Dual hesitant fuzzy Bonferroni means and its applications in decision-making

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Abstract. The Bonferroni mean (BM) was initially presented by Bonferroni and afterward more as of late summed up by Yager. The exceptionally solid characteristic of the BM is its ability to catch the interrelationship between input datum. Nevertheless, it appears that the previous work just considers the BM for aggregating crisp information rather than some other kinds of contentions. In this paper, we explore the BM under dual hesitant fuzzy datum. The dual hesitant fuzzy set (DHFS) is the all-encompassing type of intuitionistic fuzzy sets (IFSs) and hesitant fuzzy sets (HFSs). We build up an dual hesitant fuzzy BM (DHFBM) and talk about its assortment of extraordinary cases. At that point, we apply the weighted DHFBM to multicriteria dynamic. Some numerical models are given to show our outcomes.
Keywords: Bonferroni mean, dual hesitant fuzzy set, dual hesitant fuzzy Bonferroni mean (DHFBM), decision-making.

1. Introduction

In recent decades, the aggregation operators received much attention from many researchers and practitioners [1]-[4]. The aggregation operators are generally characterized into two groups. In the first group, the aggregated arguments behaves dependently, whereas, the aggregated arguments behaves independently in second group. For the first type of aggregation operators, Yager [5] developed the ordered weighted averaging (OWA) operator to reorder the arguments before the aggregation process. Later on, Chiclana *et al*. [6] and Xu and Da [7] developed the ordered weighted geometric (OWG) operator. Yager [8] also proposed the continuous ordered weighted averaging (C-OWA) operator. Moreover, Yager and Xu [9] defined the ordered weighted geometric (OWG) opera-

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tor. Many researchers explored the second type of aggregation operators and developed many novel aggregation operators, such as, the Chouqet [10] integral operator, the power aggregation operator [11] and the Heronian mean (HM) [12] operator. Yager [13] defined the power average (PA) to develop an aggregation operator. Xu and Yager [14] introduced the power geometric (PG) operator, the power ordered geometric (POG) operator and the power ordered weighted geometric (POWG). Basically, Bonferroni explored the BM (Bonferroni, [15] which can provide for the aggregation lying among the max and min operators and the logical "or" and "and" operators. The desirable characteristic of the BM is that captures the communicated interrelationship of the individual arguments. It has wide application in MADM under different fuzzy environments. Liu *et al*. [16] presented an interpretation of HM operator and proposed some new generalization instead of simple averaging operators defined earlier. Beliakov *et al*. [17] suggested some new generalized Bonferroni mean operators and applied them to multi-attribute aggregation under different environments. Xu and Yager [18] developed BM and weighted BM operators under intuitionistic fuzzy environment. Zhu *et al*. [19] introduced the BM and weighted BM aggregation operators, Zhu *et al*. [20] also developed the geometric BM combining with hesitant fuzzy information. Yu *et al*. [21] introduced the generalized hesitant fuzzy BM operator and its importance in multi-attribute group decision making.

So as to present dual hesitant fuzzy information, Zhu *et al*. [22] presented the idea of dual hesitant fuzzy sets (DHFSs), it covers the arguments with two sets of possible values, that is the membership (favorable) hesitancy function and the non-membership (unfavorable) hesitancy function, It is very much powerful appliance to represent and collect information under uncertain environment, exclusively in the decision making process, Fuzzy sets [23], intuitionistic fuzzy sets [24], hesitant fuzzy sets [25] and fuzzy multi sets [26] are the special cases of DHFSs. Many researchers developed different aggregation operators for other fuzzy information, such as IFSs, HFSs, etc. A very little research have been developed on the MADM problems for dual hesitant fuzzy environment. Therefore, we have explored the dual hesitant fuzzy Bonferroni mean (DHFBM) operator and discussed its several cases. We have also explored the weighted dual hesitant fuzzy Bonferroni mean (WDHFBM) operator. In this connection, the remainder of the paper is composed as follows. Section 2 consist of definitions and some basic concepts and special cases of BM. In Section 3, we discuss the properties and operations of DHFSs. Further, We have developed the DHFBM and the dual hesitant Bonferroni element (DHFBE), we have discussed its main properties such as, Idempotency, Monotonocity, Boundedness and Commutativity. In, Section 4, we introduce weighted dual hesitant fuzzy Bonferroni mean (WDHFBM) operator. A numerical example of real life related to decision-making is presented to demonstrate our discussion. The paper finishes with some ending remarks in Section 5.

2. Preliminaries

2.1 Bonferroni mean

The Bonferroni [15] introduced the BM as given below:

Definition 1. Suppose $\rho, \ell \geq 0$, where both of the parameters are not simultaneously 0 and $r_i (i = 1, 2, \ldots, \tau)$ be an assortment of nonnegative numbers. If

$$B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau(\tau-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{\tau} r_i^\rho r_j^\ell \right)^{\frac{1}{\rho+\ell}}, \tag{1}$$

then the BM is denoted as $B^{\rho,\ell}$.

Clearly, the BM saves the accompanying properties:

1. $B^{\rho,\ell}(0, 0, \ldots, 0) = 0$.
2. $B^{\rho,\ell}(r, r, \ldots, r) = r$, if $r_i = r$, for all i.
3. $B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) \geq B^{\rho,\ell}(c_1, c_2, \ldots, c_\tau)$ i.e. $B^{\rho,\ell}$ is monotonic, if $r_i \geq c_i$ all i.
4. $\min_i\{r_i\} \leq B^{\rho,\ell}(r_1, r_2, \ldots, r_\tau) \leq \max_i\{r_i\}$.

Additionally, we discuss several cases of BM, if $\ell = 0$, then Eq.1 becomes

$$B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^\rho \left(\frac{1}{\tau - 1} \sum_{j=1}^{\tau} r_j^0 \right) \right)^{\frac{1}{\rho+0}} = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^\rho \right)^{\frac{1}{\rho}}.$$

Dyckhoff and Pedrycz [27] called this the generalized mean operator or power-root operator. Some more case are given below.

1) When $\rho = 2$ and $\ell = 0$, then Eq.1 turns into square mean

$$B^{2,0}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau} \sum_{i=1}^{\tau} r_i^2 \right)^{\frac{1}{2}}.$$

2) When $\rho = 1$ and $\ell = 0$, then Eq.1 turns into usual average

$$B^{1,0}(r_1, r_2, \ldots, r_\tau) = \frac{1}{\tau} \sum_{i=1}^{\tau} r_i.$$

3) When $\rho \to +\infty$ and $\ell = 0$, then Eq.1 turns into max operator

$$\lim_{\rho \to +\infty} B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \max_i\{r_i\}.$$

4) When $\rho \to 0$ and $\ell = 0$, then Eq.1 turns into geometric mean

$$\lim_{\rho \to 0} B^{\rho,0}(r_1, r_2, \ldots, r_\tau) = \left(\prod_{i=1}^{\tau} r_i \right)^{\frac{1}{\tau}}.$$

5) When $\rho = 1$ and $\ell = 1$, then Eq.1 turns into square mean

$$B^{1,1}(r_1, r_2, \ldots, r_\tau) = \left(\frac{1}{\tau(\tau-1)} \sum_{i,j=1}^{\tau} r_i r_j \right)^{\frac{1}{2}}.$$

2.2 Dual hesitant fuzzy sets

In this part, we illustrate dual hesitant fuzzy set as a combination of two sets, the membership values and non-membership values, corresponding to each element in the crisp set as given below:

Definition 2. Suppose X a crisp set, Zhu *et al.*[22] presented the idea of dual hesitant fuzzy set (DHFS) on X as follows:

$$Ç = \{< x, r(x), s(x) > | x \in X\}$$

with $r(x) = \cup\{\sigma | \sigma \in r(x)\}$ and $s(x) = \cup\{\kappa | \kappa \in s(x)\}$ are two sets of few values in [0,1], $r(x), s(x)$ represents the possible membership degree and possible non-membership degree of the element $x \in X$ in the set Ç, respectively, satisfying the criteria:

$$0 \leq \sigma, \kappa \leq 1, \quad \sigma^+ + \kappa^+ \leq 1,$$

where $\sigma^+ \in r^+(x) = \bigcup_{\sigma \in r(x)} \max\{\sigma\}$, and $\kappa^+ \in s^+(x) = \bigcup_{\kappa \in s(x)} \max\{\kappa\}$ for all $x \in X$. In general, the pair $d(x) = (r(x), s(x))$ represents the dual hesitant fuzzy element (DHFE) simply written as $d = (r, s)$, with the conditions: $\sigma \in r, \kappa \in s, \sigma^+ \in r^+ = \bigcup_{\sigma \in r} \max\{\sigma\}, \kappa^+ \in s^+ = \bigcup_{\kappa \in s} \max\{\kappa\}, 0 \leq \sigma, \kappa \leq 1$, and $\sigma^+ + \kappa^+ \leq 1$.

Now, we describe some special types of DHFEs. For a DHFE, d,
(1) full vulnerability: $d = [0, 1]$; (complete ill-known, all is possible)
(2) full assurance: $d = (1, 0)$;
(3) full hesitancy: $d = (r, s)(0 < r < 1, 0 < s < 1)$;
(4) empty element: $d = \emptyset(r = \emptyset, s = \emptyset)$.

Definition 3. The complement $\overline{d}$, of a DHFE, d and $d \neq \emptyset$, is defined as:

$$\overline{d} = \begin{cases} \bigcup_{\substack{\sigma \in r \\ \kappa \in s}} \{\{\kappa\}, \{\sigma\}\}, & \text{if } s \neq \emptyset, \quad r \neq \emptyset, \\ \bigcup_{\sigma \in r} \{\{1-\sigma\}, \{\emptyset\}\}, & \text{if } s = \emptyset, \quad r \neq \emptyset, \\ \bigcup_{\kappa \in s} \{\{\emptyset\}, \{1-\kappa\}\}, & \text{if } r = \emptyset, \quad s \neq \emptyset. \end{cases}$$

Obviously, the complement function involutes as $\overline{\overline{d}} = d$.

Definition 4. Let d_1 and d_2 be two DHFEs with lower and upper bounds corresponding to r and s are r^-, r^+, s^- and s^+, respectively, where $r^- = \bigcup_{\sigma \in r} \min\{\sigma\}, r^+ = \bigcup_{\sigma \in r} \max\{\sigma\}, s^- = \bigcup_{\kappa \in s} \min\{\kappa\}$, and $s^+ = \bigcup_{\kappa \in s} \max\{\kappa\}$.

Now using the above notations, the union and intersection on d_1 and d_2 is defined as follows:

(1) $d_1 \cup d_2 = \{r \in (r_1 \cup r_2) | r \geq \max(r_1^-, r_2^-), s \in (s_1 \cap s_2) | s \leq \min(s_1^+, s_2^+)\}$;

(2) $d_1 \cap d_2 = \{r \in (r_1 \cap r_2) | r \leq \min(r_1^+, r_2^+), s \in (s_1 \cup s_2) | s \geq \max(s_1^-, s_2^-)\}$,

Furthermore, the following basic operations are valid and will be helpful in the next sections.

(1) $\oplus$-union: $d_1 \oplus d_2 = \{r_{d_1} \oplus r_{d_2}, s_{d_1} \oplus s_{d_2}\} = \bigcup_{\substack{\sigma_{d_1} \in r_{d_1} \\ \kappa_{d_1} \in s_{d_1} \\ \sigma_{d_2} \in r_{d_2} \\ \kappa_{d_2} \in s_{d_2}}} \{\{\sigma_{d_1} + \sigma_{d_2} - \sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}\kappa_{d_2}\}\}$;

(2) $\oplus$-intersection: $d_1 \otimes d_2 = \{r_{d_1} \otimes r_{d_2}, s_{d_1} \oplus s_{d_2}\} = \bigcup_{\substack{\sigma_{d_1} \in r_{d_1} \\ \kappa_{d_1} \in s_{d_1} \\ \sigma_{d_2} \in r_{d_2} \\ \kappa_{d_2} \in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1} + \kappa_{d_2} - \kappa_{d_1}\kappa_{d_2}\}\}$;

(3) $\tau d = \bigcup_{\substack{\sigma_d \in r_d \\ \kappa_d \in s_d}} \{1 - (1 - \sigma_d)^\tau, (\kappa_d^\tau)\}$;

(4) $d^\tau = \bigcup_{\substack{\sigma_d \in r_d \\ \kappa_d \in s_d}} \{(\sigma_d)^\tau, 1 - (1 - \kappa_d)^\tau\}$.

here τ is non-negative integer and above results are also $DHFEs$.

Definition 5. ([22]). Let d_1=$\{r_{d_1}, s_{d_1}\}$ and d_2=$\{r_{d_2}, s_{d_2}\}$ be any two $DHFEs$, the score and accuracy function of $d_i(i$=$1, 2)$ can be defined as S_{d_i}=$(\frac{1}{\#r}) \sum_{\sigma \in r} \sigma$ $-(\frac{1}{\#s}) \sum_{\kappa \in s} \kappa$, and $P_{d_i} = (\frac{1}{\#r}) \sum_{\sigma \in r} \sigma + (\frac{1}{\#s}) \sum_{\kappa \in s} \kappa$ is the accuracy function of $d_i (i = 1, 2)$, where $\#r$ and $\#s$ are the number of elements in r and s, respectively, then:

(i) If $S_{d_1} > S_{d_2}$, this means, d_1 greater than d_2, denoted as $d_1 \succ d_2$;

(ii) If $S_{d_1} = S_{d_2}$, then find accuracy function;

(1) If $P_{d_1} = P_{d_2}$, then d_1 and d_2 are equal, denoted as $d_1 = d_2$;

(2) If $P_{d_1} > P_{d_2}$, then d_1 is greater then d_2, written as $d_1 \succ d_2$.

3. Dual hesitant fuzzy Bonferroni mean

Firstly, we determine the relationship between the operations of $DHFEs$ listed in Theorem 1:

Theorem 1. Suppose d_1 and d_2 be two $DHFEs$, and $\tau \geq 0$, then

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$;

(2) $d_1 \otimes d_2 = d_2 \otimes d_1$;

(3) $(d_1 \otimes d_2)^\tau = d_1^\tau \otimes d_2^\tau$;

(4) $\tau(d_1 \oplus d_2) = \tau d_1 \oplus \tau d_2$.

Proof. Properties (1) and (2) are evident from Definition (4), we just produce the verification of properties (3) and (4)

(3) $(d_1 \otimes d_2)^\tau = (\bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}\})^\tau$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}\}^\tau$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}\sigma_{d_2}\}^\tau, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{\kappa_{d_1}+\kappa_{d_2}-\kappa_{d_1}\kappa_{d_2}\}^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1}-\kappa_{d_2}+\kappa_{d_1}\kappa_{d_2})^\tau\}\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1})^\tau(1-\kappa_{d_2})^\tau\}\}$

and

$d_1^\tau \otimes d_2^\tau = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}}} \{(\sigma_{d_1}^\tau), 1-(1-\kappa_{d_1})^\tau\} \otimes \bigcup_{\substack{\sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{(\sigma_{d_2}^\tau), 1-(1-\kappa_{d_2})^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{(1-(1-\kappa_{d_1})^\tau)+(1-(1-\kappa_{d_2})^\tau)-(1-(1-\kappa_{d_1})^\tau)(1-$

$(1-\kappa_{d_2})^\tau)\}\} = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}^\tau\sigma_{d_2}^\tau\}, \{1-(1-\kappa_{d_1})^\tau(1-\kappa_{d_2})^\tau\}\}.$

Therefore, we have $(d_1 \otimes d_2)^\tau = d_1^\tau \otimes d_2^\tau$.

(4) Since

$\tau(d_1 \oplus d_2) = \tau(\bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{\{\sigma_{d_1}+\sigma_{d_2}-\sigma_{d_1}\sigma_{d_2}\}, \{\kappa_{d_1}\kappa_{d_2}\}\})$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1}-\sigma_{d_2}+\sigma_{d_1}\sigma_{d_2})^\tau, (\kappa_{d_1}\kappa_{d_2})^\tau\}$

$= \{1-(1-\sigma_{d_1})^\tau(1-\sigma_{d_2})^\tau, (\kappa_{d_1})^\tau(\kappa_{d_2})^\tau\}$

and

$\tau d_1 \oplus \tau d_2$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1})^\tau+1-(1-\sigma_{d_2})^\tau-(1-(1-\sigma_{d_1})^\tau)(1-(1-\sigma_{d_2})^\tau), (\kappa_{d_1}\kappa_{d_2})^\tau\}$

$= \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}} \{1-(1-\sigma_{d_1})^\tau(1-\sigma_{d_2})^\tau, (\kappa_{d_1})^\tau(\kappa_{d_2})^\tau\}$

Therefore, we have $\tau(d_1 \oplus d_2) = \tau d_1 \oplus \tau d_2$.

With the help of Definition 1, we have combined the BM and dual hesitant fuzzy information expressed as $DHFEs$ and constructed the $DHFBM$ as below:

Definition 6. Suppose $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ be a family of $DHFEs$. For any $\rho, \ell > 0$, if

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = \left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^\rho \otimes d_j^\ell)\right)\right)^{\frac{1}{\rho+\ell}}, \tag{2}$$

then we can call $DHFB^{\rho,\ell}$ as $DHFBM$.

Using the Theorem 1 and Definition 6, we have the following important result:

Theorem 2. *Suppose $\rho, \ell > 0$, $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ be a family of $DHFEs$, then by applying $DHFBM$, their aggregated value is a $DHFE$, and*

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}\\ i\neq j}} \left\{\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1 - \sigma_{d_i}^\rho \sigma_{d_j}^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}},\right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right\}. \tag{3}$$

Proof. Applying operations (2)-(4) and as discussed in Definition 4 and Theorem 1, we have

$d_i^\rho = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \{\sigma_{d_i}^\rho, 1-(1-\kappa_{d_i})^\ell\}$,

$d_j^\ell = \bigcup_{\substack{\sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \{\sigma_{d_j}^\ell, 1-(1-\kappa_{d_j})^\ell\}$,

$$d_i^\rho \otimes d_j^\ell = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{\sigma_{d_i}^\rho \sigma_{d_j}^\ell, 1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell\right\}. \tag{4}$$

We first prove the following expression

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^\rho \otimes d_j^\ell) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^\rho\sigma_{d_j}^\ell), \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^\rho(1-\kappa_{d_j})^\ell)\right\} \tag{5}$$

by applying induction method on τ as follows:

Step 1. When $\tau = 2$, we have

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{2}(d_i^{\rho}\otimes d_j^{\ell}) = (d_1^{\rho}\otimes d_2^{\ell})\oplus(d_2^{\rho}\otimes d_1^{\ell}) = \bigcup_{\substack{\sigma_{d_1}\in r_{d_1}\\ \kappa_{d_1}\in s_{d_1}\\ \sigma_{d_2}\in r_{d_2}\\ \kappa_{d_2}\in s_{d_2}}}\Big\{1-(1-\sigma_{d_1}^{\rho}\sigma_{d_2}^{\ell})(1-\sigma_{d_2}^{\rho}\sigma_{d_1}^{\ell}),$$
$$(1-(1-\kappa_{d_1})^{\rho}(1-\kappa_{d_2})^{\ell})\times(1-(1-\kappa_{d_2})^{\rho}(1-\kappa_{d_1})^{\ell})\Big\}.$$

Step 2. Suppose Eq. 5 is true for $\tau = k$, i.e.,

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}(d_i^{\rho}\otimes d_j^{\ell}) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}}\Big\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}),\ \prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})\Big\} \tag{6}$$

then, if $\tau = k+1$, and using the operations (1) – (4) as defined in Definition 4, we have

$$\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k+1}(d_i^{\rho}\otimes d_j^{\ell}) = \Bigg(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{k}(d_i^{\rho}\otimes d_j^{\ell})\Bigg)\oplus\Bigg(\bigoplus_{i=1}^{k}(d_i^{\rho}\otimes d_{k+1}^{\ell})\Bigg)$$
$$\oplus\Bigg(\bigoplus_{j=1}^{k}(d_{k+1}^{\rho}\otimes d_j^{\ell})\Bigg). \tag{7}$$

After applying induction on k and some simple calculations we can get

$$\bigoplus_{i=1}^{k}(d_i^{\rho}\otimes d_{k+1}^{\ell})$$
$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_{k+1}}\in r_{d_{k+1}}\\ \kappa_{d_{k+1}}\in s_{d_{k+1}}}}\Big\{1-\prod_{i=1}^{k}(1-\sigma_{d_i}^{\rho}\sigma_{d_{k+1}}^{\ell}),\prod_{i=1}^{k}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_{k+1}})^{\ell})\Big\}. \tag{8}$$

In the same way, we can have

$$\bigoplus_{j=1}^{k}(d_{k+1}^{\rho}\otimes d_j^{\ell})$$

$$(9)\quad = \bigcup_{\substack{\sigma_{d_{k+1}} \in r_{d_{k+1}} \\ \kappa_{d_{k+1}} \in s_{d_{k+1}} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \prod_{j=1}^{k} (1 - \sigma_{d_{k+1}}^{\rho} \sigma_{d_j}^{\ell}), \prod_{j=1}^{k} (1 - (1 - \kappa_{d_{k+1}})^{\rho} (1 - \kappa_{d_j})^{\ell}) \right\}.$$

Thus, from Eq. 6, Eq. 8, and Eq. 9, we can further transform Eq. 9 as

$$\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{k+1} (d_i^{\rho} \otimes d_j^{\ell}) = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{k} (1 - \sigma_{d_i}^{\rho} \sigma_{d_j}^{\ell}), \right.$$

$$\left. \prod_{\substack{i,j=1 \\ i \neq j}}^{k} (1 - (1 - \kappa_{d_i})^{\rho} (1 - \kappa_{d_j})^{\ell}) \right\}$$

$$\oplus \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_{k+1}} \in r_{d_{k+1}} \\ \kappa_{d_{k+1}} \in s_{d_{k+1}}}} \left\{ 1 - \prod_{i=1}^{k} (1 - \sigma_{d_i}^{\rho} \sigma_{d_{k+1}}^{\ell}), \prod_{i=1}^{k} (1 - (1 - \kappa_{d_i})^{\rho} (1 - \kappa_{d_{k+1}})^{\ell}) \right\}$$

$$\oplus \bigcup_{\substack{\sigma_{d_{k+1}} \in r_{d_{k+1}} \\ \kappa_{d_{k+1}} \in s_{d_{k+1}} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \prod_{j=1}^{k} (1 - \sigma_{d_{k+1}}^{\rho} \sigma_{d_j}^{\ell}), \prod_{j=1}^{k} (1 - (1 - \kappa_{d_{k+1}})^{\rho} (1 - \kappa_{d_j})^{\ell}) \right\}$$

$$= \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{k+1} (1 - \sigma_{d_i}^{\rho} \sigma_{d_j}^{\ell}), \prod_{\substack{i,j=1 \\ i \neq j}}^{k+1} (1 - (1 - \kappa_{d_i})^{\rho} (1 - \kappa_{d_j})^{\ell}) \right\}.$$

This implies that, Eq. 5 is true for $\tau = k + 1$. Therefore, Eq. 5 is true for all τ. Then, combining Eq. 5 and operation (3), we can have

$$\frac{1}{\tau(\tau - 1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{\tau} (d_i^{\rho} \otimes d_j^{\ell}) \right)$$

$$(10)\quad = \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i} \\ \sigma_{d_j} \in r_{d_j} \\ \kappa_{d_j} \in s_{d_j}}} \left\{ 1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}^{\rho} \sigma_{d_j}^{\ell}) \right)^{\frac{1}{\tau(\tau-1)}}, \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - \kappa_{d_i})^{\rho} (1 - \kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}} \right\},$$

when we apply operation (4) on Eq. 11, it yields the form

$$DHFB^{\rho,\ell}(d_1, d_2 \dots, d_\tau) = \left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^{\rho}\otimes d_j^{\ell})\right)\right)^{\frac{1}{\rho+\ell}}$$

(11)
$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}}} \left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}},\right.$$
$$\left. 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right\}.$$

In addition, since

$$0 \le \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \le 1,$$
$$0 \le 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \le 1$$

from Definition 2, we have

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} + 1$$
$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$\le 1+\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} = 1$$

hence, the proof completes.

Now, further we explore some important properties of $DHFBM$.

(1) **Idempotency:** If all d_i's $(i = 1, 2, \dots, \tau)$ are same, i.e., $d_i = d = (r_d, s_d)$,

for all i, then

$$DHFB^{\rho,\ell}(d_1, d_2 \ldots, d_\tau) = DHFB^{\rho,\ell}(d, d, \ldots, d)$$

$$= \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i}}} \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}^{\rho} \sigma_{d_i}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}, \right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - (1 - \kappa_{d_i})^{\rho} (1 - \kappa_{d_i})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \right\}$$

$$= \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i}}} \left\{ (1 - (1 - \sigma_{d_i}^{\rho+\ell}))^{\frac{1}{\rho+\ell}}, 1 - (1 - (1 - (1 - \kappa_{d_i})^{\frac{1}{\rho+\ell}}))^{\frac{1}{\rho+\ell}} \right\}$$

$$= \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i}}} \left\{ (\sigma_{d_i}^{\rho+\ell})^{\frac{1}{\rho+\ell}}, 1 - ((1 - \kappa_{d_i})^{\rho+\ell})^{\frac{1}{\rho+\ell}} \right\}$$

$$= \bigcup_{\substack{\sigma_{d_i} \in r_{d_i} \\ \kappa_{d_i} \in s_{d_i}}} \{\sigma_{d_i}, \kappa_{d_i}\}$$

$$= \{h_d, g_g\} = d.$$

(2) **Monotonicity:** Let $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \ldots, \tau)$ and $\tilde{d} = (r_{\tilde{d}_i}, s_{\tilde{d}_i})(i = 1, 2, \ldots, \tau)$ be two accumulations of $DHFNs$. If for any $\sigma_{d_i} \in r_{d_i}, \kappa_{d_i} \in s_{d_i}, \sigma_{d_j} \in r_{d_j}, \kappa_{d_j} \in s_{d_j}$, we have $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$DHFB^{\rho,\ell}(d_1, d_2 \ldots, d_\tau) \leq DHFB^{\rho,\ell}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_\tau).$$

Proof. Since $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then $\sigma_{d_i}\sigma_{d_j} \leq \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j}$, for all i, j

$$\prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}} \geq \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}$$

$$1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}} \leq 1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}$$

$$\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \leq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{\tau} (1 - \sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

similarly, for non membership function, if $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$(1 - \kappa_{d_i})^{\rho}(1 - \kappa_{d_j})^{\ell} \leq (1 - \kappa_{\tilde{d}_i})^{\rho}(1 - \kappa_{\tilde{d}_j})^{\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}} \geq \prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}} \tag{12}$$

$$\geq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right).$$

Let $d = DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau)$ and $\tilde{d} = DHFB^{\rho,\ell}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_\tau)$, and let S_d and $S_{\tilde{d}}$ be the scores of d and $\tilde{d}$, respectively. Then, inequality (3.12) says $S_d \leq S_{\tilde{d}}$. Presently, the accompanying cases emerge.

Case 1. If $S_d < S_{\tilde{d}}$, then, we have from Definition 3

$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) < DHFB^{\rho,\ell}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_\tau). \tag{13}$$

Case 2. If $S_d = S_{\tilde{d}}$, then

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)$$
$$=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right).$$

Since $\sigma_{d_i} \leq \sigma_{\tilde{d}_i}$ and $\kappa_{d_i} \geq \kappa_{\tilde{d}_i}$, for all i, then

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\Bigg)$$
$$=1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_i})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\Bigg).$$

and thus

$$P_d=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}\sigma_{d_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$
$$+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)$$
$$=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{\tilde{d}_i}\sigma_{\tilde{d}_j})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{\tilde{d}_i})^{\rho}(1-\kappa_{\tilde{d}_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right)=P_{\tilde{d}}.$$

Then, by Definition 3, we get

$$DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)=DHFB^{\rho,\ell}(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_\tau) \tag{14}$$

and hence, Eq. 13 and Eq. 14 guarantees the Monotonocity.

(3) **Commutativity:** Let $d_i=(r_{d_i},s_{d_i})(i=1,2,\ldots,\tau)$ be a family of $DHFNs$. Then $DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)=DHFB^{\rho,\ell}(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_\tau)$ where $(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_\tau)$ is any alteration of $(d_1,d_2,\ldots,d_\tau)$.

Proof. Since $(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_\tau)$ is any alteration of $(d_1,d_2,\ldots,d_\tau)$, then

$$\begin{aligned}DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)&=\left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^{\rho}\otimes d_j^{\ell})\right)\right)^{\frac{1}{\rho+\ell}}\\
&=\left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(\dot{d}_i^{\rho}\otimes \dot{d}_j^{\ell})\right)\right)^{\frac{1}{\rho+\ell}}\\
&=DHFB^{\rho,\ell}(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_\tau).\end{aligned}$$

(4) **Boundedness:** Let $d_i=(r_{d_i},s_{d_i})$ $(i=1,2,\ldots,\tau)$ be a family of $DHFNs$, and let

$$\begin{aligned}d^-&=(\min_i\{\sigma_{d_i}\},\max_i\{\kappa_{d_i}\}),\\ d^+&=(\max_i\{\sigma_{d_i}\},\min_i\{\kappa_{d_i}\}).\end{aligned}$$

Then

$$d^-\leq DHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau)\leq d^+. \tag{15}$$

Proof. Since $\min_i\{\sigma_{d_i}\}\leq\sigma_{d_i}\leq\max_i\{\sigma_{d_i}\}$ and $\min_i\{\kappa_{d_i}\}\leq\kappa_{d_i}\leq\max_i\{\kappa_{d_i}\}$, for all i, then

$$\begin{aligned}&(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\leq\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell}\leq(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\\
&\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\leq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\\
&\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\geq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\end{aligned}$$

and thus

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq\left(1-\left(1-(\max_i\{\sigma_{d_i}\})^{\rho+\ell}\right)\right)^{\frac{1}{\rho+\ell}}=\max_i\{\sigma_{d_i}\}, \tag{16}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^{\rho}\sigma_{d_j}^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\geq\left(1-\left(1-(\min_i\{\sigma_{d_i}\})^{\rho+\ell}\right)\right)^{\frac{1}{\rho+\ell}}=\min_i\{\sigma_{d_i}\}. \tag{17}$$

Similarly, for non-membership function, we can have

$$(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell}\leq(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell}\leq(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\geq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(1-\min_i\{\kappa_{d_i}\})^{\rho+\ell}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}$$

$$\leq\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell})^{\frac{1}{\tau(\tau-1)}}=1-(1-\max_i\{\kappa_{d_i}\})^{\rho+\ell}$$

thus

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\geq 1-(1-(1-(1-\min_i\{\kappa_{d_i}\}))^{\rho+\ell})^{\frac{1}{\rho+\ell}}=\min_i\{\kappa_{d_i}\}, \tag{18}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^{\rho}(1-\kappa_{d_j})^{\ell})^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}$$

$$\leq 1-(1-(1-(1-\max_i\{\kappa_{d_i}\}))^{\rho+\ell})^{\frac{1}{\rho+\ell}}=\max_i\{\kappa_{d_i}\}. \tag{19}$$

Let $d = DHFB_{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = (r_d, s_d)$. Then

$$S_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma - \frac{1}{\#s}\sum_{\kappa\in s}\kappa \leq \max_i\{\sigma_{d_i}\} - \min_i\{\kappa_{d_i}\} = S_d^+, \tag{20}$$

$$S_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma - \frac{1}{\#s}\sum_{\kappa\in s}\kappa \geq \min_i\{\sigma_{d_i}\} - \max_i\{\kappa_{d_i}\} = S_d^-. \tag{21}$$

As a result, following of the three cases arise.

Case 1. If $S_d < S_d^+$ and $S_d > S_d^-$, then, from Definition 3, this implies that $d^- < DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) < d^+$.

Case 2. If $S_d = S_d^+$, then, by Eq. 18 and Eq. 19, we have $\sigma_d = \max_i\{\sigma_{d_i}\}$, $\kappa_d = \min_i\{\kappa_{d_i}\}$. Thus

$$P_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma + \frac{1}{\#s}\sum_{\kappa\in s}\kappa = \max_i\{\sigma_{d_i}\} + \min_i\{\kappa_{d_i}\} = P_d^+,$$
$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = d^+.$$

Case 3. If $S_d = S_d^-$, then, by Eq. 20 and Eq. 21, we have $\sigma_d = \min_i\{\sigma_{d_i}\}, \kappa_d = \max_i\{\kappa_{d_i}\}$. Therefore, we can have

$$P_d = \frac{1}{\#r}\sum_{\sigma\in r}\sigma + \frac{1}{\#s}\sum_{\kappa\in s}\kappa = \min_i\{\sigma_{d_i}\} + \max_i\{\kappa_{d_i}\} = P_d^-,$$
$$DHFB^{\rho,\ell}(d_1, d_2, \ldots, d_\tau) = d^-.$$

Therefore, in all the aforementioned cases, Eq. 15 holds clearly.

For various estimations of the parameters ρ and ℓ some unique cases of the $DHFBM$ arises which are discussed as given below.

Case 1. If $\ell \to 0$, then, we have

$$\lim_{\ell\to 0} DHFB^{\rho,0}(d_1, d_2, \ldots, d_\tau) = \lim_{\ell\to 0}\left(\frac{1}{\tau}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(d_i^\rho \otimes d_j^\ell)\right)\right)^{\frac{1}{\rho+\ell}} \tag{22}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}}\left\{\left(1 - \prod_{i=1}^{\tau}(1-\sigma_{d_i}^\rho)^{\frac{1}{\tau}}\right)^{\frac{1}{\rho}}, 1 - \left(1 - \prod_{i=1}^{\tau}(1-(1-\kappa_{d_i})^\rho)^{\frac{1}{\tau}}\right)^{\frac{1}{\rho}}\right\}. \tag{23}$$

This is called generalized dual hesitant fuzzy mean ($GDHFM$).

Case 2. When $\rho = 2$ and $\ell \to 0$, we get dual hesitant fuzzy square mean $(DHFSM)$.

$$\lim_{\ell\to 0} DHFB^{2,0}(d_1, d_2, \dots, d_\tau) = \left(\frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^2 \otimes d_j^0)\right)\right)^{\frac{1}{2}} \tag{24}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{\left(1-\prod_{i=1}^{\tau}(1-\sigma_{d_i}^2)^{\frac{1}{\tau}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{i=1}^{\tau}(1-(1-\kappa_{d_i})^2)^{\frac{1}{\tau}}\right)^{\frac{1}{2}}\right\}. \tag{25}$$

Case 3. If $\rho = 1$ and $\ell \to 0$, we get dual hesitant fuzzy averaging operator $(DHFA)$.

$$\lim_{\ell\to 0} DHFB^{1,0}(d_1, d_2, \dots, d_\tau) = \frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^1 \otimes d_j^0)\right) \tag{26}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}}} \left\{\left(1-\prod_{i=1}^{\tau}(1-\sigma_{d_i})^{\frac{1}{\tau}}\right), \left(\prod_{i=1}^{\tau}(\kappa_{d_i})^{\frac{1}{\tau}}\right)\right\}. \tag{27}$$

Case 4. If $\rho = 1$ and $\ell = 1$, we get dual hesitant fuzzy interrelated square mean $(DHFISM)$.

$$DHFB^{1,1}(d_1, d_2, \dots, d_\tau) = \left(\frac{1}{\tau}\left(\bigoplus_{i=1}^{\tau}(d_i^1 \otimes d_j^1)\right)\right)^{\frac{1}{2}}$$

$$= \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}\\ i\neq j}} \left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-\sigma_{d_i}^1\sigma_{d_j}^1)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{2}},\right. \tag{28}$$

$$\left. 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i})^1(1-\kappa_{d_j})^1)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{2}}\right\}.$$

4. The WDHFBM and its application in MADM

In multiple attribute decision-making process, usually, the degree of importance of attributes may not be same, therefore they should be assigned different weights. To underscore the significance of collected information, we have constructed a $WDHFBM$ as given below:

Definition 5. Let $\rho, \ell > 0$, $d_i = (r_{d_i}, s_{d_i})(i = 1, 2, \dots, \tau)$ be a family of $DHFEs$, $\omega = (\omega_1, \omega_2, \dots, \omega_\tau)^T$ be the weight vector of $d_i = (i = 1, 2, \dots, \tau)$, where ω_i represents the importance degree of d_i, satisfying $\omega_i \in [0,1]$ $(i = 1,2,\ldots,\tau)$ and $\sum_{i=1}^{\tau}\omega_i = 1$, if

$$(29)\quad WDHFB_\omega(d_1,d_2,\ldots,d_\tau) = \left(\frac{1}{\tau(\tau-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{\tau}(\omega_i d_i)^\rho \otimes (\omega_j d_j)^\ell)\right)\right)^{\frac{1}{\rho+\ell}}$$

then we can say $DHFB_\omega^{\rho,\ell}$ a $WDHFBM$.

Using Eq. 29, Next Theorem is there:

Theorem 4. *Let $\rho,\ell > 0$, $d_i = (r_{d_i}, s_{d_i})$ $(i = 1,2,\ldots,\tau)$ be a family of $DHFEs$, and $\omega = (\omega_1,\omega_2,\ldots,\omega_\tau)^T$ be the weight vector of $d_i (i = 1,2,\ldots,\tau)$, satisfying $\omega_i \in [0,1]$ $(i = 1,2,\ldots,\tau)$ and $\sum_{i=1}^{\tau}\omega_i = 1$, then according to the $WDHFB$, the collected value of the d_i's also a $DHFN$,*

$$(30)\quad WDHFB^{\rho,\ell}(d_1,d_2,\ldots,d_\tau) = \bigcup_{\substack{\sigma_{d_i}\in r_{d_i}\\ \kappa_{d_i}\in s_{d_i}\\ \sigma_{d_j}\in r_{d_j}\\ \kappa_{d_j}\in s_{d_j}\\ i\neq j}} \left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-(1-\sigma_{d_i})^{\omega_i})^\rho \times (1-(1-\sigma_{d_j})^{\omega_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}},\right.$$
$$\left. 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{\tau}(1-(1-\kappa_{d_i}^{\omega_i})^\rho(1-\kappa_{d_j}^{\omega_j})^\ell)^{\frac{1}{\tau(\tau-1)}}\right)^{\frac{1}{\rho+\ell}}\right\}.$$

Now, for the usual application of weighted aggregation operators, we describe some basic steps for $MADM$ based on the $WDHFBM$.

Step 1. For a $MADM$ problem, let $Y = \{y_1,y_2,\ldots,y_n\}$ and $C = \{c_1,c_2,\ldots,c_m\}$ be a set of n alternatives with m attributes respectively, having weight vector $\omega = \{\omega_1,\omega_2,\ldots,\omega_m\}^T$, which satisfy the condition $\omega_j > 0, j = 1,2,\ldots,m$, and $\sum_{j=1}^{m}\omega_j = 1$, where ω_j expresses the importance degree of the attribute c_j. The decision maker measure the performance of the alternative y_i corresponding to the attribute c_j by the $DHFN$ $k_{ij} = (r_{ij}, s_{ij})$, $r_{ij} = \bigcup_{\sigma_{ij}\in r_{ij}}\{\sigma_{ij}\}$ denotes the satisfaction degree of an alternative y_i corresponding the attribute c_j and $s_{ij} = \bigcup_{\kappa_{ij}\in s_{ij}}\{\kappa_{ij}\}$ denotes the dissatisfaction degree of an alternative y_i corresponding the attribute c_j such that $r_{ij}, s_{ij} \in [0,1]$, and $0 \le \sigma_{ij}^+ + \kappa_{ij}^+ \le 1$ where $\sigma_{ij}^+ \in r_{ij}^+ = \bigcup_{\sigma_{ij}\in r_{ij}}\max\{\sigma_{ij}\}$, and $\kappa_{ij}^+ \in s_{ij}^+ = \bigcup_{\kappa_{ij}\in s_{ij}}\max\{\kappa_{ij}\}$,
The dual hesitant fuzzy decision matrix $K = (k_{ij})_{n\times m}$ contains all information

Table 1: **The dual hesitant fuzzy decision matrix**

	c_1	c_2	$\dots$	c_n
y_1	(r_{11}, s_{11})	(r_{12}, s_{12})	$\dots$	(r_{1m}, s_{1m})
y_2	(r_{21}, s_{21})	(r_{22}, s_{22})	$\dots$	(r_{2m}, s_{2m})
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
y_n	(r_{n1}, s_{n1})	(r_{n2}, s_{n2})	$\dots$	(r_{nm}, s_{nm})

$k_{ij} = (r_{ij}, s_{ij})$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$. (see Table1). During the aggregation process, there may exist two types of attributes, that is, (1) the profit type attribute; and (2) the cost type attribute. If this happens, we should convert the cost type attribute preference values in terms of the benefit type attribute preference values. Then, $K = (k_{ij})_{n\times m}$ can be convert ed into the normalized matrix $R = (g_{ij})_{n\times m}$, where $g_{ij} = (t_{ij}, f_{ij}) = \begin{cases} g_{ij}, & \text{for benefit attribute} \quad c_j \\ \overline{g_{ij}}, & \text{for cost attribute} \quad c_j \end{cases}$
$i = 1, 2, \dots, n \quad j = 1, 2, \dots, m$
where $\overline{g_{ij}}$ is the complement of g_{ij} such that $\overline{g_{ij}} = (\kappa_{ij}, \sigma_{ij})$.

Step 2. Utilize the $WGHFBM$ (generally, we will use $\rho = \ell = 1$). Let r_i denotes the aggregated value of all the preference values $r_{ij}(j = 1, 2, \dots, m)$ of the ith line and corresponding to the alternative y_i.

Step 3. Utilize the steps of Definition 3 for ranking the inclusive performance values $r_i(i = 1, 2, \dots, n)$.

Step 4. Arrange all the alternatives $y_i(i = 1, 2, \dots, n)$ in descending manner according to $r_i(i = 1, 2, \dots, n)$, and then accept the optimal alternative having inclusively high performance value.

We apply our developed $WDHFBM$ aggregation operator to an $MADM$ problem.

Example. Let us examine a real life example of decision making. A Mayor of the town is planning to construct a hospital. Four feasible contractors (alternatives) are available $y_i(i = 1, 2, 3, 4)$. The decision makers examines four attributes, c_1 (economic), c_2 (functional), c_3 (operational) and c_4 (time period). Let $\omega = (0.2, 0.4, 0.1, 0.3)^T$ the weight vector of the attribute $c_j(j = 1, 2, 3, 4)$. Suppose that the preference values of the alternatives $y_i(i = 1, 2, 3, 4)$ corresponding to the attributes $c_j(j = 1, 2, 3, 4)$ are given by the $DHFEs$ $d_{ij} = (r_{ij}, s_{ij}) = (\bigcup_{\sigma_{ij}\in h_{ij}}\{\sigma_{ij}\}, \bigcup_{\kappa_{ij}\in g_{ij}}\{\kappa_{ij}\})$ where σ_{ij} illustrates the satisfaction level of the alterative y_i with attribute c_j and κ_{ij} illustrates the dissatisfaction level of the alternative y_i with attribute c_j. All d_{ij}'s $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ are involved in the dual hesitant fuzzy decision matrix $K = (k_{ij})_{4\times 4}$ (see Table 2). Now, we apply $WDHFBM$ (here, we take $\rho = \ell = 1$)

Table 2: **The dual hesitant fuzzy decision matrix**

	c_1	c_2	c_3	c_4
y_1	$\langle\{0.5, 0.6, 0.7\}, \{0.6, 0.7\}\rangle$	$\langle\{0.3, 0.4, 0.5\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2, 0.3\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2, 0.3\}, \{0.4, 0.5\}\rangle$
y_2	$\langle\{0.6, 0.7\}, \{0.1, 0.2\}\rangle$	$\langle\{0.6, 0.7\}, \{0.1, 0.2\}\rangle$	$\langle\{0.1, 0.2\}, \{0.1, 0.2\}\rangle$	$\langle\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\rangle$
y_3	$\langle\{0.4, 0.5\}, \{0.1, 0.2\}\rangle$	$\langle\{0.3, 0.4\}, \{0.1, 0.2\}\rangle$	$\langle\{0.6, 0.7\}, \{0.4, 0.5\}\rangle$	$\langle\{0.3, 0.4\}, \{0.1, 0.2\}\rangle$
y_4	$\langle\{0.2, 0.3, 0.4\}, \{0.1, 0.2\}\rangle$	$\langle\{0.2, 0.3\}, \{0.3, 0.4\}\rangle$	$\langle\{0.3, 0.4\}, \{0.3, 0.4, 0.5\}\rangle$	$\langle\{0.7, 0.8\}, \{0.2, 0.3\}\rangle$

$$d_i = (r_{d_i}, s_{d_i}) = DHFB_w^{1,1}(d_{i1}, d_{i2}, d_{i3}, d_{i4})$$

to total all the inclination esteems $d_{ij}(i = 1, 2, 3, 4)$ of the ith line and get the inclusive performance value d_i of the alternative y_i as

$$\begin{aligned} d_1 &= (\{0.0624, 0.0714, \dots, 0.1079\}, \{0.7326, 0.7487, \dots, 0.7817\}), \\ d_2 &= (\{0.1176, 0.1265, \dots, 0.1693\}, \{0.6369, 0.6623, \dots, 0.7219\}), \\ d_3 &= (\{0.1249, 0.1446, \dots, 0.1644\}, \{0.6181, 0.6444, \dots, 0.6814\}), \\ d_4 &= (\{0.1241, 0.1102, \dots, 0.1529\}, \{0.6924, 0.7123, \dots, 0.7608\}). \end{aligned}$$

Score function of all the alternatives is given as below

$$S_{d_1} = -0.6802,\ S_{d_2} = -0.5327,\ S_{d_3} = -0.5146,\ S_{d_4} = -0.5886,$$

Since $S_{d_3} > S_{d_2} > S_{d_4} > S_{d_1}$, then according to Definition 5, we can get the ranking of the $DHFNs$, which is $d_3 > d_2 > d_4 > d_1$, and then arrangements of the alternatives $y_i(i = 1, 2, 3, 4)$ in descending manner is given as $y_3 > y_2 > y_4 > y_1$. Hence, y_3 is the optimal choice.

5. Conclusion

In this manuscript, we have extended the BM under the dual hesitant fuzzy datum. We have defined some novel dual hesitant fuzzy aggregation operators, such as $DHFBM$, the $WDHFBM$, and the several special cases of the $DHFBM$. We have also discuss the basic properties, such as, idempotency, monotonicity, commutativity and boundedness in detail. The basic idea of the $WDHFBM$ in multi-attribute decision-making is that it captures the importance of attribute individually and reflects the interrelationship of the each attribute and thus gives the information about decision-making into account as much as possible. In future research, we will try to explore our BM aggregation operators to more comprehensive models like Dual Extended HFSs [27] or Expanded Dual HFSs [28]. They complement the ideas of Extended HFSs with duality.

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