Modules with quasi-continuous submodules

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Abstract. The notion of left q-ring was generalized to the big class of left π-rings. Furthermore, q-rings extended to module theory under the concept of Q-module (modules all of whose submodules are qausi-injective). In this article, the concept of QC-modules (modules all of whose submodules are qausi-continuous) are introduced as a wider class of both π-rings and Q-modules. Numerous characterizations of QC-modules have been obtained. Also, properties and results of these modules are investigated. Moreover, strong forms of QC-modules are given and studied.

Keywords: C_1-condition, C_2-condition, QC-modules, qausi-continuous modules, continuous modules.

1. Introduction

Rings with left ideals are qausi-injective (named as left q-rings) were identified by S.K. Jain et al. [6] and were extended to the modules case by the authors in [8], that is, a module M with qausi-injective submodules is called Q-module. In 1999, S.K. Jain et al. [5] introduced π-rings as a generalization of q-rings, that is, a ring R is called π-ring if every right ideal of is qausi-continuous (or π-injective).

In this work, we generalize π-rings to the category of modules and moreover we study QC-modules as a proper bigger class than the class of Q-modules. We call a module M is QC-module if every sub module of M is qausi-continuous (π-injective). Also, we introduce strong forms of QC property. A module is C-continuous if M whose submodule are continuous as well as we call a module M is strictly QC-module if M is continuous and every sub module is qausi-continuous.

In the whole of this work R will denote an associative ring with unity and M will denote a unitary left R-module. From [14], a submodule A is called Large (or essential) in M if $A \cap B \neq (0)$ where B is a non-zero sub module of M. So, C is called closed if there is a sub module B of M with C is large in B, then $C = B$. By using Zorn's lemma, every submodule of M is contained essentially in closed sub module of M. By $E(M)$ we mean that the injective hull (or envelope) of the module M and the set of all endomorphisms of M is denoted by S =End (M). Given two modules K and H, K is called H-injective if each $f \in Hom(C, K)$ where $C$is a sub module of H can be extended to $g \in Hom(H, K)$ [3]. A module K is called injective if for each module H, K is H- injective. In particular, A is called qausi-injective if A is A -injective.

Let M be a module and regard the next conditions:

C_1-condition: Let A be a submodule of M. Then there is a direct summand B of M such that $A$is large in B.

C_2- condition: If $A \cong B$(where A be a submodule of M and B is direct summand of M), then B is direct summand of M.

C_3- condition: The direct sum of any two direct summands of M with zero intersection is direct summand.

A module M is called (qausi-)continuous, if it satisfies C_1-condition and C_2-condition (C_1-condition and C_3-condition) [9]. From [14], the concepts of qausi-continuity and π-injectivity are the same. M is π-injective whenever A_1, A_2 are submodules of M with$A_1 \cap A_2 = 0$, each projection $\pi_i : A_1 \oplus A_2 \longrightarrow A_i$ $(i = 1, 2)$ lifts to an endomorphism of M. Moreover, a module N is qausi-continuous iff whenever $E(N) = \oplus_{\alpha\in\Lambda} E_\alpha$ (where E_α are submodules of $E(N)$ $\forall \alpha \in \Lambda$, then $N = (N \cap E_\alpha)$ [4]. From [12], a sub module F of M is called fully invariant if $\alpha(F) \subseteq F$ for any $\alpha \in End(M)$. Furthermore, M is called duo if it whose submodules are fully invariant [12].

2. QC-modules

Definition 2.1 A module M is called QC-module if every submodule of M is qausi-continuous (equivalently, π-injective).

In particular, a ring R is right π-ring [5] if R as right R-module is QC-module.

Remark 2.2. 1. Precisely, Q-module is QC-module. The contrary is not true generally. In fact, the set of rational numbers Q_Z is QC-module but not Q-module.

2. QC-module implies qausi-continuous. The contrary is not valid. Absolutely, $Z_{2^\infty} \oplus Z_{2^\infty}$ is qausi-continuous (it is an injective Z-module since injective class is closed under finite direct sum), although $M = Z_{2^\infty} \oplus Z_{2^\infty}$ is not QC-module since $Z_2 \oplus Z_8$ is sub module of M which is not qausi-continuous (since it is not satisfy the C_1-condition).

3. Each uniform module implies QC-module (because every uniform module is qausi-continuous and every submodule of uniform module is uniform [9], while the contrary is not valid generally. $Z/6Z$ as Z-module is QC-module and not uniform.

4. Each semi-simple module is QC-module (because semisimple module implies π-injective (qausi-continuous) and qausi-continuous property is closed

under direct summands [9]), but the contrary is not valid. Instance, $Q_Z$is not semisimple while it is QC-module.

5. It is explicit that the QC-module property is closed under submodules.

6. One can investigate QC-modules property is not closed under direct sum. Consider Z_2 and Z_4 as Z-modules, thus they are QC -modules (since they are uinform modules). But $Z_2 \oplus Z_4$ is not π-module. In fact, $A = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{0})\}$ and $B = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{2})\}$ are direct summands of $Z_2 \oplus Z_4$ with $A \cap B = 0$. Easy calculation shows that$A \oplus B = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{2}), (\overline{1}, \overline{0}), (\overline{0}, \overline{2})\}$ is not direct summand of $Z_2 \oplus Z_4$. Thus, $Z_2 \oplus Z_4$ is not qausi-continuous Z-module. So, $Z_2 \oplus Z_4$ is not QC-module.

7. An indecomposable module is qausi-continuous iff it is QC -module.

8. Directly, by (6) and Theorem 1.10 in [4], we have an analogue of Matlis theorem [7] as follows: If R is Notherian ring and M be a QC -module, then M is a direct sum of indecomposable QC -modules.

9. The classes of QC -modules and continuous modules are distinct. For instance, Z_Z is not continuous (since it is not satisfying C_2-condition) whereas Z_Z is QC-module. The contrary, one can use the same example in (2).

The next proposition generalizes Theorem 1.4 in [5] to the category of modules.

Theorem 2.3. *M is QC-module iff M whose sub modules are satisfy -condition and if C and D are sub modules of M with $C \cap D = 0$, then C and D are mutually injective.*

Proof. For the necessary part, let M be a QC -module. Directly, the submodules of M are satisfy C_1-condition. Let C, D are sub module of M with $C \cap D = 0$. By hypothesis, $C \oplus D$ is qausi-continuous and by [9] C is D –injective and D is C –injective (i.e)C and D are mutually injective. Conversely, Given a sub module T in M and $E(T) = E_1 \cap E_2$ and put $C = E_1 \cap T$ and $C = E_2 \cap T$. It is easy to inquiry that$C = E_1 \cap T$ is closed of T and by hypothesis T satisfies C_1-condition, so $T = C \oplus H$ for some sub module H of T. Let π_C and π_D be the corresponding projections maps. $D \cong \pi_H(D) \subseteq X$. Define $\alpha : \pi_H(D) \longrightarrow C$ by $\alpha(\pi_H(d)) = \pi_C(d)$ for each d in D. The map is well defined since $\pi_H : D \longrightarrow \pi_H(D)$ is an isomorphism. Now, since C is H-injective, so α extended for $\beta : H \longrightarrow C$. Put $H^* = \{h + \beta(h) | h \in H\}$ Then $T = C$. Then $T = C \oplus H*$. Also, D is essential in H^* and D is closed. Thus $D = H^*$ and therefore, $T = C \oplus D$. Hence, T is qausi-continuous and therefore M is QC -module. □

Corollary 2.4 ([5])**.** *A ring is a left π-ring iff R whose left ideals are satisfy C_1-condition and if C and D are left ideals of R with $C \cap D = 0$, then C and D are mutually injective.*

The next result asserts that we need a very restricted class of sub modules to get QC -modules.

Theorem 2.5. *M is QC-module iff each large submodule of M is qausi-continuous.*

Proof. ($\Rightarrow$). The proof id direct by the definition of QC -modules.

($\Leftarrow$) Given a submodule T in M, thus $T \oplus K$ is large submodule in M, where K a complement of N. Hence, from assumption, $T \oplus K$ is qausi-continuous and moreover T is qausi-continuous [9]. So, M is QC-module. □

Corollary 2.6 ([5])**.** *A ring R is left π -ring iff each large left ideal in R is qausi-continuous.*

By using Theorem 2.4 and Theorem 2.3, the next result is concluded.

Corollary 2.7. *M is QC-module iff any large submodule in M satisfy C_1-condition and if C and D are large submodules of M with $C \cap D = 0$, then C andD are mutually injective.*

Forthcoming Theorem 1.1.6 in [8], Q-modules is equivalent to qausi-injective modules whose essential sub modules are fully invariant. Here, we have the next result as a characterization of QC-modules.

Firstly, we need the following well-known result for qausi-continuous modules.

Lemma 2.8 ([9])**.** *A module W is qausi-continuous if and only if $e(W) \subseteq W$ for each idempotent endomorphism e of $E(W)$.*

Theorem 2.9. *M is QC-module iff M is qausi-continuous with any essential submodule is fully invariant in M relative to idempotent elements of $S = End(M)$.*

Proof. ($\Rightarrow$) Directly, by QC-module property, we have M is qausi-continuous module. Consider A as an essential sub module of M and let $g : M \longrightarrow M$ be idempotent homomorphism. But the injective envelope $E(M)$ is qausi-continuous module, so g extended to an idempotent endomorphism of $E(M)$ [14]. Now, since N is essential, thus by [14] $E(A) = E(M)$ and A is qausi-continuous (since M is QC-module), thus $g(N) \subseteq N$.

($\Leftarrow$) Given A a large sub module in M. Let τ be an idempotent endomorphism of A. But A is large sub module in M, so $E(M) = E(M)$ and moreover τ is idempotent endomorphism of $E(M)$. But, by assumption, M implies qausi-continuous module, thus $\tau(M) \subseteq (M)$ (i.e) $\tau \in End(M)$. But A (by hypothesis) is fully invariant sub module of M relative to idempotent elements. Hence, $\tau(A) \subseteq (A)$. Thus, A is qausi-continuous module. So M satisfy QC –module (from Theorem 2.6). □

Following [1], sub module T is called projection-invariant in M if $\alpha(T) \subseteq T$ for all idempotent endomorphism α of M as well as if every sub module of M is projection-invariant, then M is said to satisfy $(*)$ condition.

Corollary 2.10. *A module M satisfy $(*)$ condition is qausi-continuous iff M is QC -module.*

Since every Duo module satisfy $(*)$ condition [1], so the next result arises.

Corollary 2.11. *A duo module is qausi-continuous iff it is QC-module.*

The next corollary provide a rich source of examples of QC-modules.

Corollary 2.12. *A commutative ring is a π-ring iff it is a qausi-continuous.*

Following [9], if a module N isomorphic to $N \oplus N$, then N is called purely infinite. Here, we call M is fully purely infinite if its sub modules are purely infinite.

Proposition 2.13. *A fully purely infinite module is Q-module iff it is QC -module.*

Proof. It is obvious by corollary 2.12 in [9]. □

3. Strong forms of QC-modules

Definition 3.1. A module M is called is C-module whenever any submodule in M is continuous.

Definition 3.2. M is called strictly QC-module whenever M is continuous and their sub modules of M are qausi-continuous.

Remarks 3.3. 1. The following implications are direct by using the above definitions: C-modules $\Rightarrow$ Strictly QC - modules $\Rightarrow$ QC -modules. The opposite of these implications are not true generally. Indeed, Z_Z is QC-module while Z_Z is not strictly QC-module because it is not continuous. Moreover, Q_Z is strictly QC-module which it is not satisfy C-module.

2. Precisely, each semi-simple module is C-module.

3. Unlike QC -modules, uniform modules need not be C-modules. For example, $Q$is uniform Z-module which it is not C-module.

4. The class of C-modules is need not be closed under direct sum property. $Z_2 \oplus Z_8$ as Z-module is not C-module while Z_2 and Z_8 as Z-modules are C-modules.

5. As like QC -modules, the C-module property is closed under sub modules without auxiliary conditions.

By using the similar criteria in the proof of Theorem 2.5, we get a characterization of C-modules.

Proposition 3.4. *M is QC-module if and only if every large submodule of M continuous.*

Observe which Z_Z is large of Q_Z and Q_Z is continuous which it is not C-module.

Definition 3.5. (1) A module M is said to be fully C_1-module if sub modules of M are satisfy C_1-condition.

(2) A module M is said to be fully C_2-module if sub modules of M are satisfy C_2-condition.

Remarks 3.6. 1. It is obvious that$M$ is C-module if and only if it is fully C_1 -module and fully C_2 -module.

2. Each QC-module is fully C_1-module while the contrary is not valid generally. Let $T = \begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}$ where Z_2 is the set of integers modulo 2 . Then, T as T-module is fully C_1-module but it is not QC-module since it is not qausi-continuous. Note that T_T does not satisfy fully C_2 module.

3. There is a QC-module which it is not fully C_2-module. For instance,Z_Z is not fully C_2- module whereas it achieve QC-module. Recall that an module M is direct-injective if for each direct summand D in M with inclusion mapping $i_D : D \longrightarrow M$ and any monomorphism $\lambda : D \longrightarrow M$ there exist an endomorphism $\psi : M \longrightarrow M$ such that$\psi \circ \lambda = i_D$[11]. Following Theorem 7.13 in [11], direct-injective modules and C_2-modules are equivalent. So the next result is direct.

Proposition 3.7. *A module M is fully C_2-module if and only M whose sub modules are direct-injective.*

Proposition 3.8. *A module $M$is fully C_2-module if and only if M whose essential sub modules are direct-injective.*

Proof. The necessary part is immediate. For the sufficient part, let T be a submdodule in M. Thus, a complement submodule H of T in M is exist such that $T \oplus H$ is large sub module of M (by Zorn,s Axiom). By hypothesis, $T \oplus H$ is direct-injective. By [11], T is direct-injective. So, by proposition (3.7), M is fully C_2-module. □

By using the similar attitude of the proof of proposition (2.6), we get a useful equivalent statement of fully C_1-modules.

Proposition 3.9. *A module M is fully C_1-module iff large sub modules of M are satisfy C_1-condition.*

On can obtain another express of Theorem 1.3 such the following:

Theorem 3.10. *A module M is QC-module if and only if M is fully C_1-module and if C and D are sub modules of M with $C \cap D = 0$, then C and D are mutually injective.*

We mentioned that every Q-module is C-module. In the next result, we give a condition under it the converse is true.

Proposition 3.11. Over commutative Noetherian, the concepts of Q-modules and C-modules are equivalent.

Proof. Since over a commutative Noetherian ring qausi-injectivity and continuity for modules are equivalent [13]. □

From [10], recall that a sub module A of M is stable if $\alpha(A) \subseteq A$ for any $\alpha \in Hom(A, M)$. Moreover, if every submodule of M is stable, then M is called fully stable.

Following [2], a module M is called has SC_1-condition if for each submodule A of M there is a stable direct summand B such that A is large in B. Also, M is called has SC_3-condition if whenever S_1 and S_2 are summands of S_2 with $S_1 \cap S_2$, $S_1 \oplus S_2$ is summand of M. Moreover, if M satisfy both SC_1-condition and SC_3-condition, then $M$is called strongly qausi-continuous.

As strong form of QC-modules arises as follows:

Definition 3.12. A module M is called amply QC-module if every sub module of M is strongly qausi-continuous. A ring R is called amply QC-ring if R as R-module is amply QC-module.

Remarks 3.13. 1. Directly, amply QC-module is QC-module while the contrary is not true generally. The vector space$V = R \oplus R$ as R-module is QC-module since it is semisimple whereas it is not amply QC-module because Vdoesn't have SC_2-condition since $0 \oplus R \cong R$ but $0 \oplus R$ is not stable.

2. Precisely, uniform module is amply QC-module. The contrary is not valid. $Z/10Z$ is amply QC-module whereas it not uniform.

3. Contrary QC-modules, semisimple modules need not be amply QC-modules. One can take the same example in (1).

4. Q-modules and amply QC-modules are various classes. Q_Z is amply QC-module whilst Q_Z is not satisfy Q-module. Also, $V = R \oplus R$ as R-module is Q-module whereas it is not amply QC-module.

5. Similar to QC-modules, amply QC-modules have hereditary property.

6. Identical to QC-modules, the direct sum of amply QC-modules not necessary amply QC-module. $Z_{p^\infty} \oplus Z_{p^\infty}$ as Z-module is not amply QC-module whereas Z_{p^∞} as Z-module is amply QC-module.

Proposition 3.14. *M is amply QC-module if and only if every large sub module of M is strongly qausi-continuous.*

Proof. $\Rightarrow$ Straightforward.

$\Leftarrow$ Given a submodule A of M. Then there a complement sub module B with $A \oplus B$ is large in M. From assumption, $A \oplus B$ is strongly qausi-continuous. From [2], A is strongly qausi-continuous and so M is amply QC-module. □

From [2], a module whose direct summand are stable is called SS-module. We use this concept to introduce the next concept:

Definition 3.15. *A module M is called fully SS-module if every sub module of M is SS-module. Uniform modules, fully stable modules and commutative rings are explicitly examples of fully SS-modules.*

Proposition 3.16. *A module M is fully SS-module if and only if any sub module T in M with $T = S \oplus F$,$Hom\,(S, F) = 0$.*

Proof. Given a direct summand T of a sub module N in M. Thus $N = D \oplus C$where C be a sub module of N. By assumption, $Hom\,(D, C) = 0$. Take the projection homomrphism $\rho : N = D \oplus C \longrightarrow C$ with $ker\,(\rho) = S$ .Let $\eta \in Hom\,(D, N)$. Then, $\rho\eta\,(D) = \rho\,(\eta\,(D)) = 0$ and so $\eta\,(D) \subseteq D$. Therefore, D is stable as well as N is SS-module and it is fully SS-module. Conversely, suppose T is a sub module of M such that $T = S \oplus F$. By hypothesis, T is SS-module and so S is stable sub module of T. Let $\xi \in Hom\,(S, F)$. Then, ξ can be extended to a homomrphism $\psi : T \longrightarrow F$ such that $\psi\,(s + f) = \xi\,(s)$. Since S is stable sub module of T, thus $\psi\,(s) = \xi\,(s) \in S$ for each $s \in S$. But $\xi\,(s) \in F$ and $S \cap F = 0$, thus $\xi = 0$. Therefore, $Hom\,(S, F) = 0$. □

Definition 3.17. (a) If sub modules of M are satisfy SC_1-condition, then M is called fully SC_1-condition.

(b) A module M is called fully SC_2-condition if every sub module of M has SC_2-condition.

Clearly, M is amply QC-module if and only if M is fully SC_1-condition and M is fully SC_3-condition.

Lemma 3.18. (1) *M is fully SC_1 -condition if and only if M is fully SS-module and M is fully C1-condition.*

(2) *M is fully SC_3-condition if and only if M is fully SS-module and M is fully C_3-condition.*

Proof. (1) $\Rightarrow$ Given a direct summand K of a sub module N in M. By hypothesis, N has SC_2-condition and snice K is closed so K is stable. By, definitions, fully SC_1-condition implies fully C_1-condition.

$\Leftarrow$ From Lemma 2.3.4 in [2], SC_1-condition and C_1-condition are equivalent under SS-module. So the necessary part is immediate.

(2) $\Rightarrow$ Given a direct summand C of a sub module N in M. Consider a direct summand $D = (0)$ of N. So $C \cap D = 0$. But N has SC_3-condition (because M is fully SC_2-module), so $C \oplus D = C$ is stable direct summand of N. So, M is fully SS-module.

$\Leftarrow$ It is obvious from Lemma (2.3.6) in [2]. □

From Lemma (3.18), the next result is achieved directly.

Proposition 3.19. *M is amply QC-module if and only if M is fully SS-module and M is QC-module.*

Corollary 3.20. *Let R commutative ring. Then the next statements are equivalent:*

I. *R is amply QC-module.*

II. *R is strongly qausi-continuous ring.*

III. *R is qausi-continuous.*

IV. *R is QC-module.*

Corollary 3.21. *Every commutative q-ring is amply QC-ring.*

Corollary 3.22. *The following are equivalent:*

I. *M is amply QC-module:*

II. *M is fully SC_1-condition and M is fully C_3-condition.*

III. *M is fully C_1-condition and M is fully SC_3-condition.*

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Accepted: June 24, 2020