homogeneous hyperbolic system of time-Space-fractional PDEs with variable coefficients:*

$$\begin{aligned}
&U_t^{(\alpha)}(x,t)+A(x,t)\,U_x^{(\beta)}(x,t)+B(x,t)\,U(x,t)\\
&=F(x,t),0<\alpha,\beta\le 1,x,t\ge 0,
\end{aligned}\tag{44}$$

subject to the following initial condition:

$$U(x,0)=\begin{pmatrix}0\\x^{\beta}\end{pmatrix}, \tag{45}$$

where

$$A(x,t)=\begin{pmatrix}x^{\beta}&0\\0&x^{\beta}\end{pmatrix},\quad B(x,t)=\begin{pmatrix}1&t^{\alpha}\\t^{\alpha}&1\end{pmatrix},$$
$$F(x,t)=\begin{pmatrix}t^{\alpha}x^{\beta}E_{\alpha}(t^{\alpha})+\left(t^{\alpha}+t^{\alpha}x^{\beta}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\right)t^{\alpha}E_{\beta}\left(x^{\beta}\right)\\(2+\Gamma(1+\beta))\,x^{\beta}E_{\alpha}(t^{\alpha})+t^{3\alpha}E_{\beta}\left(x^{\beta}\right)\end{pmatrix},$$

and the exact solution is

$$U(x,t)=\begin{pmatrix}t^{2\alpha}E_{\beta}\left(x^{\beta}\right)\\x^{\beta}E_{\alpha}(t^{\alpha})\end{pmatrix}, \tag{46}$$

where $E_{\alpha}(t)$ is the Mittag-Leffler function defined by the following expansion [30]:

$$E_{\alpha}(t)=\sum_{m=0}^{\infty}\frac{t^{m}}{\Gamma(1+m\alpha)}. \tag{47}$$

All symbolic and numerical calculations were done using Mathematica 7 through a low-Ram PC. Since the Mittag-Leffler function is an infinite expansion, it was difficult to perform the calculations using the Mittag-Leffler function as it is. For this the 5th-truncated of the expansion in Eq. (47) was used through the calculations.

Like the previous applications, assume that the solution of the IVP (44) and (45) has a matrix expansion as in Eq. (19). According to the initial conditions in Eq. (45), $H_0(x)=H(x)=\begin{pmatrix}0\\x^{\beta}\end{pmatrix}$. To determine the rest coefficients of the series Eq. (19), we need to compute the basic functions in Eq. (27) and use them in the recurrence relation in Eq. (31). The first few coefficients are given in the following vectors:

$$H_0(x)=\begin{pmatrix}0\\x^{\beta}\end{pmatrix},$$
$$H_1(x)=\begin{pmatrix}0\\x^{\beta}\end{pmatrix},$$
$$H_2(x)=\begin{pmatrix}\mathit{\Gamma}(1+2\alpha)\,E_{\beta}\left(x^{\beta}\right)\\x^{\beta}\end{pmatrix},$$

$$H_3(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}, \quad H_4(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}, \quad H_5(x) = \begin{pmatrix} 0 \\ x^\beta \end{pmatrix}. \tag{48}$$

So, the 5th-approximation of the solution of the IVP (44) and (45) can be expressed as follows:

$$\begin{aligned} U_5(x,t) = {} & \begin{pmatrix} 0 \\ x^\beta \end{pmatrix} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix} \frac{t^\alpha}{\Gamma(1+\alpha)} + \begin{pmatrix} \Gamma(1+2\alpha)E_\beta(x^\beta) \\ x^\beta \end{pmatrix} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix} \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix} \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \begin{pmatrix} 0 \\ x^\beta \end{pmatrix} \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}. \end{aligned} \tag{49}$$

Obvious, there is a pattern between the terms of Eq. (49) which gives us the exact solution as in Eq. (46).

The mathematical behavior of the solution of the IVP (44) and (45) are illustrated next by plotting the 3-dimensional space figures of the 5th-approximation of the two components of the vector solution in Eq. (49) for different cases. Figures 1 (a), (b) and (c) show the 5th-approximate solution, $(U_1)_5(x,t)$ and $(U_2)_5(x,t)$, when $(\alpha,\beta) = (0.7, 0.4)$, $(\alpha,\beta) = (0.4, 0.7)$ and $(\alpha,\beta) = (0.9, 1)$, respectively on the square $[0,1] \times [0,1]$. Figure 1 (d) shows the exact solution expressed by Eq. (46) for $(\alpha,\beta) = (0.9, 1)$.

Figure 1 (c) and (d) show that the 5th-approximation of the solution of the IVP (44) and (45) is excellent compared to the exact solution, as well as in previous cases, which have not been documented in order not to increases the numbers of graphs. It is known that by increasing the number of terms in the series, the accuracy of the solution increases and thus the error of solution reduces, therefore we can reduce the error of the solution by calculating more coefficients of the matrix expansion solution in Eq. (19).

In the next application, the exact solution is unknown. Therefore, we are trying to find the exact solution or an appropriate approximation of the solution.

Application 4.4. *Consider the following non-homogeneous hyperbolic system of time-fractional PDEs with variable coefficients:*

$$\begin{aligned} & U_t^{(\alpha)}(x,t) - A(x,t)\, U_x^{(1)}(x,t) - B(x,t)\, U(x,t) \\ = {} & F(x,t), 0 < \alpha \leq 1, x \in \mathbb{R}, t \geq 0, \end{aligned} \tag{50}$$

subject to the following initial condition:

$$U(x,0) = \begin{pmatrix} x+1 \\ e^x \end{pmatrix}, \tag{51}$$

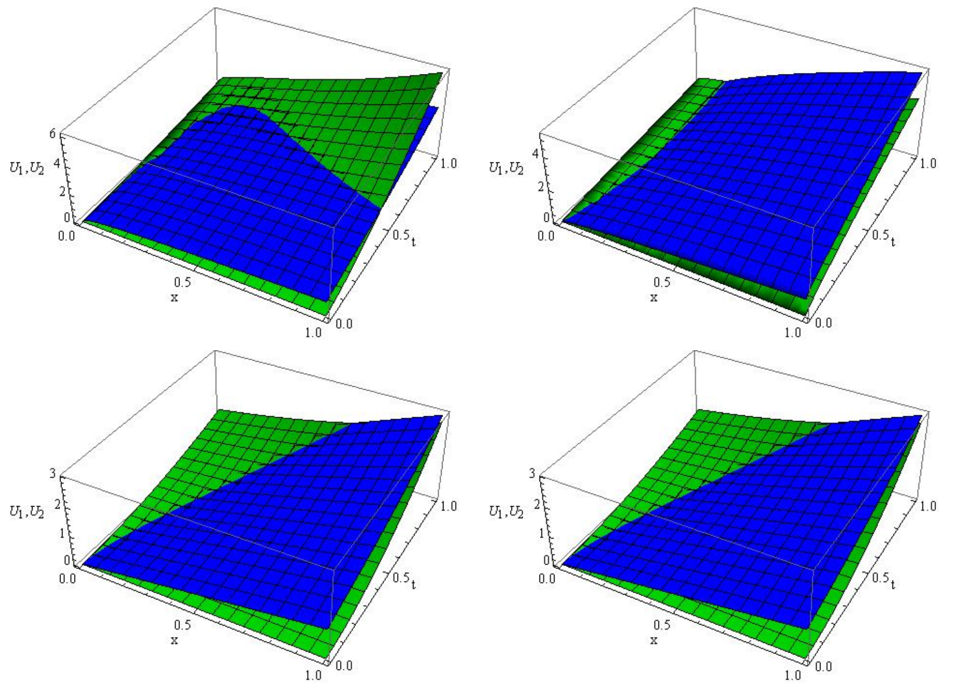

Figure 1: The surface graphs of the 5th-approximate solutions of $U_1(x,t)$ and $U_2(x,t)$ in Eq. (49) and the exact solution of $U_1(x,t)$ and $U_2(x,t)$ in Eq. (46): (a) $(\alpha,\beta)=(0.7,0.4)$, (b) $(\alpha,\beta)=(0.4,0.7)$, (c) $(\alpha,\beta)=(0.9,1)$, (d) exact solution when $(\alpha,\beta)=(0.9,1)$.

where

$$A(x,t)=\begin{pmatrix} 0 & -e^{-x}\left(e^{x}+(x-1)E_{\alpha}(t^{\alpha})+(1+x)t^{\alpha}\right) \\ \Gamma(1+\alpha)-e^{x}x-t^{\alpha}x & 0 \end{pmatrix},$$

$$B(x,t)=\begin{pmatrix} t^{\alpha} & 1 \\ 0 & x \end{pmatrix},\quad F(x,t)=\begin{pmatrix} xE_{\alpha}(t^{\alpha}) \\ 0 \end{pmatrix}.$$

Similar to the previous applications, the recurrence relation in Eq. (31) gives us the coefficients of the matrix expansion solution of the IVP (50) and (51). The first seven coefficients of the expansion Eq. (19) for this application are given by the following vector functions:

$$H_0(x)=\begin{pmatrix} x+1 \\ e^{x} \end{pmatrix},$$

$$H_1(x)=\begin{pmatrix} 1 \\ \Gamma(1+\alpha) \end{pmatrix},$$

$$H_2(x)=\begin{pmatrix} 1+\Gamma(1+\alpha) \\ 0 \end{pmatrix},$$

$$H_3(x) = \begin{pmatrix} 1 + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \\ 0 \end{pmatrix},$$

$$H_4(x) = \begin{pmatrix} 1 + \frac{(1+\Gamma(1+\alpha))\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)} \\ 0 \end{pmatrix},$$

$$H_5(x) = \begin{pmatrix} 1 + \frac{(\Gamma(1+\alpha)+\Gamma(1+2\alpha))\Gamma(1+4\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)} \\ 0 \end{pmatrix},$$

$$H_6(x) = \begin{pmatrix} 1 + \frac{(\Gamma(1+2\alpha)+(1+\Gamma(1+\alpha))\Gamma(1+3\alpha))\Gamma(1+5\alpha)}{\Gamma(1+2\alpha)\Gamma(1+4\alpha)} \\ 0 \end{pmatrix},$$

$$H_7(x) = \begin{pmatrix} 1 + \frac{(\Gamma(1+\alpha)\Gamma(1+3\alpha)+(\Gamma(1+\alpha)+\Gamma(1+2\alpha))\Gamma(1+4\alpha))\Gamma(1+6\alpha)}{\Gamma(1+3\alpha)\Gamma(1+5\alpha)} \\ 0 \end{pmatrix}. \tag{52}$$

Thus, the 7th-approximate solution of the IVP (50) and (51) can be expressed as follows:

$$\begin{aligned} U_7(x,t) = {} & H_0(x) + H_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + H_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + H_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + H_4(x)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + H_5(x)\frac{t^{5\alpha}}{\Gamma(1+5\alpha)} \\ & + H_6(x)\frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + H_7(x)\frac{t^{7\alpha}}{\Gamma(1+7\alpha)}. \end{aligned} \tag{53}$$

In order to examine the approximate solution as in Eq. (53), we need to find the norm of residual error vector ($\|\mathrm{RES}(x,t)\|$) for different values of t and x in the region $[0,1] \times [0,1]$, where the residual error vector is defined by

$$\begin{aligned} \mathrm{RES}_k(x,t) = {} & (U_k)_t^{(\alpha)}(x,t) - A(x,t)(U_k)_x^{(1)}(x,t) \\ & - B(x,t)\,U_k(x,t) - F(x,t). \end{aligned} \tag{54}$$

Tables 1 and 2 show that the values of $\|\mathrm{RES}_6(x,t)\|$ and $\|\mathrm{RES}_7(x,t)\|$, respectively, for different values of α. The data in the tables indicate that the norm of the residual error of the obtained approximate solution decreases as $(x,t) \to (0,0)$ as well as when $\alpha \to 1$. This indicates that the convergence of the bivariate fractional MPS Eq. (19) depends on t, x and α as illustrated in Corollary 2.1. As we know, we can reduce the error in the matrix expansion solution Eq. (19) as we increase the number of terms of the expansion. As we can see from the data in Tables 1 and 2, the 7th approximation gave us a solution better than the 6th approximation. Anyway, it can be said that the RPSM is good at providing an accurate approximate solution of a hyperbolic system of time-fractional PDEs with variable coefficients.

Table 1: The values of $\|\text{RES}_6\,(x,t)\|$ for different values of α.

$(\boldsymbol{x},\boldsymbol{t})$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
$(0.0, 0.0)$	0.000000	0.000000	0.000000
$(0.2, 0.2)$	3.5563×10^{-3}	1.6882×10^{-4}	7.6648×10^{-6}
$(0.4, 0.4)$	4.8228×10^{-2}	5.1980×10^{-3}	5.3168×10^{-4}
$(0.6, 0.6)$	2.2559×10^{-1}	3.9524×10^{-2}	6.5249×10^{-3}
$(0.8, 0.8)$	6.8021×10^{-1}	1.6888×10^{-1}	3.9295×10^{-2}
$(1.0, 1.0)$	1.6103×10^{0}	5.2523×10^{-1}	1.5995×10^{-1}

Table 2: The values of $\|\text{RES}_7\,(x,t)\|$ for different values of α.

$(\boldsymbol{x},\boldsymbol{t})$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
$(0.0, 0.0)$	0.000000	0.000000	0.000000
$(0.2, 0.2)$	9.6895×10^{-4}	2.6675×10^{-5}	6.82732×10^{-7}
$(0.4, 0.4)$	1.9432×10^{-2}	1.3932×10^{-3}	9.2535×10^{-5}
$(0.6, 0.6)$	1.1395×10^{-1}	1.4361×10^{-2}	1.6690×10^{-3}
$(0.8, 0.8)$	4.0283×10^{-1}	7.5982×10^{-2}	1.3162×10^{-2}
$(1.0, 1.0)$	1.0780×10^{0}	2.7858×10^{-1}	6.5901×10^{-2}

5. Conclusions

We have found that the exact solution for the hyperbolic system of time-fractional partial differential equations with variable coefficients is available if the solution was a linear combination of power functions or if it a composite of an elementary function and a power function. In case the exact solution is not available, a good approximation of the solution can be obtained. The RPSM is an effective and accurate technique in determining the values of coefficients for the series solution. By this work, we have presented a solution that may be missing for this kind of problem and we have opened the way for researchers to provide other ways to solve this class of equations. Moreover, we can use the proposed method to solve other types of systems of Differential equations such as the temporal-fractional Drinfeld–Sokolov–Wilson System, the epidemiological model, and others.

Acknowledgment

The author expresses his appreciation and thanks to Al-Balqa Applied University for granting him a sabbatical leave which made this research possible.

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Accepted: December 28, 2020