Gumbel-exponential distribution

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Abstract. In this paper we introduce a new distribution denoted (GE) which depends on the Exponential and Gumbel distributions. Some of its mathematical properties like moment generated function, hazard function, quantile functions, mode, median, variance, the r-th moment about the mean and origin of skewness and kurtosis, a simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. A real data is adapted to illustrate the importance of the new distribution.
Keywords: Gumbel distribution, exponential distribution, T-X family, moments, estimation, real data.

1. Introduction

Statistical distributions are commonly applied to describe real world phenomena. There are always ways for developing it, they are either more flexible or fitting specific real world scenarios. This has researchers are motivated for seeking and developing new and more flexible distributions. Many classes of generalized distributions have been developed and applied to describe various phenomena. In statistical literature, researchers have introduced new distributions by adding one or more parameters.

Can we use other distributions with support as a generator to derive different classes of distributions?

This question is to introduce a new technique to derive families of distributions using any probability density function as a generator. A new method is bing proposed to generate these families of continuous distributions. They consider a random variable X called "the transformer" whith is used to transform another variable T "the transformed", this result is the family of distributions

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$T-X$. The well-known generators are the Exponentiated-exponential family introduced by Gupta and Kundu [14]. Nadarajah (2006) enunciated the Exponentiated Gumbel distribution . Cordeiro and al (2011) have suggested the Kumaraswamy Gumbel distribution [22]. the Weibull-G family of probability distribution has been advanced by Bourguignon and al [19]. to generate new families, Alzaatreh and al (2013) has introduced a method [3], in wich they have proposed an Exponential $T-X$ family of distribution [4] and Weibull-Pareto distribution with applications [2].

2. Method of the transform-transformer

Alzaatreh and al [3], have introduced a method to generate new families labelled $T-X$ that satisfies some conditions.

Definition 2.1. *Let X be a random variable with density f, F would function accumulated, let T be a continuous random variable with the use of the probability density $r(x)$ defined on $[a,b]$, $G(x)$ the function of the accumulated distribution of the new family of distribution defined as*

$$G(x)=\int_a^{W(F(x))} r(t)dt, \tag{1}$$

where $W(F(x))$ satisfies the following conditions

$$\begin{cases} W(F(x))\in[a,b] \\ W(F(x)) \text{ is differentiable and monotonically non decreasing} \\ \text{if } x\to-\infty, W(F(x))\to a \text{ and } x\to\infty, W(F(x))\to b \end{cases} \tag{2}$$

The probability density function $g(x)$ is related to the accumulated function distribution $G(x)$ connected to the previous given by the relation

$$g(x)=\left\{\frac{d}{dx}W(F(x))\right\} r\left\{w(F(x))\right\}, \tag{3}$$

where $R(t)$ is the function of accumulated distribution of the a random variable.

3. Gumbel-exponential distribution "GE"

In this section we introduce a new family of the distribution generated by a random variable T. We have $T\in(-\infty,\infty)$ a random variable that follows the Gumbel distribution [25].

Theorem 3.1. *The PDF for Gumbel exponential(GE) distribution is defined by*

$$g(x)=\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1}\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right), \mathbf{x}>\mu. \tag{4}$$

Proof. Let be $T \in (-\infty, \infty)$ a random variable that follows the Gumbel distribution, so that $r(t)$ the density probability function

$$r(t) = \frac{1}{\delta} \exp(-\frac{t-\mu}{\delta}) \exp(-\exp(-\frac{t-\mu}{\delta})),\ t \in \mathbb{R},$$
$$R\,(x) = \exp\left(-\exp(-\frac{x-\mu}{\sigma})\right), si\ x \in \mathbb{R}.$$

Let the random variable X follow the exponential distribution, that the distribution function is

$$F(x) = 1 - e^{-\frac{x-\mu}{\delta}}, \quad \mathbf{x} \geq \mu.$$

Therefore,

$$w(F(x) = \log\left[-\log(1 - F(x))\ \right] = \log(\frac{x-\mu}{\delta}).$$

We find

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\left\{w(F(x))\right\}$$

and

$$G(x) = \int_{\mu}^{W(F(x))} r(t)dt = R\left\{W(F(x))\right\},$$
$$G(x) = \int_{\mu}^{W(F(x))} r(t)dt = R\left\{W(F(x))\right\}$$
$$= \int_{\mu}^{W(F(x))} \frac{1}{\delta} \exp(-\frac{t-\mu}{\delta}) \exp(-\exp(-\frac{t-\mu}{\delta}))dt$$
$$= \exp\left(-(\frac{\delta}{x-\mu})^{\frac{1}{\delta}} \exp(-\frac{\mu}{\delta})\right) - \frac{1}{e}$$

we drift $G(x)$, we find, the density function of probability $g(x)$ is given by

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2} \left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1} \exp\left(-e^{-\frac{\mu}{\delta}} \left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right), \mathbf{x} > \mu. \tag{5}$$

□

3.1 Quantile function

The quantile function of X is $G_x^{-1}(\gamma),\ 0 < \gamma < 1$. The quantile function of the distribution GE is

$$G^{-1}(\gamma) = \frac{\delta}{\left(-\ln\left(\gamma + \frac{1}{e}\right) / \exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{6}$$

for $\gamma = 0.25, \gamma = 0.50$, and $\gamma = 0.75$, the three quartiles of the distribution GE are

$$\varphi_1 = G_x^{-1}(\frac{1}{4}) = \frac{\delta}{\left(-\ln\left(\frac{1}{4} + \frac{1}{e}\right) / \exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{7}$$

$$\text{median } \varphi_2 = G_x^{-1}\left(\frac{1}{2}\right) = \frac{\delta}{\left(-\ln\left(\frac{1}{2}+\frac{1}{e}\right)/\exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu, \tag{8}$$

$$\varphi_3 = G_x^{-1}\left(\frac{3}{4}\right) = \frac{\delta}{-\left(\ln\left(\frac{3}{4}+\frac{1}{e}\right)/\exp(\frac{-\mu}{\delta})\right)^{\delta}} + \mu. \tag{9}$$

3.2 Hazard function

The hazard function of GE distribution is

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2} z^{\frac{1}{\delta}+1}}{\left(1+\frac{1}{e}\right)\exp(e^{-\frac{\mu}{\delta}} z^{\frac{1}{\delta}})}. \tag{10}$$

By simple computation, we obtain

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}(\frac{\delta}{x-\mu})^{\frac{1}{\delta}-1}\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right)}{1-\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right)+\frac{1}{e}} \tag{11}$$

Mutiplying by $\exp(e^{-\frac{\mu}{\delta}}(\frac{\delta}{x-\mu})^{\frac{1}{\delta}})$, we obtain

$$h(x) = \frac{\frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1}}{(1+\frac{1}{e})z-1}, \tag{12}$$

where $z = \exp(e^{-\frac{\mu}{\delta}}(\frac{\delta}{x-\mu})^{\frac{1}{\delta}})$.

3.3 Moments and function generating moments

3.3.1 Function generating moments

The moment Generating function of a continuous random variable X is given by

$$M_X(t) = \int_{\mu}^{\infty} \exp(tx)g(x)dx = \int_{\mu}^{\infty}\left(\sum_{i=0}^{\infty}\frac{t^i x^i}{i!}g(x)\right)dx, \tag{13}$$

$$M_X(t) = \sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i) = \frac{t^i}{i!}k\sum_{k=0}^{n}C_k^n\delta^k\mu^{n-k}\Gamma\left(1-n\delta\right). \tag{14}$$

3.3.2 Moments

The first moment of X can be written as

$$E(X) = \int_{\mu}^{\infty} xg(x)dx = k\left(\mu+\delta\Gamma\left(1-\delta\right)\right), \quad k = \exp\left(-\frac{\mu}{\delta}+\exp(-\frac{\mu}{\delta})\right) \tag{15}$$

with

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{(x-\mu)^2}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}-1}\exp\left(-e^{-\frac{\mu}{\delta}}\left(\frac{\delta}{x-\mu}\right)^{\frac{1}{\delta}}\right), \mathbf{x} > \mu.$$

By variable change

$$y = \left[\frac{\delta}{x-\mu}\right]^{\delta^{-1}}, \tag{16}$$

$$y^{\delta} = \frac{\delta}{x-\mu}. \tag{17}$$

We obtain

$$x = \delta y^{-\delta} + \mu, \tag{18}$$

$$dx = -\delta^2 y^{-\delta-1} dy. \tag{19}$$

We obtain a new $g(x)$

$$g(x) = \frac{e^{-\frac{\mu}{\delta}}}{\delta^2} y^{\delta} \exp(-ye^{-\frac{\mu}{\delta}}) \tag{20}$$

We can calculate the moment of ordre1 E(x)

$$E(X) = \int_{\mu}^{\infty} xg(x)dx = k\delta \int_0^{\infty} y^{-\delta}\exp(-y)dy + k\mu \int_0^{\infty} \exp(-y)dy, \tag{21}$$

where

$$\int_0^{\infty} y^{-\delta}\exp(-y)dy = k\int_0^{\infty} y^{(1-\delta)-1}\exp(-y)dy = \Gamma(1-\delta). \tag{22}$$

By consequence

$$E(X) = k\left(\mu + \delta\Gamma\left(1-\delta\right)\right). \tag{23}$$

The second order moment of the GE distribution is

$$E(X^2) = k\left(\mu + 2\delta\mu\Gamma(\delta-1) + \delta^2\Gamma(1-2\delta)\right). \tag{24}$$

The nth moment on the origin

$$E(X^n) = \int_{\mu}^{\infty} x^n g(x)dx = k\int_0^{\infty} (\delta y^{-\delta} + \mu)^n \exp(-y)dy, \tag{25}$$

$$E(X^n) = \sum_{k=0}^{n} C_k^n \delta^k \mu^{n-k} \int_0^{\infty} y^{-k\delta} dy = k\sum_{k=0}^{n} C_k^n \delta^k \mu^{n-k}\Gamma\left(1-n\delta\right), \tag{26}$$

for $k = 1, \ldots, n$.

The variance of the GE distribution is

$$V(X) = \Gamma(1-\delta)(2\delta\mu - \Gamma(1-\delta) - 2\mu) + \mu(1-\mu) + \delta^2\Gamma(1-2\delta), \tag{27}$$

$$\begin{aligned} V(X) &= E(x^2) - E(x)^2 \\ &= k\left(\left[\mu + 2\delta\mu\Gamma(\delta-1) + \delta^2\Gamma(1-2\delta)\right] - \left[\mu + \Gamma(1-\delta)\right]^2\right) \\ &= k\left(\Gamma(1-\delta)(2\delta\mu - \Gamma(1-\delta) - 2\mu) + \mu(1-\mu) + \delta^2\Gamma(1-2\delta)\right). \end{aligned}$$

The coefficients of skewness and kurtosis of the GE distribution are

$$\begin{aligned} skewness = E(x^3) &= \sum_{k=0}^{n} C_3^k \delta^k \mu^{3-k} \int_0^\infty y^{-3\delta} dy \\ &= k\sum_{k=0}^{n} C_3^k \delta^k \mu^{3-k} \Gamma(1-k\delta), \end{aligned} \tag{28}$$

$$skewness = k\left(\mu^3 + 3\delta\mu^2\Gamma(1-\delta) + 3\delta^2\mu\Gamma(1-2\delta) + \delta^3\Gamma(1-3\delta)\right),$$

$$kurtosis = E(x^4) = k\sum_{k=0}^{n} C_4^k \delta^k \mu^{4-k} \Gamma(1-4\delta), \tag{29}$$

$$\begin{aligned} kurtosis = k\big(&\mu^4 + 4\delta\mu^3\Gamma(1-\delta) + \delta^2\mu^2\Gamma(1-2\delta) \\ &+ 4\delta^3\mu\Gamma(1-3\delta) + \delta^4\Gamma(1-4\delta)\big). \end{aligned} \tag{30}$$

3.4 Entropy

The entropy (Rényi, 1961),of a random variable X is a measure of variation of uncertainty , that of the GE distribution is given by

$$\begin{aligned} I_R(s) &= \frac{1}{1-s}\{\ln(g^s(x)dx\}, \quad s > 0, s \neq 1 \\ &= \frac{ke^{-\frac{s\mu}{\delta}}}{1-s} \ln\left\{\delta^{-2s+2} s^{-\delta s+\delta-s}\Gamma(\delta s + s - \delta)\right\}, \ s > 0, s \neq 1. \end{aligned}$$

Moreover, the Shannon entropy (1948) defined by $E\{-\ln(g(x))\}$, this is a special case derived from $\lim_{s\to 1} I_R(s)$.

3.5 Reliability

The measure of reliability of industrial components has many applications especially in the area of lifetime testing and engineering,The component fails at the instant that the random stress X_2 applied to it exceeds the random strength $X1$, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = P[X_2 < X_1]$ is a measure of component reliability. We derive the reliability

R when X_1 and X_2 have independent $GE(\delta_1;\mu)$ and $GE(\delta_2;\mu)$ distributions with the same shape parameter μ and fixed parameter δ. The reliability is defined

$$R=\int_0^{\infty} g_1(x)G_2(x)dx=-k(2-\delta_1)\left(-\exp\left((\frac{\delta_2}{x-\mu})^{\frac{1}{\delta}}*\exp(\frac{-\mu}{\delta_2})\right)-\frac{1}{e}\right),$$

$k=\exp\left(-\frac{\mu}{\delta_1}+\exp(-\frac{\mu}{\delta_1})\right)$.

3.6 Mean deviations

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean m and median M, the mean deviation from the mean $D(m)$ and the mean deviation from the median $D(M)$, can be written as

$$D(m)=\int_{\mu}^{\infty}|x-m|\,g(x)dx=2mG(x)-2\int_{\mu}^{m}xg(x)dx,$$
$$D(M)=\int_{\mu}^{\infty}|x-M|\,g(x)dx=m-2\int_{\mu}^{M}xg(x)dx,$$

where $\int_{\mu}^{m}xg(x)dx=\delta\left[\Gamma\left((1-\delta),m\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu\left[\exp(m)-\exp(\mu)\right]$

We obtain, where $k=e^{-\frac{\mu}{\delta_1}+\exp(-\frac{\mu}{\delta_1})}$

$$D(m)=2mG(x)-2\delta k\left[\Gamma\left((1-\delta),m\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu k\left[\exp(m)-\exp(\mu)\right],$$
$$D(M)=m-2\delta k\left[\Gamma\left((1-\delta),M\right)-\Gamma\left((1-\delta),\mu\right)\right]+\mu k\left[\exp(M)-\exp(\mu)\right].$$

3.7 Maximum likelihood estimates

Let X_i follows $GE(\delta,\mu), i=1,....,n$, random variables, the maximum likelihood function $l(x,\mu,\delta)$ is

$$L(\theta)=lnl(x,\mu,\delta)=\sum(\frac{-\mu}{\delta})-2n\log(x-\mu)+n(\frac{1}{\delta}-1)\log\delta$$
$$-(\frac{1}{\delta}-1)\sum\log(x_i-\mu)+\sum\left((\frac{1}{x_i-\mu})*\exp(\frac{\mu}{\delta})\right). \tag{31}$$

The derivatives of $L(\theta)$ with respect to δ and μ are

$$\frac{\partial L(\theta)}{\partial\delta}=\sum(\frac{\mu}{\delta^2})-\frac{n}{\delta^2}\log\delta+n\frac{1}{\delta}(\frac{1}{\delta}-1)-\frac{\mu}{\delta^2}\sum_{i=1}^{n}\left(\frac{\mu}{\delta^2}(\frac{\delta}{x_i-\mu})*\exp(\frac{\mu}{\delta})\right), \tag{32}$$

$$\frac{\partial L(\theta)}{\partial\mu}=-\sum(\frac{1}{\delta})\frac{2n}{(x-\mu)}+(\frac{1}{\delta}-1)\sum\frac{1}{(x_i-\mu)}$$
$$-\sum\frac{1}{(x_i-\mu)^2}*\exp(\frac{\mu}{\delta})-\frac{1}{\delta^2}\sum(\frac{1}{x_i-\mu})*\exp(\frac{\mu}{\delta}). \tag{33}$$

The two equations can not be solved directly, we must used the Fisher scoring method. We have

$$\left[\begin{matrix}\frac{\partial L(\theta)}{\partial\delta^2} & \frac{\partial^2 L(\theta)}{\partial\delta\partial\mu}\\ \frac{\partial^2 L(\theta)}{\partial\mu\partial\delta} & \frac{\partial L(\theta)}{\partial\mu^2}\end{matrix}\right]_{\substack{\hat{\delta}=\delta_0\\ \hat{\mu}=\mu_0}} \left[\begin{matrix}\hat{\delta}=\delta_0\\ \hat{\mu}=\mu_0\end{matrix}\right] = \left[\begin{matrix}\frac{\partial L(\theta)}{\partial\delta}\\ \frac{\partial L(\theta)}{\partial\mu}\end{matrix}\right]_{\substack{\hat{\delta}=\delta_0\\ \hat{\mu}=\mu_0}}. \tag{34}$$

The equation (42) can be solved iteratively where δ_0, μ_0 are the initial values of δ, μ.

3.8 Existence and uniquess of ML parameters

Lemma 3.1. *there exists a compact subset $K \equiv K\ (\eta) \subset (0,\infty)\ (0,\infty)$.such that*

$$\{(\delta,\mu) : L(\theta) \geq -\eta\} \subset K. \tag{35}$$

Theorem 3.2. *Suppose that $X_i \backsim GE\ (\delta,\mu), i = 1, ..., n$, then the MLEs of parameters δ and μ of Gumbel Exponential distribution exist and are unique.*

Proof. We need only to show that the MLEs of parameters δ and μ uniquely exist. According to the results of Màkelàinen et al. (1981), in order to show the existence and uniqueness of the MLEs of δ and μ, it is sufficient to verify the following two conditions:

(i) For any given $\eta > 0$, (49) holds

(ii) The Hessian matrix of $L(\theta)$

$$H = \left[\begin{matrix}\frac{\partial L(\theta)}{\partial\delta^2} & \frac{\partial^2 L(\theta)}{\partial\delta\partial\mu}\\ \frac{\partial^2 L(\theta)}{\partial\mu\partial\delta} & \frac{\partial L(\theta)}{\partial\mu^2}\end{matrix}\right]$$

is negative definite at every point $(\delta,\mu) \in (0,\infty)$. □

Condition i is certainly satisfied by Lemma 1. Therefore, to prove the theorem, we need only to show ii. Then, $x^t H x < 0$ and H is negative definite, where $x^t = (x_1\ x_2)$.

4. Simulation

In this section, we investigate the behaviour of the ML estimators for a finite sample size (n). A simulation study consisting of the following steps is being carried out for each (δ,μ,n),where n=$100, 500, 1000$.

- Choose the initial values of δ_0, μ_0 for the corresponding elements of the parameter vector $\theta = (\delta,\mu)$ to specify GE(δ,μ) distribution.

- Choose sample size n.
- Generate N independent samples of size n from $GE(\delta, \mu)$ distribution.
- Compute the ML estimate $\hat{\theta}_n$ of θ_0 for each of the N samples.
- Compute the mean of the obtained estimators over all N samples

$$Averagebias = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_i - \theta_0 \right) \tag{36}$$

and the average square error

$$MSE\left(\theta\right) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_i - \theta_0 \right)^2 \tag{37}$$

Table 03

N	$biais(\delta)\quad \delta = 0.01$	$biais(\mu)\quad \delta = 0.01$	$biais(\delta)\ \delta = 0.3$	$biais(\mu)\ \delta = 0.3$
$n = 100$	-0.004220959	-0.0495	-0.294221	-0.00425
$n = 500$	-0.005392097	-0.0499	-0.2953921	-0.0499
$n = 1000$	-0.005784495	-0.04995	-0.2957845	-0.04995

Table 04

N	$biais(\delta)\ \delta = 0.8$	$biais(\mu)\delta = 0.8$	$biais(\delta)\delta = 0.25$	$biais(\mu)\ \delta = 0.25$
$n = 100$	-0.794221	-0.099	-0.195784	-0.099
$n = 500$	-0.7953921	-0.0998	-0.1953921	-0.0998
$n = 1000$	-0.7957845	-0.0999	-0.2457845	-0.0999

Table 05

N	$biais(\delta)\delta = 0.5$	$biais(\mu)\delta = 0.5$	$biais(\delta)\delta = 0.75$	$biais(\mu)\delta = 0.75$
$n = 100$	-0.494221	-0.7425	-0.744221	-0.7425
$n = 500$	-0.4953921	-0.7485	-0.7453921	-0.7485
$n = 1000$	-0.4957845	-0.74925	-0.7457845	-0.74925

Tables 03,04 and 05 : represent average bias of the estimations feigned with $\mu = 0.05, \mu = 0.1, 0.75$

N	$MSE(\delta)\delta = 0.01$	$MSE(\mu)\delta = 0.01$	$MSE(\delta)\ \delta = 0.3$	$MSE(\mu)\ \delta = 0.3$
$n = 100$	$1.78165e - 05$	0.00245025	0.08656597	0.00245025
$n = 500$	$0.907471e - 05$	0.002419001	0.08125649	0.00241001
$n = 1000$	$0.346038e - 05$	0.00241000	0.0118847	0.00140003

Table 06

N	$MSE(\delta)\ \delta = 0.8$	$MSE(\mu)\ \delta = 0.8$	$MSE(\delta)\delta = 0.25\ )$	$MSE(\mu)\delta = 0.25$
$n = 100$	0.6307869	0.009801	0.03833157	0.00998001
$n = 500$	0.6316486	0.009196004	0.03817807	0.00996004
$n = 1000$	0.00630273	0.00098001	0.001041002	0.008001

Table 07

N	$MSE(\delta)\ \delta = 0.5$	$MSE(\mu)\ \delta = 0.5$	$MSE(\delta)\ \delta = 0.75$	$MSE(\mu)\ \delta = 0.75$
$n = 100$	0.2442544	0.5513063	0.5561945	0.5613756
$n = 500$	0.12454133	0.25602523	0.35556094	0.45602523
$n = 1000$	0.02458023	0.05613756	0.05561945	0.05613756

Table 08

Tables 06,07 and 08: Represent MSE of the feigned estimations with $\mu = 0.05, \mu = 0.1, \mu = 0.75$

Discussion

Previous tables shows how the bias and mean squared errors vary with respect to n. The mean squared errors for each parameter decrease to zero as $n \to \infty$. These numerical results coincide with the established theoretical results.

5. Application

In this section, we show present that the GE distribution may be a best model for some distributions, for example Nadarajah [22] presented the exponential distribution of Gumbel. Persson and Ryden [24] have examined this before distribution. in oceanographic application. In addition, Pinheiro and Ferrari [23], compare several generalizations of the Gumbel distribution.

We provied an applicability of GE distribution by considering real data used by different researchers. Application to waiting times in a queue, and compare them with different distribution.In each case, the parameters are estimated by maximum likelihood

Illustration. Application to waiting times in a queue.

We consider 100 observations on waiting time as a real example that happens before the customer received service in a bank. The data set represents the waiting time (mins) of one hundred (100) bank customers before service is being rendered, This data has previously been used by Ghitany et al (2008). These data are

$$0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.2, 2.23, 0.26, 0.31, 0.73,$$
$$0.52, 4.98, 6.97, 9.02, 13.29, 0.4, 2.26, 3.57, 5.06, 7.09, 11.98, 4.51, 2.07,$$
$$0.22, 13.8, 25.74, 0.5, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 19.13, 6.54, 3.36,$$
$$0.82, 0.51, 2.54, 3.7, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 1.76, 8.53, 6.93,$$
$$0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 3.25, 12.03, 8.65,$$
$$0.39, 10.34, 14.83, 34.26, 0.9, 2.69, 4.18, 5.34, 7.59, 10.66, 4.5, 20.28, 12.63,$$
$$0.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 6.25, 2.02, 22.69,$$
$$0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 8.37, 3.36, 5.49,$$
$$0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 12.02, 6.76,$$
$$0.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.1, 1.46, 4.4, 5.85, 2.02, 12.07$$

The distribution of GE is adjusted to the data,In each case, the parameters are estimated by maximum likelihood, we adapt (GE) distributions, Exponential Pareto (EP), Exponential (E), Lindley Exponential (LE) and Lomax-Gumbel (LG). The required calculations are carried out using the software R-softwar.

Distribution	*Parameters*
Gumbel- Exponential	$\hat{\delta} = 0.0004061848, \hat{\mu} = 0.009$
Exponential- Pareto	$\hat{\alpha} = 0.801,\ \hat{k} = 1.5137,\ \hat{\lambda} = 0.0183$
Exponential	$\hat{\theta} = 0.101$
Lindley -Exponential	$\hat{\theta} = 2.6501, \hat{\lambda} = 0.152$
Lomax-Gumbel	$\hat{\gamma} = 0.577, \hat{\alpha} = 0.125$

We used statistical tools to compare models and choose the best possible distribution for the dataset among other distributions such as (AIC) Akaike Information Criterion, (CAIC) information criterion corrected Akaike and BIC (Bayesian InformationCriterion) are described as follows

$$AIC = -2LL + 2p$$

$$BIC = -2LL + p\log(n)$$

$$CAIC = 2LL + \frac{2np}{n-p-1}$$

Distribution	$-LL$	AIC	$CIAC$	BIC
Gumbel- Exponential	74.20096	152.4019	152.5256	157.6122
Exponential -Pareto	312.1154	628.2308	628.372	633.9192
Exponential	329.00	660.00	662.1237	660.00
Lindley -Exponential	317.005	638.01	638.1337	643.2203
Lomax-Gumbel	416.379	836.757	836.8817	842.446

Discussion

After a visual reading of the numerical results of estimated parameters and statistical tools from CIA CAIC and BIC, we note that the GE model more powerful than other popular models, we can conclude that the GE model is simple to manipulate and hence best fits the data among all the models considered.

Conclusion

We have proposed the new distribution based on the Gubmel and exponential distribution with discussion some properties including moment generated function, hazard functions, quantil functions, mode, median, mean ,variance, entropy, maximum Likelihood Estimates. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. We have also illustrated the application of GE distribution to real data used by researchers earlier. By comparing GE distribution with other popular models, we conclude GE is distribution best model for the prediction and may be the most flexible for generated this data. Thereafter, it would be interesting to use other data, also we can in our future research propose other distribution has more parameters.

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