Implicit hybrid block methods for solving second, third and fourth orders ordinary differential equations directly

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Abstract. Strong numerical algorithm for simultaneously solving second, third and fourth orders of ordinary differential equations with given initial conditions is considered in this paper. Linear Multi-step collocation approach was applied in the derivation of the new method with the use of power series approximate solution as interpolation polynomial. The numerical integrators that formed the block were derived by evaluating the continuous scheme along with its derivatives at the non-interpolating points within the selected interval of integration. The basic properties of the method were investigated properly. In order to examine the effectiveness of the block, the methods were tested on some second, third and fourth orders ordinary differential equations and the results generated proved its effectiveness over existing methods in terms of accuracy.

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1. Introduction

The numerical solution of second, third and fourth orders initial value problems of ordinary differential equations (ODEs) directly using a linear multistep hybrid block method is considered in this study.

Higher order ODEs have extensive usage in engineering and physical sciences. Very often, these problems do not have an analytical solution, and this has necessitated the use of numerical method to approximate their solutions. Previously, higher order ODEs were solved by reducing them to their equivalent system of first order ordinary differential equations and thereafter suitable numerical method for first order ODE would be applied to solve the systems. However, it is observed in Lambert and Watson [10] and Awoyemi [1] that this reduction approaches has serious problems which include wastage of human effort, computational burden and non-economization of computer time. A P-stable linear multistep method for direct solution of second order ODEs was developed by Awoyemi [2] which was implemented in predictor-corrector mode. It was observed that the predictors have reducing order of accuracy which affects the efficiency of the predictor-corrector method. In addition, the implementation requires much human effort and computer time.

To cater for these setbacks encountered in the reduction and predictor-corrector approaches, and also bring about improvement on the accuracy of numerical method Jator [9], Ogunware et al. [13], Skwame et al. [18] amongst others developed block methods for solving higher order ordinary differential equations directly in which the accuracy is better than the reduction of order to system of first order ordinary differential equations.

Also, various authors such as Omole and Ogunware [15], Adeyeye and Omar [3], Akinfewa et al. [5] respectively developed hybrid block methods for the direct solution of second, third and fourth order ODEs. These hybrid block methods, while retaining certain characteristics of the continuous linear multistep method, share with Runge-Kutta's Methods the property of utilizing data at further points, other than the step point. The block method was found to be efficient in handling higher order ODEs and it was also cost effective.

Therefore, in this research, the development and application of a two-step linear multistep method with four hybrid points implemented in block mode for the solution of second, third and fourth orders ODEs concurrently which is rare in numerical analysis works will be our target.

2. Research methodology

We considered power series as an approximate solution to higher order ODEs to be of the form

$$y(x) = \sum_{j=0}^{\eta+\varphi} a_j x^j \tag{1}$$

where $a_j^{'s}$ are parameters to be determined, η and ψ are the respective number of distinct collocation and interpolation points.

The fourth, fifth and sixth derivatives of (1) are obtained as

$$y^{'v}(x) = \sum_{j=4}^{\eta+\varphi} j(j-1)(j-2)(j-3)a_j x^{j-4}, \tag{2}$$

$$y^{v}(x) = \sum_{j=5}^{\eta+\varphi} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}, \tag{3}$$

$$y^{v'}(x) = \sum_{j=6}^{\eta+\varphi} j(j-1)(j-2)(j-3)(j-4)(j-5)a_j x^{j-6} \tag{4}$$

The approximate power series is being interpolated at $x = x_{n+j}, j = 0, \frac{1}{2}, 1, \frac{3}{2}$ While the fourth derivative is being collocated at all the grid and off grid points, $x = x_{n+j}$, where $j = 0, \frac{1}{4}, \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, 2$. The fifth and sixth derivatives are collocated at $x = x_{n+u}$ where $u = 2$. These equations are then combined to generate a structure of non-linear system of equations which are solved using Gaussian Elimination Method to obtain $a_j, j = 0, 1, 2, ..., 12$. The obtained values of $a_j^{'s}$ are then substituted to the power series to give a continuous hybrid formula of the form:

$$\begin{aligned} &\alpha_0(z)y_n + \alpha_{\frac{1}{2}}(z)y_{n+\frac{1}{2}} + \alpha_1(z)y_{n+1} + \alpha_{\frac{3}{2}}(z)y_{n+\frac{3}{2}} \\ &= h^4\left(\sum_{j=0}^{k} \beta_j(z)f_{n+j} + \beta_v(z)f_{n+v}\right) + h^5\gamma_2(z)g_{n+2} + h^6\psi_2(z)m_{n+2}, \end{aligned} \tag{5}$$

where $v = \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{3}{2}$ is the hybrid points.

Using the transformation

$$z = \frac{(x - x_{n+1})}{h}, dz = \frac{1}{h}dx \tag{6}$$

(see, Adeyefa and Kuboye [4]).

The coefficients of y_{n+j}, g_{n+j} and m_{n+j} are obtained in terms of z as follows:

$$\alpha_0(z) = \left(-\frac{11}{3}z - \frac{4}{3}z^3 + 4z^2 + 1\right),$$

$$\alpha_{\frac{1}{2}}(z) = (4z^3 - 10z^2 + 6z),$$
$$\alpha_1(z) = (-4z^3 + 8z^2 - 3z),$$
$$\alpha_{\frac{3}{2}}(z) = \left(\frac{4}{3}z^3 - 2z^2 + \frac{2}{3}z\right),$$

$$\begin{aligned}
\beta_0(z) &= h^4(\frac{304987}{127733760}z^2 + \frac{10471}{201600}z^8 - \frac{7}{9900}z^11 \\
&- \frac{9113}{100800}z^7 + \frac{17}{160}z^6 - \frac{1811}{90720}z^9 + \frac{149}{30240}z^{10} + \frac{1}{22275}z^12 \\
&+ \frac{1}{24}z^4 - \frac{52411}{212889600}z - \frac{372689}{29030400}z^3 - \frac{299}{3600}z^5), \\
\beta_{\frac{1}{4}}(z) &= h^4(\frac{1474751}{4862025}z^2 + \frac{10496}{231525}z^{11} - \frac{1008128}{231525}z^6 + \frac{76736}{64827}z^9 \\
&- \frac{1024606}{1620675}z^3 + \frac{2430208}{540225}z^7 - \frac{495616}{1620675}z^{10} \\
&- \frac{2141}{46305}z - \frac{1555136}{540225}z^8 - \frac{2048}{694575}z^{12} + \frac{11264}{5145}z^5), \\
\beta_{\frac{1}{2}}(z) &= h^4(\frac{5579221}{107775360}z^2 - \frac{589397}{35925120}z - \frac{5356}{8505}z^7 + \frac{3757}{8505}z^8 - \frac{64}{8019}z^{11} \\
&- \frac{16}{81}z^5 + \frac{1912}{3645}z^6 - \frac{2957}{15309}z^9 - \frac{126881}{4898880}z^3 + \frac{64}{120285}z^{12} + \frac{38}{729}z^{10}), \\
\beta_1(z) &= h^4(\frac{189881}{3193344}z^2 - \frac{5599}{5040}z^8 - \frac{593}{3780}z^{10} + \frac{38}{1485}z^{11} - \frac{269}{270}z^6 + \frac{29}{54}z^9 \\
&- \frac{8}{4455}z^{12} + \frac{1}{3}z^5 + \frac{251}{180}z^7 - \frac{94799}{5322240}z - \frac{667}{10080}z^3), \\
\beta_{\frac{5}{4}}(z) &= h^4(-\frac{9472}{18225}z^{11} - \frac{11264}{2025}z^5 + \frac{865216}{42525}z^8 + \frac{61952}{3645}z^6 \\
&- \frac{40507}{76545}z^2 + \frac{2048}{54675}z^{12} + \frac{299266}{382725}z^3 \\
&+ \frac{2641}{18225}z - \frac{149248}{6072}z - \frac{149248}{6075}z^9 + \frac{11264}{3645}z^{10}), \\
\beta_{\frac{3}{2}}(z) &= h^4(\frac{576599}{19958400}z^2 - \frac{1457}{190080}z - \frac{41981}{907200}z^3 - \frac{1058}{4725}z^{10} - \frac{2183}{1575}z^8 \\
&- \frac{64}{22275}z^{12} + \frac{2564}{1575}z^7 + \frac{32}{825}z^{11} - \frac{248}{225}z^6 + \frac{407}{567}z^9 + \frac{16}{45}z^5), \\
\beta_2(z) &= h^4(\frac{676023449}{53770106880}z^3 - \frac{158927665}{21508042752}z^2 + \frac{12739553}{40007520}z^6 \\
&+ \frac{155082617}{373403520}z^8 - \frac{45187}{444528}z^5 + \frac{19774007}{280052640}z^{10} - \frac{37038373}{168031584} \\
&+ \frac{3581}{3750705}z^{12} - \frac{691067}{55010340}z^{11} - \frac{89199919}{186701760}z^7 + \frac{107543549}{56330588160}), \\
\gamma_2(z) &= h^5(\frac{17275933}{9388431360}z^2 - \frac{34}{130977}z^{12} - \frac{21372}{197568}z^8 - \frac{26023}{317520}z^6
\end{aligned}$$

$$+\frac{491}{145530}z^{11}+\frac{25745}{444528}z^{9}+\frac{61043}{493920}z^{7}$$
$$-\frac{42187}{89413632}z-\frac{452701}{142248960}z^{3}+\frac{23}{882}z^{5}-\frac{1543}{82320}z^{10}),$$
$$\psi_2(z)=h^6(\frac{2}{93555}z^{12}-\frac{181429}{1341204480}z^{2}-\frac{865}{190512}z^{9}+\frac{473}{317520}z^{10}-\frac{2011}{211680}z^{7}$$
$$+\frac{3551}{423360}z^{8}+\frac{439}{12773376}z+\frac{14537}{60963940}z^{3}+\frac{71}{11340}z^{6}-\frac{17}{62370}z^{11}-\frac{1}{504}z^{5}).$$

Evaluating the continuous method at the end point i.e. at $z=3$ gives equation (7), the discrete scheme.

$$\begin{aligned}y_{n+2}&=4y_{n+\frac{1}{2}}-y_n-6y_{n+1}+4y_{n+\frac{3}{2}}\\&+h^4(\frac{199}{4064256}h^2m_{n+2}+\frac{18595}{28449792}hg_{n+2}+\frac{353}{1935360}f_n\\&+\frac{10349}{241920}f_{n+1}-\frac{45059093}{17923368960}f_{n+2}+\frac{2741}{233280}f_{n+\frac{1}{2}}\\&-\frac{326}{324135}f_{n+\frac{1}{4}}+\frac{1049}{60480}f_{n+\frac{3}{2}}-\frac{22}{3645}f_{n+\frac{5}{4}}).\end{aligned}\tag{7}$$

The evaluation of the first, second and third derivatives of the continuous scheme at all the points give:

$$\begin{aligned}y'_n&=\frac{1}{281652940800h}(9679950h^6m_{n+2}-132889050h^5g_{n+2}\\&-69339753h^4f_n-5016763080h^4f_{n+1}+537717745h^4f_{n+2}\\&-4620872480h^4f_{n+\frac{1}{2}}-1183887360h^4f_{n+\frac{1}{4}}-2158924320h^4f_{n+\frac{3}{2}}\\&+3710414848h^4f_{n+\frac{5}{4}}-1032727449600y_n-844958822400y_{n+1}\\&+1689917644800y_{n+\frac{1}{2}}+187768627200y_{n+\frac{3}{2}}),\end{aligned}\tag{8}$$

$$\begin{aligned}y'_{n+\frac{1}{4}}&=\frac{1}{1514166209740800h}(5049383850h^6m_{n+2}-70104884850h^5g_{n+2}\\&-20273486625h^4f_n-3169618739280h^4f_{n+1}+286356165745h^4f_{n+2}\\&-4468957482080h^4f_{n+\frac{1}{2}}-1454169860352h^4f_{n+\frac{1}{4}}\\&-1188753517728h^4f_{n+\frac{3}{2}}+2129134577920h^4f_{n+\frac{5}{4}}\\&-2902151902003200y_n-37854155243500y_{n+1}\\&-2649790867046400y_{n+\frac{1}{2}}+126180517478400y_{n+\frac{3}{2}}),\end{aligned}\tag{9}$$

$$\begin{aligned}y'_{n+\frac{1}{2}}&=\frac{1}{11829423513600h}(193391730h^6m_{n+2}-2656692990h^5g_{n+2}\\&-691972659h^4f_n-101125304280h^4f_{n+1}+10754937355h^4f_{n+2}\\&-64309810880h^4f_{n+\frac{1}{2}}-1210871808h^4f_{n+\frac{1}{4}}\end{aligned}\tag{10}$$

$$- 43218197568h^4 f_{n+\frac{3}{2}} + 74156314624h^4 f_{n+\frac{5}{4}} + 7886282342400y_n - 23658847027200y_{n+1} + 11829423513600y_{n+\frac{1}{2}} + 3943141171200y_{n+\frac{3}{2}}),$$

$$y'_{n+1} = \frac{1}{5914711756800h}(95075190h^6 m_{n+2} - 1307370330h^5 g_{n+2} - 332553249h^4 f_n - 60445065240h^4 f_{n+1} + 5297215865h^4 f_{n+2} - 22146439840h^4 f_{n+\frac{1}{2}} - 1735299072h^4 f_{n+\frac{1}{4}} - 21367275552h^4 f_{n+\frac{3}{2}} + 35647238144h^4 f_{n+\frac{5}{4}} + 1971570585600y_n + 5914711756800y_{n+1} + 11829423513600y_{n+\frac{1}{2}} + 3943141171200y_{n+\frac{3}{2}}), \tag{11}$$

$$y'_{n+\frac{5}{4}} = \frac{1}{378541552435200h}(1605581775h^6 m_{n+2} - 21968673300h^5 g_{n+2} - 5893116957h^4 f_n - 1559150194770h^4 f_{n+1} + 88585440655h^4 f_{n+2} - 377189979320h^4 f_{n+\frac{1}{2}} - 33117526080h^4 f_{n+\frac{1}{4}} - 350423274600h^4 f_{n+\frac{3}{2}} + 199383013312h^4 f_{n+\frac{5}{4}} + 31545129369600y_n + 662447716761600y_{n+1} - 94635388108800y_{n+\frac{1}{2}} + 725537975500800y_{n+\frac{3}{2}}), \tag{12}$$

$$y'_{n+\frac{3}{2}} = \frac{1}{563305881600h}(19029150h^6 m_{n+2} - 261497250h^5 g_{n+2} - 67281165h^4 f_n - 14884437960h^4 f_{n+1} + 1059057365h^4 f_{n+2} - 4365108160h^4 f_{n+\frac{1}{2}} + 366391296h^4 f_{n+\frac{1}{4}} - 4337153856h^4 f_{n+\frac{3}{2}} + 4625223680h^4 f_{n+\frac{5}{4}} + 375537254400y_n + 3379835289600y_{n+1} - 1689917644800y_{n+\frac{1}{2}} - 2065454899200y_{n+\frac{3}{2}}), \tag{13}$$

$$y'_{n+2} = \frac{1}{1182942351360h}(184130730h^6 m_{n+2} - 2251516302h^5 g_{n+2} - 787768443h^4 f_n - 191311144920h^4 f_{n+1} + 7060477075h^4 f_{n+2} - 51022210400h^4 f_{n+\frac{1}{2}} - 4350799872h^4 f_{n+\frac{1}{4}} - 94177998432h^4 f_{n+\frac{3}{2}} + 17829941248h^4 f_{n+\frac{5}{4}} + 4337455288320y_n + 22475904675840y_{n+1} - 16561192919040y_{n+\frac{1}{2}} - 10252167045120y_{n+\frac{3}{2}}), \tag{14}$$

$$y''_n = \frac{1}{2957355878400h}(800101890h^6 m_{n+2} - 10883837790h^5 g_{n+2} - 14122423035h^4 f_n - 351697588200h^4 f_{n+1} + 43705107875h^4 f_{n+2} - 306187648480h^4 f_{n+\frac{1}{2}} - 163095662592h^4 f_{n+\frac{1}{4}} - 170876266848h^4 f_{n+\frac{3}{2}} \tag{15}$$

$$+ 284547092480h^4 f_{n+\frac{5}{4}} - 23658847027200y_n - 47317694054400y_{n+1} + 59147117568000y_{n+\frac{1}{2}} + 11829423513600y_{n+\frac{3}{2}}),$$

$$y''_{n+\frac{1}{4}} = \frac{1}{1514166209740800h}(139595784930h^6 m_{n+2} - 1925300342070h^5 g_{n+2} - 281037000951h^4 f_n - 77587505561160h^4 f_{n+1} + 7819909232575h^4 f_{n+2} - 66405992542400h^4 f_{n+\frac{1}{2}} - 2454354155520h^4 f_{n+\frac{1}{4}} - 31790742943680h^4 f_{n+\frac{3}{2}} + 55383061866496h^4 f_{n+\frac{5}{4}} - 9084997258444800y_n - 1514166209740800y_{n+1} - 21198326936371200y_{n+\frac{1}{2}} + 3028332419481600y_{n+\frac{3}{2}}), \tag{16}$$

$$y''_{n+\frac{1}{2}} = \frac{1}{844958822400h}(6804630h^6 m_{n+2} - 94852170h^5 g_{n+2} - 11032497h^4 f_n - 3341315880h^4 f_{n+1} + 388600025h^4 f_{n+2} - 13894659520h^4 f_{n+\frac{1}{2}} - 2141835264h^4 f_{n+\frac{1}{4}} - 1624262976h^4 f_{n+\frac{3}{2}} + 3021197312h^4 f_{n+\frac{5}{4}} + 3379835289600y_n - 3379835289600y_{n+1} - 6759670579200y_{n+\frac{1}{2}}), \tag{17}$$

$$y''_{n+1} = \frac{1}{422479411200h}(4493790h^6 m_{n+2} - 61552890h^5 g_{n+2} - 15361353h^4 f_n - 8183707560h^4 f_{n+1} + 248347825h^4 f_{n+2} - 519172640h^4 f_{n+\frac{1}{2}} - 135254016h^4 f_{n+\frac{1}{4}} - 974193696h^4 f_{n+\frac{3}{2}} + 507179008h^4 f_{n+\frac{5}{4}} + 3379835289600y_{n+1} + 1689917644800y_{n+\frac{1}{2}} + 31689917644800y_{n+\frac{3}{2}}), \tag{18}$$

$$y''_{n+\frac{5}{4}} = \frac{1}{1514166209740800h}(15072008130h^6 m_{n+2} - 2069280738630h^5 g_{n+2} - 533157946335h^4 f_n - 110021504229000h^4 f_{n+1} + 8370764537575h^4 f_{n+2} - 34837218234560h^4 f_{n+\frac{1}{2}} - 2868160444416h^4 f_{n+\frac{1}{4}} - 33290701327296h^4 f_{n+\frac{3}{2}} + 57035703961600h^4 f_{n+\frac{5}{4}} + 3028332419481600y_n + 21198326936371200y_{n+1} - 1514166209740800y_{n+\frac{1}{2}} + 9084997258444800y_{n+\frac{3}{2}}), \tag{19}$$

$$\begin{aligned}
y''_{n+\frac{3}{2}} = &\frac{1}{5914711756800h^2}(1248748830h^6 m_{n+2} - 17218844370h^5 g_{n+2} \\
&- 4341936501h^4 f_n - 1010022811560h^4 f_{n+1} + 70006820125h^4 f_{n+2} \\
&- 278412276800h^4 f_{n+\frac{1}{2}} + 23978004480h^4 f_{n+\frac{1}{4}} \\
&- 31031028840h^4 f_{n+\frac{3}{2}} + 153638711296h^4 f_{n+\frac{5}{4}} + 23658847027200y_n \\
&+118294235136000y_{n+1} - 94635388108800y_{n+\frac{1}{2}} - 47317694054400y_{n+\frac{3}{2}}),
\end{aligned} \tag{20}$$

$$\begin{aligned}
y''_{n+2} = &\frac{1}{591471175680h^2}(77010066h^6 m_{n+2} - 212236290h^5 g_{n+2} \\
&- 833110299h^4 f_n - 221571499464h^4 f_{n+1} \\
&- 11786692445h^4 f_{n+2} - 55440752864h^4 f_{n+\frac{1}{2}} \\
&- 4585697280h^4 f_{n+\frac{1}{4}} - 154370645856h^4 f_{n+\frac{3}{2}} \\
&+ 8135938048h^4 f_{n+\frac{5}{4}} + 4731769405440y_n + 18927077621760y_{n+1} \\
&- 16561192919040y_{n+\frac{1}{2}} - 7097654108160y_{n+\frac{3}{2}}),
\end{aligned} \tag{21}$$

$$\begin{aligned}
y'''_n = &\frac{1}{44808422400h^3}(64108170h^6 m_{n+2} - 855604890h^5 g_{n+2} \\
&- 3451472829h^4 f_n - 17790010560h^4 f_{n+1} + 3380117245h^4 f_{n+2} \\
&- 6963229280h^4 f_{n+\frac{1}{2}} - 15451803648h^4 f_{n+\frac{1}{4}} \\
&- 12441153312h^4 f_{n+\frac{3}{2}} + 19111235584h^4 f_{n+\frac{5}{4}} \\
&- 358467379200y_n - 1075402137600y_{n+1} \\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{22}$$

$$\begin{aligned}
y'''_{n+\frac{1}{4}} = &\frac{1}{1911826022400h^3}(366131430h^6 m_{n+2} - 5234701290h^5 g_{n+2} \\
&- 2607159807h^4 f_n - 336774289320h^4 f_{n+1} + 21890686505h^4 f_{n+2} \\
&- 584711715840h^4 f_{n+\frac{1}{2}} - 151059723264h^4 f_{n+\frac{1}{4}} \\
&- 98085004416h^4 f_{n+\frac{3}{2}} + 190219875328h^4 f_{n+\frac{5}{4}} \\
&- 15294608179200y_n - 37854155243500y_{n+1} \\
&- 45883824537600y_{n+\frac{1}{2}} + 15294608179200y_{n+\frac{3}{2}}),
\end{aligned} \tag{23}$$

$$\begin{aligned}
y'''_{n+\frac{1}{2}} = &\frac{1}{44808422400h^3}(26411490h^6 m_{n+2} - 361080090h^5 g_{n+2} \\
&- 109437237h^4 f_n - 11780732880h^4 f_{n+1} + 1455672965h^4 f_{n+2}
\end{aligned}$$

$$\begin{aligned}
&- 6030488800h^4 f_{n+\frac{1}{2}} - 1295308800h^4 f_{n+\frac{1}{4}} - 5756637600h^4 f_{n+\frac{3}{2}}\\
&+ 9724209152h^4 f_{n+\frac{5}{4}} + 358467379200y_n - 1075402137600y_{n+1}\\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{24}$$

$$\begin{aligned}
y'''_{n+1} = &\frac{1}{14936140800h^3}(7466130h^6 m_{n+2} - 103046370h^5 g_{n+2}\\
&- 24742305h^4 f_n - 4351435200h^4 f_{n+1} + 418983905h^4 f_{n+2}\\
&- 1481595360h^4 f_{n+\frac{1}{2}} - 144175104h^4 f_{n+\frac{1}{4}} - 1719928224h^4 f_{n+\frac{3}{2}}\\
&+ 3280506880h^4 f_{n+\frac{5}{4}} + 119489126400y_n + 358467379200y_{n+1}\\
&- 358467379200y_{n+\frac{1}{2}} + 119489126400y_{n+\frac{3}{2}}),
\end{aligned} \tag{25}$$

$$\begin{aligned}
y'''_{n+\frac{5}{4}} = &\frac{1}{5735478067200h^3}(2210022990h^6 m_{n+2} - 30091755330h^5 g_{n+2}\\
&- 8412998013h^4 f_n - 2305452973320h^4 f_{n+1} + 120717845765h^4 f_{n+2}\\
&- 545465052160h^4 f_{n+\frac{1}{2}} + 46980946944h^4 f_{n+\frac{1}{4}}\\
&- 453102056064h^4 f_{n+\frac{3}{2}} + 276995253248h^4 f_{n+\frac{5}{4}} + 45883824537600y_n\\
&+137651473612800y_{n+1} - 137651473612800y_{n+\frac{1}{2}} + 45883824537600y_{n+\frac{3}{2}}),
\end{aligned} \tag{26}$$

$$\begin{aligned}
y'''_{n+\frac{3}{2}} = &\frac{1}{44808422400h^3}(23051070h^6 m_{n+2} - 324208710h^5 g_{n+2}\\
&- 71189307h^4 f_n - 17196565680h^4 f_{n+1} + 1347045995h^4 f_{n+2}\\
&- 4363344160h^4 f_{n+\frac{1}{2}} + 406923264h^4 f_{n+\frac{1}{4}}\\
&- 9354943584h^4 f_{n+\frac{3}{2}} - 4374243328h^4 f_{n+\frac{5}{4}}\\
&+ 375537254400y_n + 1075402137600y_{n+1}\\
&- 1075402137600y_{n+\frac{1}{2}} - 358467379200y_{n+\frac{3}{2}}),
\end{aligned} \tag{27}$$

$$\begin{aligned}
y'''_{n+2} = &\frac{1}{44808422400h^3}(39271050h^6 m_{n+2} - 1095912090h^5 g_{n+2}\\
&- 52593219h^4 f_n - 19005053760h^4 f_{n+1}\\
&+ 11447376125h^4 f_{n+2} - 4051406240h^4 f_{n+\frac{1}{2}}\\
&- 276811776h^4 f_{n+\frac{1}{4}} - 22760722656h^4 f_{n+\frac{3}{2}} - 1029812224h^4 f_{n+\frac{5}{4}}\\
&+ 358467379200y_n + 1075402137600y_{n+1}\\
&+ 1075402137600y_{n+\frac{1}{2}} + 358467379200y_{n+\frac{3}{2}}).
\end{aligned} \tag{28}$$

These schemes (the discrete scheme and its derivatives) are combined together in matrix form and by using the matrix inversion, a block method of the following form is produced (see, Lee et al. [12])

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-\frac{1}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{5}{4}} \\ y_{n-\frac{3}{2}} \\ y_n \end{bmatrix}
$$

$$
+ h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}
\begin{bmatrix} y'_{n-\frac{1}{4}} \\ y'_{n-\frac{1}{2}} \\ y'_{n-1} \\ y'_{n-\frac{5}{4}} \\ y'_{n-\frac{3}{2}} \\ y'_n \end{bmatrix}
$$

$$
+ h^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}
\begin{bmatrix} y''_{n-\frac{1}{4}} \\ y''_{n-\frac{1}{2}} \\ y''_{n-1} \\ y''_{n-\frac{5}{4}} \\ y''_{n-\frac{3}{2}} \\ y''_n \end{bmatrix}
+ h^3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{48} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{125}{384} \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{16} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \end{bmatrix}
\begin{bmatrix} y'''_{n-\frac{1}{4}} \\ y'''_{n-\frac{1}{2}} \\ y'''_{n-1} \\ y'''_{n-\frac{5}{4}} \\ y'''_{n-\frac{3}{2}} \\ y'''_n \end{bmatrix}
$$

$$
+ h^4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{25610353}{249141657600} \\ 0 & 0 & 0 & 0 & 0 & \frac{41273}{36495360} \\ 0 & 0 & 0 & 0 & 0 & \frac{71161}{6652800} \\ 0 & 0 & 0 & 0 & 0 & \frac{646970875}{29896998912} \\ 0 & 0 & 0 & 0 & 0 & \frac{37772}{985600} \\ 0 & 0 & 0 & 0 & 0 & \frac{38}{405} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{1}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{5}{4}} \\ f_{n-\frac{3}{2}} \\ f_n \end{bmatrix}
$$

$$
+ h^4 \begin{bmatrix}
\frac{9485369}{7969417600} & -\frac{6131177}{63063982080} & \frac{569345}{3737124864} & -\frac{2246441}{9853747200} & \frac{206153}{1297612800} & -\frac{437315029541}{9690663742341120} \\
\frac{127942}{53482275} & -\frac{53929}{35925120} & \frac{13423}{5806080} & -\frac{967}{280665} & \frac{95581}{39916800} & -\frac{59482693}{87625359360} \\
\frac{1823872}{53482275} & -\frac{1138}{120285} & \frac{2447}{99792} & -\frac{152192}{4209975} & \frac{62}{2475} & -\frac{262184651}{36966948480} \\
\frac{2610574375}{35050143755} & -\frac{28121875}{2675441664} & \frac{1473284375}{26159874048} & -\frac{72754625}{919683072} & \frac{25285625}{467140608} & -\frac{3278324715625}{215348083163136} \\
\frac{182547}{1320550} & -\frac{439}{98560} & \frac{2619}{22528} & -\frac{53}{350} & \frac{25299}{246400} & -\frac{7753787}{270448640} \\
\frac{3817472}{10696455} & \frac{1792}{40095} & \frac{1648}{4455} & -\frac{16384}{40095} & \frac{256}{891} & -\frac{497456}{6417873}
\end{bmatrix}
$$

$$
\cdot \begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{5}{4}} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} + h^5 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{80778437}{6991820881920} \\ 0 & 0 & 0 & 0 & 0 & \frac{652247}{3755372544} \\ 0 & 0 & 0 & 0 & 0 & \frac{532249}{293388480} \\ 0 & 0 & 0 & 0 & 0 & \frac{59798346875}{15382005940224} \\ 0 & 0 & 0 & 0 & 0 & \frac{282357}{38635520} \\ 0 & 0 & 0 & 0 & 0 & \frac{90292}{4584195} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+1} \\ g_{n+\frac{5}{4}} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \end{bmatrix}
$$

$$
(29) \qquad + h^6 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{9634379}{10987147100160} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1963}{149022720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1153}{8382528} \\ 0 & 0 & 0 & 0 & 0 & -\frac{71834375}{244158824448} \\ 0 & 0 & 0 & 0 & 0 & -\frac{6093}{11038720} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{2695} \end{bmatrix} \begin{bmatrix} m_{n+\frac{1}{4}} \\ m_{n+\frac{1}{2}} \\ m_{n+1} \\ m_{n+\frac{5}{4}} \\ m_{n+\frac{3}{2}} \\ m_{n+2} \end{bmatrix}
$$

writing out the block explicitly, we have

$$
\begin{aligned}
y_{n+\frac{1}{4}} = {} & y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{1}{384}h^3y'''_n + h^4(\frac{25610353}{249141657600}f_n \\
& + \frac{9485369}{79659417600}f_{n+\frac{1}{4}} - \frac{6131177}{63063982080}f_{n+\frac{1}{2}} + \frac{569345}{3737124864}f_{n+1} \\
& - \frac{2246441}{9853747200}f_{n+\frac{5}{4}} + \frac{206153}{1297612800}f_{n+\frac{3}{2}} \\
& - \frac{437315029541}{9690663742341120}f_{n+2} + \frac{80778437}{6991820881920}hg_{n+2} \\
& - \frac{9634379}{10987147100160}h^2m_{n+2}),
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} = {} & y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + h^4(\frac{41273}{36495360}f_n + \frac{127942}{53482275}f_{n+\frac{1}{4}} \\
& - \frac{53929}{35925120}f_{n+\frac{1}{2}} + \frac{13423}{5806080}f_{n+1} - \frac{967}{280665}f_{n+\frac{5}{4}} + \frac{95581}{39916800}f_{n+\frac{3}{2}} \\
& - \frac{59482693}{87625359360}f_{n+2} + \frac{652247}{3755372544}hg_{n+2} - \frac{1963}{149022720}h^2m_{n+2}),
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
y_{n+1} = {} & y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4(\frac{71161}{6652800}f_n + \frac{1823872}{53482275}f_{n+\frac{1}{4}} \\
& - \frac{1138}{120285}f_{n+\frac{1}{2}} + \frac{2447}{99792}f_{n+1} - \frac{152192}{4209975}f_{n+\frac{5}{4}} \\
& + \frac{62}{2475}f_{n+\frac{3}{2}} - \frac{262184651}{36966948480}f_{n+2} \\
& + \frac{532249}{293388480}hg_{n+2} - \frac{1153}{8382528}h^2m_{n+2}),
\end{aligned} \tag{32}
$$

$$\begin{aligned} y_{n+\frac{5}{4}} &= y_n + \frac{5}{4}hy'_n + \frac{25}{32}h^2y''_n + \frac{125}{384}h^3y'''_n \\ &+ h^4(\frac{646970875}{29896998912}f_n + \frac{2610574375}{35050143744}f_{n+\frac{1}{4}} \\ &- \frac{28121875}{2675441664}f_{n+\frac{1}{2}} + \frac{1473284375}{26159874048}f_{n+1} - \frac{72754625}{919683072}f_{n+\frac{5}{4}} \\ &+ \frac{25285625}{467140608}f_{n+\frac{3}{2}} - \frac{3278324715625}{215348083163136}f_{n+2} \\ &+ \frac{59798346875}{15382005940224}hg_{n+2} - \frac{71834375}{244158824448}h^2m_{n+2}), \end{aligned} \tag{33}$$

$$\begin{aligned} y_{n+\frac{3}{2}} &= y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n + h^4(\frac{37773}{985600}f_n + \frac{182547}{1320550}f_{n+\frac{1}{4}} \\ &- \frac{439}{98560}f_{n+\frac{1}{2}} + \frac{2619}{22528}f_{n+1} - \frac{53}{350}f_{n+\frac{5}{4}} + \frac{25299}{246400}f_{n+\frac{3}{2}} \\ &- \frac{7753787}{270448640}f_{n+2} + \frac{282357}{38635520}hg_{n+2} - \frac{6093}{11038720}h^2m_{n+2}), \end{aligned} \tag{34}$$

$$\begin{aligned} y_{n+2} &= y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4(\frac{38}{405}f_n + \frac{3817472}{10696455}f_{n+\frac{1}{4}} \\ &+ \frac{1792}{40095}f_{n+\frac{1}{2}} + \frac{1648}{4455}f_{n+1} - \frac{16384}{40095}f_{n+\frac{5}{4}} + \frac{256}{891}f_{n+\frac{3}{2}} \\ &- \frac{496456}{6417873}f_{n+2} + \frac{90292}{4584195}hg_{n+2} - \frac{4}{2695}h^2m_{n+2}). \end{aligned} \tag{35}$$

Substituting the above equations (30)-(35) into the equations (8)-(28) yield equation (36)-(53):

$$\begin{aligned} y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + \frac{1}{32}h^2y'''_n + h^3(\frac{338231}{22889889792}h^2m_{n+2} + \frac{156033833}{801146142720}hg_{n+2} \\ &+ \frac{80269559}{54499737600}f_n + \frac{1758583}{681246720}f_{n+1} - \frac{6983495549}{9176764907520}f_{n+2} \\ &- \frac{3071041}{1839366144}f_{n+\frac{1}{2}} + \frac{157195609}{73021132800}f_{n+\frac{1}{4}} \\ &+ \frac{4567571}{1703116800}f_{n+\frac{1}{4}} - \frac{4567571}{1703116800}f_{n+\frac{3}{2}} - \frac{22153493}{5748019200}f_{n+\frac{5}{4}}, \end{aligned} \tag{36}$$

$$\begin{aligned} y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{1}{8}h^2y'''_n + h^3(-\frac{42151}{447068160}h^2m_{n+2} + \frac{486287}{391184640}hg_{n+2} \\ &+ \frac{803171}{10644480}f_n + \frac{175921}{10644480}f_{n+1} - \frac{87079903}{17923368960}f_{n+3} - \frac{754387}{71850240}f_{n+\frac{1}{2}} \\ &- \frac{349981}{17827425}f_{n+\frac{1}{4}} + \frac{227993}{13305600}f_{n+\frac{1}{2}} - \frac{34589}{1403325}f_{n+\frac{5}{4}}, \end{aligned} \tag{37}$$

$$
\begin{aligned}
y'_{n+1} = {} & y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3(-\frac{647}{1397088}h^2m_{n+2} + \frac{299153}{48898080}hg_{n+2} \\
& + \frac{112859}{3326400}f_n + \frac{725}{8316}f_{n+1} - \frac{2683189}{112021056}f_{n+2} - \frac{3697}{280665}f_{n+\frac{1}{2}} \\
& - \frac{2710816}{17827425}f_{n+\frac{1}{4}} + \frac{4423}{51975}f_{n+\frac{3}{2}} - \frac{24896}{200475}f_{n+\frac{5}{4}},
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
y'_{n+\frac{5}{4}} = {} & y'_n + \frac{5}{4}hy''_n + \frac{25}{32}h^2y'''_n + h^3(-\frac{9266875}{11444944896}h^m_{n+2} + \frac{214931875}{20028653568}hg_{n+2} \\
& + \frac{29669125}{544997376}f_n + \frac{47745625}{272498688}f_{n+1} - \frac{19333410625}{458838245376}f_{n+2} + \frac{13608125}{1839366144}f_{n+\frac{1}{2}} \\
& + \frac{598077625}{1839366144}f_{n+\frac{1}{4}} + \frac{10316125}{68124672}f_{n+\frac{3}{2}} - \frac{51867125}{229920768}f_{n+\frac{5}{4}},
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
y'_{n+\frac{3}{2}} = {} & y'_n + \frac{3}{2}hy''_n + \frac{9}{8}h^2y'''_n + h^3(-\frac{351}{275968}h^2m_{n+2} + \frac{652833}{38635520}hg_{n+2} \\
& + \frac{629667}{7884800}f_n + \frac{123201}{394240}f_{n+1} - \frac{3269271}{49172480}f_{n+2} + \frac{123}{2816}f_{n+\frac{1}{2}} \\
& + \frac{203877}{660275}f_{n+\frac{1}{4}} + \frac{118791}{492800}f_{n+\frac{3}{2}} - \frac{687}{1925}f_{n+\frac{5}{4}},
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
y'_{n+2} = {} & y'_n + 2hy''_n + 2h^2y'''_n + h^3(-\frac{548}{218295}h^2m_{n+2} + \frac{50738}{1528065}hg_{n+2} \\
& + \frac{7558}{51975}f_n + \frac{7648}{10395}f_{n+1} - \frac{226126}{1750329}f_{n+2} + \frac{45824}{280665}f_{n+\frac{1}{2}} \\
& + \frac{10338304}{17827425}f_{n+\frac{1}{4}} + \frac{27392}{51975}f_{n+\frac{3}{2}} - \frac{966656}{1403325}f_{n+\frac{5}{4}},
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
y''_{n+\frac{1}{4}} = {} & y''_n + \frac{1}{4}hy'''_n + h^2(-\frac{1865797}{10404495360}h^2m_{n+2} + \frac{172242689}{72831467520}hg_{n+2} \\
& + \frac{72668371}{4954521600}f_n + \frac{1303619}{41287680}f_{n+1} - \frac{84837118099}{9176764907520}f_{n+2} - \frac{8706799}{418037760}f_{n+\frac{1}{2}} \\
& + \frac{1628609}{55319040}f_{n+\frac{1}{4}} + \frac{72169}{2211840}f_{n+\frac{3}{2}} - \frac{3069139}{65318400}f_{n+\frac{5}{4}},
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
y''_{n+\frac{1}{2}} = {} & y''_n + \frac{1}{2}hy'''_n + h^2(-\frac{5971}{13547520}h^2m_{n+2} + \frac{1637243}{284497920}hg_{n+2} \\
& + \frac{130541}{3870720}f_n + \frac{7319}{96768}f_{n+1} - \frac{268600651}{11948912640}f_{n+2} - \frac{2557}{60480}f_{n+\frac{1}{2}} \\
& - \frac{185951}{1620675}f_{n+\frac{1}{4}} + \frac{23927}{302400}f_{n+\frac{3}{2}} - \frac{193}{1701}f_{n+\frac{5}{4}},
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
y''_{n+1} = {} & y''_n + hy'''_n + h^2(-\frac{73}{63504}h^2m_{n+2} + \frac{33937}{2222640}hg_{n+2} \\
& + \frac{10919}{151200}f_n + \frac{163}{630}f_{n+1} - \frac{16822367}{280052640}f_{n+2} + \frac{1292}{25515}f_{n+\frac{1}{2}} \\
& - \frac{17408}{60025}f_{n+\frac{1}{4}} + \frac{1028}{4725}f_{n+\frac{3}{2}} - \frac{41984}{127575}f_{n+\frac{5}{4}},
\end{aligned} \tag{44}
$$

$$y''_{n+\frac{5}{4}} = y''_n + \frac{5}{4}hy'''_n + h^2(-\frac{1121875}{693633024}h^2m_{n+2} + \frac{313973125}{14566293504}hg_{n+2} + \frac{2600725}{28311552}f_n + \frac{11148125}{24772608}f_{n+1} - \frac{52028243125}{611784327168}f_{n+2} + \frac{3169375}{27869184}f_{n+\frac{1}{2}} + \frac{12413875}{33191424}f_{n+\frac{1}{4}} + \frac{963875}{3096576}f_{n+\frac{3}{2}} - \frac{45925}{96768}f_{n+\frac{5}{4}}, \tag{45}$$

$$y''_{n+\frac{3}{2}} = y''_n + \frac{3}{2}hy'''_n + h^2(-\frac{1047}{501760}h^2m_{n+2} + \frac{97899}{3512320}hg_{n+2} + \frac{79923}{716800}f_n + \frac{11601}{17920}f_{n+1} - \frac{16257883}{147517440}f_{n+2} + \frac{1187}{6720}f_{n+\frac{1}{2}} + \frac{5499}{12005}f_{n+\frac{1}{4}} + \frac{921}{2240}f_{n+\frac{3}{2}} - \frac{299}{525}f_{n+\frac{5}{4}}, \tag{46}$$

$$y''_{n+2} = y''_n + 2hy'''_n + h^2(-\frac{2}{735}h^2m_{n+2} + \frac{4744}{138915}hg_{n+2} + \frac{712}{4725}f_n + \frac{992}{945}f_{n+1} - \frac{112958}{972405}f_{n+2} + \frac{512}{1701}f_{n+\frac{1}{2}} + \frac{1015808}{1620675}f_{n+\frac{1}{4}} + \frac{512}{675}f_{n+\frac{3}{2}} - \frac{32768}{42525}f_{n+\frac{5}{4}}, \tag{47}$$

$$y'''_{n+\frac{1}{4}} = y'''_n + h(-\frac{1611667}{1300561920}h^2m_{n+2} + \frac{29782007}{1820786688}hg_{n+2} + \frac{48548753}{619315200}f_n + \frac{3419713}{15482880}f_{n+1} - \frac{73396589569}{1147095613440}f_{n+2} - \frac{70189}{466560}f_{n+\frac{1}{2}} + \frac{55145099}{207446400}f_{n+\frac{1}{4}} + \frac{1095161}{4838400}f_{n+\frac{3}{2}} - \frac{762857}{2332800}f_{n+\frac{5}{4}}, \tag{48}$$

$$y'''_{n+\frac{1}{2}} = y'''_n + h(-\frac{2137}{2540160}h^2m_{n+2} + \frac{2453}{222264}hg_{n+2} + \frac{45109}{604800}f_n + \frac{8111}{60480}f_{n+1} - \frac{48111107}{1120210560}f_{n+2} + \frac{607}{29160}f_{n+\frac{1}{2}} - \frac{605726}{1620675}f_{n+\frac{1}{4}} + \frac{5639}{37800}f_{n+\frac{3}{2}} - \frac{3818}{18225}f_{n+\frac{5}{4}}, \tag{49}$$

$$y'''_{n+1} = y'''_n + h(-\frac{613}{317520}h^2m_{n+2} + \frac{11555}{444528}hg_{n+2} + \frac{11897}{151200}f_n + \frac{1301}{1890}f_{n+1} - \frac{28981681}{280052640}f_{n+2} + \frac{928}{3645}f_{n+\frac{1}{2}} - \frac{543232}{1620675}f_{n+\frac{1}{4}} + \frac{1856}{4725}f_{n+\frac{3}{2}} - \frac{11776}{18225}f_{n+\frac{5}{4}}, \tag{50}$$

$$y'''_{n+\frac{5}{4}} = y'''_n + h(-\frac{472375}{260112384}h^2m_{n+2} + \frac{44320375}{1820786688}hg_{n+2} + \frac{1944505}{224772608}f_n + \frac{2474125}{3096576}f_{n+1} - \frac{22134914125}{229419122688}f_{n+2} + \frac{23375}{93312}f_{n+\frac{1}{2}} + \frac{2793475}{8297856}f_{n+\frac{1}{4}} + \frac{69025}{193536}f_{n+\frac{3}{2}} - \frac{44305}{93312}f_{n+\frac{5}{4}}, \tag{51}$$

(52)
$$\begin{aligned} y'''_{n+\frac{3}{2}} &= y'''_n + h(-\frac{61}{31360}h^2 m_{n+2} + \frac{289}{10976}hg_{n+2} \\ &+ \frac{1761}{22400}f_n + \frac{1749}{2240}f_{n+1} - \frac{1459001}{13829760}f_{n+2} + \frac{91}{360}f_{n+\frac{1}{2}} \\ &+ \frac{20154}{60025}f_{n+\frac{1}{4}} + \frac{681}{1400}f_{n+\frac{3}{2}} - \frac{74}{225}f_{n+\frac{5}{4}}, \end{aligned}$$

(53)
$$\begin{aligned} y'''_{n+2} &= y'''_n + h(-\frac{11}{19845}h^2 m_{n+2} - \frac{149}{27783}hg_{n+2} \\ &+ \frac{739}{9450}f_n + \frac{776}{945}f_{n+1} + \frac{3151273}{17503290}f_{n+2} + \frac{896}{3645}f_{n+\frac{1}{2}} \\ &+ \frac{548864}{1620675}f_{n+\frac{1}{4}} + \frac{3712}{4725}f_{n+\frac{3}{2}} - \frac{8192}{18225}f_{n+\frac{5}{4}}. \end{aligned}$$

3. Analysis of the properties of the block method

3.1 Order and error constant of the block

Following Fatunla [6] and Lambert [11], we define the local truncation error associated with linear multistep method to be the linear difference operator

(54)
$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x+jh) - h^4 \beta_j j'^v(x+jh).$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (7) above as a Taylor series about x to obtain the expression

(55)
$$L[y(x);h] = C_0 y(x) + C_1 hy' + ..., C_q h^q y^q(x) + ...,$$

where the constant coefficient $C_q, q = 0, 1, ...$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j, C_1 = \sum_{j=0}^{k} j\alpha_j, ..., C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - (q-4)! \sum_{j=1}^{k} j^{q-4}\beta_j\right].$$

According to Henrici [8], we say that our block is of uniform order $p = [9, 9, 9, 9, 9, 9]$ and error constants given by the vector

$$\begin{aligned} C_{13} = [&\frac{138668197}{7521989255862681600}, \frac{503747}{1836423158169600}, \frac{71801}{25107347865600}, \\ &\frac{12690929375}{2106156991641550848}, \frac{1333}{119957094400}, \frac{163}{5064318720}]^T. \end{aligned}$$

3.2 Zero stability of the block method

Given the general form of block method

$$A^{(0)} Y_m = A^{(i)} Y_{m-1} + h^{\mu}[B^{(i)} F_m + B^{(0)} F_{m-1}].$$

A block method is said to be zero stable, if the roots

$$det[\lambda A^{(0)} - A^{(i)}] = 0$$

of the first characteristic polynomial satisfy $|\lambda| \leq 1$ and for the roots with $|\lambda| \leq 1$, the multiplicity must not exceed the order of the differential equation (see, Adeyefa and Kuboye [4] and Omar and Kuboye [15]). For our method

$$\left[z \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \right] = 0,$$

$$A = z^6 - z^6 = 0, z = 0, 0, 0, 0, 0, 0$$

Hence, the block is zero stable.

3.3 Region of absolute stability

The stability nature of the method is found in the footstep of Awoyemi [2] as shown in Figure 1.

The region of absolute stability of our method is A-stable, because the region consists of the complex plane outside the enclosed figure (see, Yakubu et al. [19]).

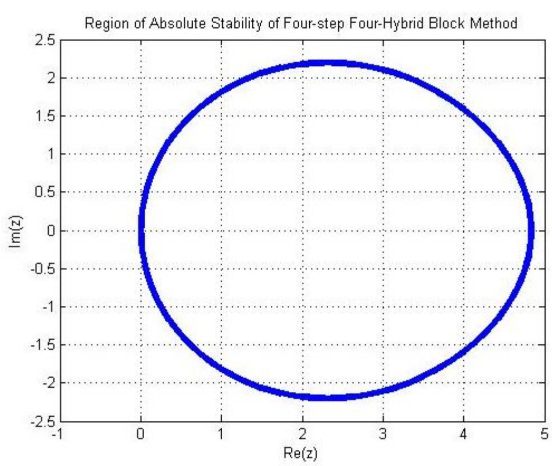

Figure 1: Showing the region of absolute stability of the method

3.4 Consistency and convergence

Our new hybrid block method is consistent since the order of each of the method is greater than 1.

Theorem 1: Convergence

According to Lambert [11], the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Also, Akinfewa et al. [5], corroborated Lambert's stanceon convergence as thus; convergence = consistency + zero-stability. Hence the new hybrid block method is convergent.

4. Numerical results

Here in this segment, the performance of the new hybrid method is seen on modeled second, third and fourth order ODEs. The results obtained from the test Problems are shown in tabular form. All the test problems are solved using our written code in Maple.

$Error = |Exact\ solution - Computed\ solution|$.

Problem 1: Real-life Problem (Cooling of a body)

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$ or the otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6In4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulation of the Problem

$$y'' = \frac{(-y')}{3}, y(0) = 60, y'(0) = -\frac{80}{9}, h = 0.1$$

Analytical Solution

$$y(x) = \frac{80}{3}e^{-(\frac{1}{3})x} + \frac{100}{3}$$

Table 1. Showing the comparison of the result for problem 1 with the error in Obarhua [14]

X	Exact	Computed	Error	Error in Obarhua[14]
0.1	59.125762679520157387	59.125762679520157388	1.10E-18	2.344791 E -13
0.2	58.280186267509806339	58.280186267509806339	0.000000	2.202682 E -13
0.3	57.462331147625588617	57.462331147625588619	2.10E-18	3.935749 E -12
0.4	56.671288507811932106	56.671288507811932107	1.10E-18	2.704951 E -12
0.5	55.906179330416375307	55.906179330416375308	1.10E-18	7.599112 E -12
0.6	55.166153415412849564	55.166153415412849565	1.10E-18	1.569518 E -12
0.7	54.450388435647511050	54.450388435647511050	0.000000	2.756872 E -12
0.8	53.758089023057298472	53.758089023057298472	0.000000	4.375392 E -11
0.9	53.088485884845809762	53.088485884845809760	2.10E-18	6.474571 E -11
1.0	52.440834948634380011	52.440834948634380010	1.10E-18	9.100178 E -11

In this result displayed above, the new two-step implicit hybrid block method produced more accurate result for solving problem 1 than the method of Obarhua [14].

Problem 2. Consider the third order oscillatory differential equation

$$y'''(t) = -4y'(t) + t,\ y(0) = y'(0) = 0,\ y''(0) = 1$$

with the exact solution given by:

$$y(t) = \left(\frac{3}{16}\right)(1 - cos2t) + \left(\frac{1}{8}\right)t^2.$$

Table 2. Showing the comparison of the result for problem 2 with the error in Sunday [17]

x	Exact	Computed	Error	Error in Sunday[17]
0.1	0.0049875166547671941600	0.0049875166547671940759	8.4100E-20	8.3209 E-13
0.2	0.019801063624459046980	0.019801063624459045988	9.9200E-19	3.4752 E -12
0.3	0.043999572204435319270	0.043999572204435312050	7.2200E-18	7.8178 E -12
0.4	0.076867491997406483580	0.076867491997406460536	2.3044E-17	1.3681 E -11
0.5	0.11744331764972380299	0.11744331764972374178	6.1210E-17	2.0825 E -11
0.6	0.16455792103562370419	0.16455792103562357206	1.3213E-16	2.8962 E -11
0.7	0.21688116070620482401	0.21688116070620456653	2.5748E-16	3.7764 E -11
0.8	0.27297491043149163616	0.27297491043149118204	4.5412E-16	4.6879 E -11
0.9	0.33135039275495382287	0.33135039275495307253	7.5034E-16	5.5941 E -11
1.0	0.39052753185258919756	0.39052753185258803045	1.16711E-15	6.4592 E -11

Our new method generated better result than the method of Sunday [17] in solving third order oscillatory differential equation as seen in Table 2 above.

Problem 3. The new method is used to solve a physical problem that arising from ship dynamics. In particular, this problem describes how the sinusoidal wave of frequency Ω passes along a ship or an offshore structure as stated by Familua and Omole [7]. This leads to a fourth order ordinary differential equation defined as follows:

$$y^4 + 3y'' + y(2 + \epsilon\cos(\Omega t)) = 0,\ t\ f\ 0.$$

Is subjected to the following initial conditions:

$$y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0,\ h = \frac{1}{320},$$

where $\epsilon = 0$ for the existence of the theoretical solution: $y(t) = 2\cos t - \cos(t\sqrt{2})$.

Table 3. Showing the comparison of the result for problem 3 with the error in Familua and Omole [7]

X	Exact	Computed	Error	Error in Familua and Omole [7]
0.003125	0.99999999999205272181	0.99999999999205272179	2.10E-20	6.685763E-13
0.006250	0.99999999987284392123	0.99999999987284392118	5.10E-20	1.458489E-11
0.009375	0.99999999935627549414	0.99999999935627549419	5.10E-20	1.082968E-10
0.001250	0.99999999796552658062	0.99999999796552658062	0.00000	3.917803E-10
0.015625	0.99999999503306753347	0.99999999503306753346	1.10E-20	1.025145E-09
0.018750	0.99999998970067947569	0.99999998970067947573	4.10E-20	2.217319E-09
0.021875	0.99999998091947944412	0.99999998091947944411	1.10E-20	4.226068E-09
0.025000	0.99999996744995111889	0.99999996744995111890	1.10E-20	7.358019E-09
0.028125	0.99999994786198113959	0.99999994786198113956	3.10E-20	1.196868E-08
0.031250	0.99999992053490100516	0.99999992053490100515	1.10E-20	1.846249E-08

In solving fourth order ODE in Problem 3, our new hybrid block method yielded far better result than the method of Familua and Omole [7] as shown in Table 3.

5. Discussion of results

In Table 1, it is observed that our new two-step hybrid block method performs far better than the result generated by Obarhua [14]. In Problem 1, it was observed that the body gets cooler and the temperature falling below one degree Celsius as the step length of our scheme is being reduced which make the temperature of the body in the room to satisfy our differential equation. The result generated by our new scheme when applied to problem 2, an oscillatory third order ODE as displayed in table 2 reveals that our method produces better performance over that of Sunday [17]. And table 3 shows the supremacy of our method over that of Familua and Omole [7] in solving a fourth order ODE arising from ship dynamics in Problem 3.

6. Conclusion

The new multi-derivative block method for the numerical solution of second, third and fourth orders ODEs was developed in this paper. The method is stable, convergent and consistent. The main benefit of the method over the existing numerical approaches is its knack for solving directly and effectively three different orders of differential equations namely: second, third and fourth-orders IVPs of ordinary differential equations. To prove the effectiveness of the new hybrid block method, it is used to solve some ordinary differential equations of order two,three and four. The generated results confirm the supremacy of our new block method over the existing methods in relations to error as shown in Tables 1-3 above.

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