The Gel'fand spaces of discrete Beurling algebras on $\mathbb{Z}_+^2$ and $\mathbb{Z}^2$

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Abstract. It is well known that the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ can be identified, respectively, with some closed disc and closed annulus in the complex plane $\mathbb{C}$ (see [6, P.104]). Therefore, it is natural to investigate the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$. Surprisingly, their Gel'fand spaces are isomorphic to the union (possibly, uncountable) of product of closed discs and closed annuli in $\mathbb{C}^2$, respectively.
Keywords: Beurling algebras, complex homomorphism, Gel'fand space, Gel'fand transform, holomorphic functions.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ denote the sets of integers, non-negative integers, and positive integers, respectively. Then, the set $\mathbb{Z}_+^2$ (respectively, $\mathbb{Z}^2$) is a unital abelian semigroup (respectively, group) with respect to co-ordinatewise addition. Let $S = \mathbb{Z}_+^2$ or $\mathbb{Z}^2$. A *weight* on S is a strictly positive function $\omega : S \longrightarrow (0, \infty)$ satisfying the submultiplicativity $\omega(s+t) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Let $l^1(S, \omega)$ be the set of all functions $f : S \longrightarrow \mathbb{C}$ such that $\sum\{|f(s)|\omega(s) : s \in S\} < \infty$. For $f, g \in l^1(S, \omega)$, the *convolution product* $f * g$ of f and g is defined as

$$f * g(s) = \sum\{f(u)g(v) : u, v \in S \text{ and } u + v = s\}, \qquad (s \in S).$$

Then $l^1(S, \omega)$ is a unital, commutative Banach algebra with respect to pointwise linear operations, the convolution product, and the norm $\|f\|_\omega = \sum\{|f(s)|\omega(s) :$

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$s \in S\}$. The Banach algebras $l^1(\mathbb{Z}_+, \omega)$ and $l^1(\mathbb{Z}, \omega)$ have been studied extensively in Banach algebra theory as well as in Fourier analysis [1, 2, 3, 4]. For $r_+ \geq r_- > 0$ and $r \geq 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$ and $\Gamma(r_-, r_+) = \{z \in \mathbb{C} : r_- \leq |z| \leq r_+\}$. In particular, the Gel'fand space of $\Delta(l^1(\mathbb{Z}_+, \omega))$ and $\Delta(l^1(\mathbb{Z}, \omega))$ can be identified with the closed disc $\mathbb{D}_r$ and the closed annulus $\Gamma(r_-, r_+)$, respectively, where $0 \leq r < \infty$ and $0 < r_- \leq r_+ < \infty$. It is quite natural to believe that, the Gel'fand spaces of $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$ might be isomorphic to either finite product of closed discs or annuli, respectively. However, it turns out that this is not true in general. In this article, we prove that their Gel'fand space could be an infinite (possibly, uncountable) union of products of closed discs and closed annuli, respectively.

Notations: Throughout we shall consider the following notations:

(1) Let $\mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}; \mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}; \mathbb{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$;

(2) $e_1 = (1, 0), e_2 = (0, 1)$, and $e = (1, 1)$.

2. Some results on weights

If ω is a weight on $\mathbb{Z}_+$, then there is only one non-negative real number ρ_ω (which is so-called *ω-spectral radius*) associated with the weight ω and it is equal to the limit $\lim_{n\to\infty} \omega(n)^{1/n} = \inf_{n\in\mathbb{N}} \omega(n)^{1/n}$ [6, P.103]. On the other hand, if ω is a weight on $\mathbb{Z}_+^2$, then $\omega_1(m) = \omega(m, 0)$ and $\omega_2(n) = \omega(0, n)$ both are weights on $\mathbb{Z}_+$ and $\omega(m, n) \leq \omega_1(m)\omega_2(n)$, for all $m, n \in \mathbb{Z}_+$. In the case of weight ω on $\mathbb{Z}_+^2$, we can define several numbers related to ω as follow. However, none of these numbers is related to the numbers ρ_{ω_1} and ρ_{ω_2} related to the weights ω_1 and ω_2 defined by ω as above.

Definition 2.1. *Let ω be a weight on $\mathbb{Z}_+^2$. Then define*

$$\begin{aligned}
\beta_{1,k} &= \inf\{\omega(m,k)^{1/m} : m \in \mathbb{N}\} \ \ (k \in \mathbb{Z}_+) \\
\beta_{2,k} &= \inf\{\omega(k,n)^{1/n} : n \in \mathbb{N}\} \ \ (k \in \mathbb{Z}_+) \\
\rho_2 &= \inf\{\omega(n,n)^{1/n} : n \in \mathbb{N}\} \\
\beta_1 &= \inf\{\omega(m,n)^{1/m} : m \in \mathbb{N}, n \in \mathbb{Z}_+\} \\
\beta_2 &= \inf\{\omega(m,n)^{1/n} : m \in \mathbb{Z}_+, n \in \mathbb{N}\} \\
\mu_2 &= \inf\{\omega(m,n)^{\frac{1}{m+n}} : m,n \in \mathbb{Z}_+ \textit{ and } (m,n) \neq (0,0)\} \\
\nu_2 &= \inf\{\omega(m,n)^{1/mn} : m,n \in \mathbb{Z}_+ \textit{ and } mn \neq 0\} \\
\sigma_1 &= \inf_{n\in\mathbb{Z}_+} \left[\liminf_{m\to\infty} \omega(m,n)^{1/m}\right] \\
\sigma_2 &= \inf_{m\in\mathbb{Z}_+} \left[\liminf_{n\to\infty} \omega(m,n)^{1/n}\right]
\end{aligned}$$

If ω is a weight on $\mathbb{Z}^2$, then, further, define

$$\begin{aligned}
\alpha_{1,k} &= \sup\{\omega(-m,k)^{-1/m} : m \in \mathbb{N}\} \ \ (k \in \mathbb{Z}_-) \\
\alpha_{2,k} &= \sup\{\omega(k,-n)^{-1/n} : n \in \mathbb{N}\} \ \ (k \in \mathbb{Z}_-)
\end{aligned}$$

$$\begin{aligned}
\rho_1 &= \sup\{\omega(-n,-n)^{-1/n} : n \in \mathbb{N}\}\\
\alpha_1 &= \sup\{\omega(-m,n)^{-1/m} : m \in \mathbb{N}, n \in \mathbb{Z}_-\}\\
\alpha_2 &= \sup\{\omega(m,-n)^{-1/n} : m \in \mathbb{Z}_-, n \in \mathbb{N}\}\\
\mu_1 &= \sup\{\omega(m,n)^{\frac{1}{m+n}} : m,n \in \mathbb{Z}_- \text{ and } (m,n) \neq (0,0)\}\\
\nu_1 &= \sup\{\omega(m,n)^{1/mn} : m,n \in \mathbb{Z}_- \text{ and } mn \neq 0\}\\
\gamma_1 &= \sup_{n\in\mathbb{Z}_-}\left[\limsup_{m\to-\infty}\omega(m,n)^{1/m}\right]\\
\gamma_2 &= \sup_{m\in\mathbb{Z}_-}\left[\limsup_{n\to-\infty}\omega(m,n)^{1/n}\right]
\end{aligned}$$

Note that it may happen that $\alpha_1 = \alpha_2 = \infty$. For example, consider $\omega(m,n) = e^{m+n}$ $(m,n \in \mathbb{Z})$.

Lemma 2.1. *Let ω be a weight on $\mathbb{Z}_+^2$ and $k \in \mathbb{Z}_+$. Then*

(i) $\beta_i \leq \beta_{i,k} \leq \beta_{i,0}$ $(i = 1,2)$;

(ii) $\beta_{1,0} = \lim_{m\to\infty}\omega(m,0)^{1/m}$ *and* $\beta_{2,0} = \lim_{n\to\infty}\omega(0,n)^{1/n}$;

(iii) $\rho_2 = \lim_{n\to\infty}\omega(n,n)^{1/n}$;

(iv) $\beta_i \leq \sigma_i \leq \beta_{i,0}$ $(i = 1,2)$.

Proof. (i) It is clear from the definition that $\beta_i \leq \beta_{i,k}$ ($i = 1,2$ and $k \in \mathbb{Z}_+$). Now, let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\omega(k_0,0)^{1/k_0} < \beta_{1,0}+\varepsilon$. Write $m \in \mathbb{N}$ in the form $m = p(m)k_0 + q(m)$, where $p(m), q(m) \in \mathbb{Z}_+$ and $q(m) < k_0$. Then

$$\frac{1}{k_0} = \lim_{m\to\infty}\left(\frac{p(m)}{m} + \frac{q(m)}{k_0 m}\right) = \lim_{m\to\infty}\frac{p(m)}{m} = \lim_{m\to\infty}\frac{p(m)-1}{m}.$$

Now,

$$\begin{aligned}
\omega(m,k)^{1/m} &= \omega(p(m)k_0+q(m),k)^{1/m}\\
&\leq \omega(p(m)k_0,k)^{1/m}\omega(q(m),0)^{1/m}\\
&\leq \omega(k_0,k)^{1/m}\omega(k_0,0)^{\frac{p(m)-1}{m}}\omega(q(m),0)^{1/m} \qquad (1)
\end{aligned}$$

This implies $\liminf_{m\to\infty}\omega(m,k)^{1/m} \leq \omega(k_0,0)^{\frac{1}{k_0}}$, and hence $\beta_{1,k} \leq \beta_{1,0} + \epsilon$. Since $\epsilon > 0$ is arbitrary, $\beta_{1,k} \leq \beta_{1,0}$. Similarly, $\beta_{2,k} \leq \beta_{2,0}$.

(ii) Substituting $k = 0$ in the inequality (1), we get

$$\begin{aligned}
\omega(m,0)^{1/m} &\leq \omega(k_0,0)^{\frac{p(m)}{m}}\omega(q(m),0)^{1/m}\\
\Rightarrow \limsup_{m\to\infty}\omega(m,0)^{1/m} &\leq \omega(k_0,0)^{\frac{1}{k_0}} < \beta_{1,0}+\epsilon.
\end{aligned}$$

But $\beta_{1,0} \leq \liminf_{m\to\infty}\omega(m,0)^{1/m}$. Hence $\beta_{1,0} = \lim_{m\to\infty}\omega(m,0)^{1/m}$. Similarly, we can show that $\beta_{2,0} = \lim_{n\to\infty}\omega(0,n)^{1/n}$.

(iii) Define $\gamma(n) = \omega(n, n)$ $(n \in \mathbb{Z}_+)$. Then γ is a weight on $\mathbb{Z}_+$ and

$$\rho_2 = \inf\{\gamma(n)^{1/n} : n \in \mathbb{N}\} = \lim_{n\to\infty} \gamma(n)^{1/n} = \lim_{n\to\infty} \omega(n, n)^{1/n}.$$

(iv) Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \sigma_1 &= \inf_{n\in\mathbb{N}} \left(\liminf_{m\to\infty} \omega(m, n)^{1/m}\right) \le \liminf_{m\to\infty} \omega(m, k)^{1/m} \\ &\le \liminf_{m\to\infty} \omega(m, 0)^{1/m}\omega(0, k)^{1/m} = \beta_{1,0}. \end{aligned}$$

Clearly $\beta_1 \le \sigma_1 \le \beta_{1,0}$. Similarly, $\beta_2 \le \sigma_2 \le \beta_{2,0}$. □

Lemma 2.2. *Let ω be a weight on $\mathbb{Z}^2$ and let $k \in \mathbb{Z}_-$. Then*

(i) $\alpha_{i,0} \le \alpha_{i,k} \le \alpha_i$ $(i = 1, 2)$*;*

(ii) $\alpha_{1,0} = \lim_{m\to\infty} \omega(-m, 0)^{-1/m}$ *and* $\alpha_{2,0} = \lim_{n\to\infty} \omega(0, -n)^{-1/n}$*;*

(iii) $\rho_1 = \lim_{n\to\infty} \omega(-n, -n)^{-1/n}$*;*

(iv) $\alpha_{i,0} \le \gamma_i \le \alpha_i$ $(i = 1, 2)$.

Proof. (i) Note that $\beta_i \le \beta_{i,k} \le \beta_{i,0}$ is proved in Lemma 2.1(i). Similarly, we can prove that $\alpha_{i,0} \le \alpha_{i,k} \le \alpha_i$ $(i = 1, 2)$. Statements (ii), (iii), and (iv) can be proved as per the proofs of Lemma 2.1(ii), Lemma 2.1(iii), and Lemma 2.1(iv), respectively. □

Remark. We make following remarks for a weight ω on $\mathbb{Z}^2_+$.

(1) In general, $\lim_{m\to\infty} \lim_{n\to\infty} \omega(m, n)^{\frac{1}{m+n}}$ exists, but $\lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{m+n}}$ may not exist.

Similarly, $\lim_{m\to\infty} \lim_{n\to\infty} \omega(m, n)^{\frac{1}{mn}}$ exists but not $\lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{mn}}$.

(2) In general, $\mu_2 \ne \lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{m+n}}$ and $\nu_2 \ne \lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{mn}}$.

(3) Suppose that the double limits of $\omega(m, n)^{\frac{1}{m+n}}$ and $\omega(m, n)^{\frac{1}{mn}}$ exist. In this case, we do not know whether $\mu_2 = \lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{m+n}}$ and $\nu_2 = \lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{mn}}$.

Lemma 2.3. *Let ω_1 and ω_2 be weights on $\mathbb{Z}_+$. Define $\omega(m, n) = \omega_1(m)\omega_2(n)$ for $m, n \in \mathbb{Z}_+$. Then*

(i) $\lim_{m\to\infty} \omega(m, k)^{1/m} = \beta_{1,0}$ *and* $\lim_{n\to\infty} \omega(k, n)^{1/n} = \beta_{2,0}$*, for all* $k \in \mathbb{Z}_+$*; but they may not be equal to* $\beta_{1,k}$ *and* $\beta_{2,k}$*, respectively;*

(ii) $\lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{m+n}}$ *may not exist;*

(iii) $\lim_{m,n\to\infty} \omega(m, n)^{\frac{1}{mn}}$ *may not exist.*

Proof. (i) Let $k \in \mathbb{N}$. Then, by Lemma 2.1(ii), we have

$$\begin{aligned} \beta_{1,0} &= \lim_{m\to\infty} \omega(m,0)^{\frac{1}{m}} \\ &= \lim_{m\to\infty} \omega_1(m)^{\frac{1}{m}}\omega_2(0)^{\frac{1}{m}} \\ &= \lim_{m\to\infty} \omega_1(m)^{\frac{1}{m}}\omega_2(k)^{\frac{1}{m}} \\ &= \lim_{m\to\infty} \omega(m,k)^{\frac{1}{m}}. \end{aligned}$$

Now, $\beta_{1,0}$ may not be equal to $\beta_{1,k}$. For example, $\omega(m,n) = \omega_1(m)\omega_2(n)$ $(m, n \in \mathbb{Z}_+)$, where $\omega_1(m) = e^{-m}$ and $\omega_2(n) = e^{-n}$. Then $\beta_{1,0} = \lim_{m\to\infty} \omega(m,0)^{1/m} = e^{-1}$, while for $k \in \mathbb{N}$,

$$\beta_{1,k} = \inf\{\omega(m,k)^{1/m} : m \in \mathbb{N}\} = \inf\{e^{-1}e^{-k/m} : m \in \mathbb{N}\} = e^{-k-1}.$$

Similarly, we can show that $\beta_{2,0} = \lim_{n\to\infty} \omega(k,n)^{\frac{1}{n}} \neq \beta_{2,k}$.

An example for statements (ii) and (iii) is $\omega(m,n) = \omega_1(m)\omega_2(n)$ $(m, n \in \mathbb{Z}_+)$, where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. □

Lemma 2.4. *Let ω_1 and ω_2 be weights on $\mathbb{Z}$. Define $\omega(m,n) = \omega_1(m)\omega_2(n)$ for $m, n \in \mathbb{Z}$. Then*

(i) $\lim_{m\to\infty} \omega(-m,k)^{-1/m} = \alpha_{1,0}$ *and* $\lim_{n\to\infty} \omega(k,-n)^{-1/n} = \alpha_{2,0}$, *for all $k \in \mathbb{Z}_-$; but they may not be equal to $\alpha_{1,k}$ and $\alpha_{2,k}$, respectively;*

(ii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{m+n}}$ *may not exist;*

(iii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{mn}}$ *may not exist.*

Proof. Statement (i) can be proved as per the proof of Lemma 2.3(i) and by taking the weight $\omega(m,n) = \omega_1(m)\omega_2(n)$ $(m, n \in \mathbb{Z})$, where $\omega_1(m) = e^m$ and $\omega_2(n) = e^n$. A counter example for (ii) and (iii) is $\omega(m,n) = \omega_1(m)\omega_2(n)$ $(m, n \in \mathbb{Z})$, where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. □

Lemma 2.5. *Let ω_1 and ω_2 be weights on $\mathbb{Z}_+$.*

Define $\omega(m,n) = \max\{\omega_1(m), \omega_2(n)\}$ for $m, n \in \mathbb{Z}_+$. Then

(i) $\lim_{m\to\infty} \omega(m,k)^{1/m} = \beta_{1,0}$ *and* $\lim_{n\to\infty} \omega(k,n)^{1/n} = \beta_{2,0}$, *for all $k \in \mathbb{Z}_+$; but they may not be equal to $\beta_{1,k}$ and $\beta_{2,k}$, respectively;*

(ii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{m+n}}$ *may not exist;*

(iii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{mn}}$ *may not exist.*

Proof. (i) Let $k \in \mathbb{Z}_+$. Then $\omega(m,k) = \max\{\omega_1(m), \omega_2(k)\}$. This implies that $\omega(m,k)^{1/m} = \max\{\omega_1(m)^{1/m}, \omega_2(k)^{1/m}\}$. Therefore

$$\begin{aligned}\lim_{m\to\infty} \omega(m,k)^{1/m} &= \max\{\lim_{m\to\infty} \omega_1(m)^{1/m}, \lim_{m\to\infty} \omega_2(k)^{1/m}\}\\ &= \max\{\lim_{m\to\infty} \omega_1(m)^{1/m}, \lim_{m\to\infty} \omega_2(0)^{1/m}\}\\ &= \lim_{m\to\infty} \omega(m,0)^{1/m} = \beta_{1,0}\end{aligned}$$

Similarly, we have $\lim_{n\to\infty} \omega(k,n)^{1/n} = \beta_{2,0}$. Next we give an example exhibiting that these limits may not be equal to $\beta_{1,k}$ and $\beta_{2,k}$, respectively. For example, let $\omega(m,n) = \max\{e^{-m^2}, e^{-n^2}\}$ $(m,n \in \mathbb{Z}_+)$. Then it is clear that $\lim_{m\to\infty} \omega(m,k)^{1/m} = \max\{\lim_{m\to\infty}(e^{-m^2})^{1/m}, 1\} = 1$, while

$$\begin{aligned}\beta_{1,k} &= \inf\{\omega(m,k)^{1/m} : m \in \mathbb{N}\}\\ &= \inf\{\max\{e^{-m^2}, e^{-k^2}\}^{1/m} : m \in \mathbb{N}\}\\ &= \inf\{e^{-1}, e^{\frac{-2^2}{2}}, \ldots, e^{\frac{-k^2}{k}}, e^{\frac{-k^2}{k+1}}, e^{\frac{-k^2}{k+2}}, \ldots\} = e^{-k}.\end{aligned}$$

Similarly, we can prove that $\lim_{n\to\infty} \omega(k,n)^{1/n} = 1$ and $\beta_{2,k} = e^{-k}$.

An example for (ii) and (iii) is $\omega(m,n) = \max\{\omega_1(m), \omega_2(n)\}$ $(m,n \in \mathbb{Z}_+)$, where $\omega_1(m) = 1$ and $\omega_2(n) = e^n + e^{-n}$. □

Lemma 2.6. *Let ω_1 and ω_2 be weights on $\mathbb{Z}$.*

Define $\omega(m,n) = \max\{\omega_1(m), \omega_2(n)\}$ for $m,n \in \mathbb{Z}$. Then

(i) $\lim_{m\to\infty} \omega(-m,k)^{-1/m} = \alpha_{1,0}$ and $\lim_{n\to\infty} \omega(k,-n)^{-1/n} = \alpha_{2,0}$, for all $k \in \mathbb{Z}_-$; but they may not be equal to $\alpha_{1,k}$ and $\alpha_{2,k}$ respectively;

(ii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{m+n}}$ may not exist;

(iii) $\lim_{m,n\to\infty} \omega(m,n)^{\frac{1}{mn}}$ may not exist.

Proof. (i) As in the proof of Lemma 2.5(i), we have $\lim_{m\to\infty} \omega(-m,k)^{-1/m} = \alpha_{1,0}$. Now we show that it may not be equal to $\alpha_{1,k}$. For example, let $\omega(m,n) = \max\{e^m, e^n\}$ $(m,n \in \mathbb{Z})$. Then

$$\lim_{m\to\infty} \omega(-m,k)^{-1/m} = \max\{\lim_{m\to\infty}(e^{-m})^{1/m}, 1\}^{-1} = 1,$$

$$\begin{aligned}\text{while } \alpha_{1,k} &= \sup\{\omega(-m,k)^{-1/m} : m \in \mathbb{N}\}\\ &= \sup\{\max\{e^{-m}, e^k\}^{-1/m} : m \in \mathbb{N}\}\\ &= \sup\{e, e^{\frac{-k}{(-k+1)}}, e^{\frac{-k}{(-k+2)}}, \ldots\} = e.\end{aligned}$$

Similarly, we can show that $\lim_{n\to\infty} \omega(k,-n)^{-1/n} = 1 = \alpha_{2,0}$ but $\alpha_{2,k} = e$ $(k \leq -1)$.

The statements (ii) and (iii) follow from example given in Lemma 2.5. □

Remark. We note the following:

(1) If ω_1 and ω_2 are weights on $\mathbb{Z}_+$ or $\mathbb{Z}$, then $\omega(m,n) = \min\{\omega_1(m), \omega_2(n)\}$ may not be a weight on $\mathbb{Z}_+^2$ or $\mathbb{Z}^2$.

(2) Let ω_1 and ω_2 be two weights on $\mathbb{Z}_+$ or $\mathbb{Z}$. Then Lemma 2.5 and Lemma 2.6 are also true for the weight $\omega = \omega_1 + \omega_2$ because $\max\{\omega_1(m), \omega_2(n)\} \le \omega_1(m) + \omega_2(n) \le 2\max\{\omega_1(m), \omega_2(n)\}$, for all $m, n \in \mathbb{Z}_+$ or $\mathbb{Z}$.

3. The Gel'fand spaces of $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$

The Russian mathematician Gel'fand developed the Gel'fand theory for commutative Banach algebras. He showed the importance of the Gel'fand space in the theory of commutative Banach algebras [5]. So it is always interesting to identify the Gel'fand space with some subset of $\mathbb{C}^n$. From this point of view, in this section, we have made an attempt to identify the Gel'fand spaces of the Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$. Such an attempt was made by Dabhi for $\omega \ge 1$ on $\mathbb{Z}^2$ [2]. Though we could identify them with some subsets of $\mathbb{C}^2$, they are found to be quite complicated for some weights. We start with the following two simple lemmas:

Lemma 3.1. *Let $z, w \in \mathbb{C}^\bullet$ and ω be a weight on $\mathbb{Z}_+^2$. Suppose that there exists a constant $M > 0$ such that $|z|^m |w|^n \le M\omega(m,n)$ $(m, n \in \mathbb{Z}_+)$. Then*

$$|z|^m |w|^n \le \omega(m,n) \; (m,n \in \mathbb{Z}_+). \tag{2}$$

Similar statement holds true for a weight on $\mathbb{Z}^2$.

Proof. Let $k \in \mathbb{N}$. Then $(|z|^m |w|^n)^k = |z|^{km} |w|^{kn} \le M\omega(km, kn) \le M\omega(m,n)^k$. Hence $|z|^m |w|^n \le M^{\frac{1}{k}} \omega(m,n)$. Since k is arbitrary, $|z|^m |w|^n \le \omega(m,n)$, for all m, n. Therefore, $|z|^m |w|^n \le M\omega(m,n) \Rightarrow |z|^m |w|^n \le \omega(m,n)$. □

Lemma 3.2. *Let ω be a weight on $\mathbb{Z}_+^2$. Then μ_2 is the largest number satisfying the following*

$$|z|^m |w|^n \le \omega(m,n) \; (z, w \in \mathbb{D}_{\mu_2}; m, n \in \mathbb{Z}_+).$$

Proof. Let $r \ge 0$ such that $|z|^m |w|^n \le \omega(m,n)$ $(z, w \in \mathbb{D}_r; m, n \in \mathbb{Z}_+)$. Take $z = w = r \in \mathbb{D}_r$. Then $r^{m+n} \le \omega(m,n)$ $(m, n \in \mathbb{Z}_+)$. So that $r \le \omega(m,n)^{\frac{1}{m+n}}$, for all $(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. Hence $r \le \mu_2$. □

Notations: We set the following notations:

1. Let ω be a weight on $\mathbb{Z}_+^2$. Then define

   (a) $\mathbb{S}_0(\omega) = \{(z,w) \in \mathbb{C}^{\bullet 2} : |z|^m |w|^n \le \omega(m,n) \; (m,n \in \mathbb{Z}_+)\}$.
   (b) $\mathbb{S}_1(\omega) = \{(z,0) \in \mathbb{C}^2 \setminus \{(0,0)\} : |z|^m \le \omega(m,0) \; (m \in \mathbb{Z}_+)\}$.
   (c) $\mathbb{S}_2(\omega) = \{(0,w) \in \mathbb{C}^2 \setminus \{(0,0)\} : |w|^n \le \omega(0,n) \; (n \in \mathbb{Z}_+)\}$.
   (d) $\mathbb{G}(\omega) = \mathbb{S}_0(\omega) \cup \mathbb{S}_1(\omega) \cup \mathbb{S}_2(\omega) \cup \{(0,0)\}$.

2. Let $z, w \in \mathbb{C}^{\bullet}$. Then, for $f \in l^1(\mathbb{Z}_+^2, \omega)$, define

$$\begin{aligned}
\phi_{z,w}(f) &= \sum \{f(m,n) z^m w^n : m, n \in \mathbb{Z}_+\} \\
\phi_{z,0}(f) &= \sum \{f(m,0) z^m : m \in \mathbb{Z}_+\} \\
\phi_{0,w}(f) &= \sum \{f(0,n) w^n : n \in \mathbb{Z}_+\} \\
\phi_{0,0}(f) &= f(0,0).
\end{aligned}$$

The set $\mathbb{G}(\omega)$ will be called the *Gel'fand set* of ω. It is routine to check that if (z, w) belongs to the Gel'fand set $\mathbb{G}(\omega)$, then $\phi_{z,w}$ is a complex homomorphism on $l^1(\mathbb{Z}_+^2, \omega)$. The following result asserts that if $\varphi \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$, then there exists a unique pair $(z, w) \in \mathbb{G}(\omega)$ such that $\varphi = \phi_{z,w}$.

Theorem 3.1. *The Gel'fand space $\Delta(l^1(\mathbb{Z}_+^2, \omega))$ is homeomorphic to $\mathbb{G}(\omega)$.*

Proof. Define $\Lambda : \Delta(l^1(\mathbb{Z}_+^2, \omega)) \longrightarrow \mathbb{C}^2$ as $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))$.

First, we show that the range of Λ is contained in $\mathbb{G}(\omega)$. Fix $\varphi \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$. Let $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2})) = (z, w)$. If $z = w = 0$, then clearly $\Lambda(\varphi) \in \mathbb{G}(\omega)$. Assume that $z \neq 0$ and $w \neq 0$. Then, for $m, n \in \mathbb{Z}_+$,

$$|z|^m |w|^n = |\varphi(\delta_{e_1})|^m |\varphi(\delta_{e_2})|^n = |\varphi(\delta_{(m,n)})| \leq \|\delta_{(m,n)}\|_\omega = \omega(m, n).$$

Hence $\Lambda(\varphi) = (z, w) \in \mathbb{S}_0(\omega) \subset \mathbb{G}(\omega)$. Now assume that $z \neq 0$ and $w = 0$. Then, for $m \in \mathbb{Z}_+$,

$$|z|^m = |\varphi(\delta_{e_1})|^m = |\varphi(\delta_{(m,0)})| \leq \omega(m, 0).$$

So, $\Lambda(\varphi) = (z, 0) \in \mathbb{S}_1(\omega) \subset \mathbb{G}(\omega)$. Similarly, if $z = 0$ and $w \neq 0$, then it can be shown that $\Lambda(\varphi) = (0, w) \in \mathbb{S}_2(\omega) \subset \mathbb{G}(\omega)$. Thus $\Lambda : \Delta(l^1(\mathbb{Z}_+^2, \omega)) \longrightarrow \mathbb{G}(\omega)$.

Next, we show that Λ is one-one. Let $\varphi_1, \varphi_2 \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ such that $\Lambda(\varphi_1) = \Lambda(\varphi_2)$. Then $\varphi_1(\delta_{e_1}) = \varphi_2(\delta_{e_1})$ and $\varphi_1(\delta_{e_2}) = \varphi_2(\delta_{e_2})$. Let $f \in l^1(\mathbb{Z}_+^2, \omega)$. Then

$$\begin{aligned}
\varphi_1(f) &= \sum_{m,n \in \mathbb{Z}_+} f(m,n) \varphi_1(\delta_{(m,n)}) = \sum_{m,n \in \mathbb{Z}_+} f(m,n) \varphi_1(\delta_{e_1})^m \varphi_1(\delta_{e_2})^n \\
&= \sum_{m,n \in \mathbb{Z}_+} f(m,n) \varphi_2(\delta_{e_1})^m \varphi_2(\delta_{e_2})^n = \sum_{m,n \in \mathbb{Z}_+} f(m,n) \varphi_2(\delta_{(m,n)}) = \varphi_2(f).
\end{aligned}$$

Thus, $\varphi_1 = \varphi_2$, i.e. Λ is one-one.

Next, we show that Λ is onto. Let $(z, w) \in \mathbb{G}(\omega)$. Then it is easy to see that $\phi_{z,w} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ and $\Lambda(\phi_{z,w}) = (z, w)$. Thus, Λ is onto.

Finally, we show that Λ is a homeomorphism. Consider a convergent net $\varphi_\alpha \longrightarrow \varphi$ in $\Delta(l^1(\mathbb{Z}_+^2, \omega))$. By the definition of Gel'fand topology [6, P.52], $\varphi_\alpha(f) \longrightarrow \varphi(f)$, for all $f \in l^1(\mathbb{Z}_+^2, \omega)$. In particular, $\varphi_\alpha(\delta_{e_1}) \longrightarrow \varphi(\delta_{e_1})$ and $\varphi_\alpha(\delta_{e_2}) \longrightarrow \varphi(\delta_{e_2})$. Hence $\Lambda(\varphi_\alpha) = (\varphi_\alpha(\delta_{e_1}), \varphi_\alpha(\delta_{e_2})) \longrightarrow (\varphi(\delta_{e_1}), \varphi(\delta_{e_2})) = \Lambda(\varphi)$. So Λ is continuous. Since $l^1(\mathbb{Z}_+^2, \omega)$ has identity, the Gel'fand space is compact [6, P.52]. Since $\mathbb{G}(\omega)$ is Hausdorff, Λ is a homeomorphism. $\square$

Theorem 3.2. *Let ω be a weight on $\mathbb{Z}_+^2$. Then*

(i) $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subseteq \mathbb{G}(\omega)$;

(ii) $\mathbb{G}(\omega) \subseteq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;

(iii) $\mathbb{S}_0(\omega) \subseteq \mathbb{D}_{\sigma_1} \times \mathbb{D}_{\sigma_2}$;

(iv) $(\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{s_2}) \cup (\mathbb{D}_{s_1} \times \mathbb{D}_{\beta_{2,0}}) \subseteq \mathbb{G}(\omega)$ *for some* $s_1, s_2 \geq 0$.

Proof. (i) Let $(z,w) \in \mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2}$. Then $|z| \leq \mu_2$ and $|w| \leq \mu_2$. This implies that $|z| \leq \omega(m,n)^{\frac{1}{m+n}}$ and $|w| \leq \omega(m,n)^{\frac{1}{m+n}}$, for all $(m,n) \neq (0,0)$. Thus $|z|^m|w|^n \leq \omega(m,n)$. Hence $(z,w) \in \mathbb{G}(\omega)$.

(ii) Let $(z,w) \in \mathbb{G}(\omega)$. It is easy to check, if $(z,w) \in \{(0,0)\} \cup \mathbb{S}_1(\omega) \cup \mathbb{S}_2(\omega)$. Now, if $(z,w) \in \mathbb{S}_0(\omega)$, then $|z|^m|w|^n \leq \omega(m,n)$, for all $m,n \in \mathbb{Z}_+$. This implies $|z|^m \leq \omega(m,0)$, for all $m \in \mathbb{Z}_+$. Therefore $|z| \leq \beta_{1,0}$. Similarly, $|w| \leq \beta_{2,0}$.

(iii) Let $(z,w) \in \mathbb{S}_0(\omega)$. Then, as usual, $|z|^m|w|^n = |\varphi(\delta_{(m,n)})| \leq \omega(m,n)$, for all $m,n \in \mathbb{Z}_+$. Fix any $k \in \mathbb{Z}_+$. Then $|z|^m|w|^k \leq \omega(m,k)$ $(m \in \mathbb{N})$. This implies that

$$|z| = \liminf_{m\to\infty} |z||w|^{\frac{k}{m}} \leq \liminf_{m\to\infty} \omega(m,k)^{\frac{1}{m}}.$$

Since k is arbitrary, we get $|z| \leq \sigma_1$. Similarly, we have $|w| \leq \sigma_2$. Hence $(z,w) \in \mathbb{D}_{\sigma_1} \times \mathbb{D}_{\sigma_2}$.

(iv) Define $s_2 = \sup\{|w| : (\beta_{1,0}, w) \in \mathbb{G}(\omega)\} \geq 0$. Let $|z| \leq \beta_{1,0}$ and $|w| \leq s_2$. By definition of s_2, we have $(\beta_{1,0}, w) \in \mathbb{G}(\omega)$. Hence $\mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{s_2} \subset \mathbb{G}(\omega)$. Similarly, we can define s_1 and prove that $\mathbb{D}_{s_1} \times \mathbb{D}_{\beta_{2,0}} \subset \mathbb{G}(\omega)$. □

Theorem 3.3. *Let ω be a weight on $\mathbb{Z}_+^2$. Then there is a subset $\{(r_i, s_i) : i \in \Lambda\}$ of $\mathbb{R}_+^2$ such that*

$$\mathbb{G}(\omega) = \cup_{i\in\Lambda}(\mathbb{D}_{r_i} \times \mathbb{D}_{s_i}).$$

Proof. Let us define $\mathcal{D} = \{(r,s) \in \mathbb{R}_+^2 : \phi_{r,s} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))\}$. For $(r,s), (u,v) \in \mathcal{D}$, we have $(r,s) \leq (u,v)$ if and only if $r \leq u$ and $s \leq v$. Then $(\mathcal{D}, \leq)$ is a partially ordered set. Let $M = \{(r_i, s_i) : i \in \Lambda\}$ be the set of all maximal elements of $\mathcal{D}$. Now, we shall prove that the set M is non-empty. Let $\mathcal{C} \subset \mathcal{D}$ be any chain in $\mathcal{D}$. Set $(\bar{r}, \bar{s}) = \sup\{(r,s) : (r,s) \in \mathcal{C}\}$. We claim that $(\bar{r}, \bar{s}) \in \mathcal{D}$.

Case (i). $\bar{r} = 0$ and $\bar{s} = 0$. This implies that $\mathcal{C} = \{(0,0)\}$ and $(0,0) \in \mathcal{D}$.

Case (ii). $\bar{r} > 0$ and $\bar{s} = 0$. Then there exists a sequence $(u_n, 0) \in \mathcal{C}$ such that $u_n \longrightarrow \bar{r}$ and $\phi_{u_n,0} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$. Now,

$$\phi_{\bar{r},0}(\delta_{(m,n)}) = \bar{r}^m = \lim_{n\to\infty} u_n^m = \lim_{n\to\infty} \phi_{u_n,0}(\delta_{(m,n)}).$$

Thus $\phi_{u_n,0}(\delta_{(m,n)}) \longrightarrow \phi_{\bar{r},0}(\delta_{(m,n)})$. This implies that $\phi_{u_n,0}(f) \longrightarrow \phi_{\bar{r},0}(f)$, for all $f \in l^1(\mathbb{Z}_+^2, \omega)$ with *suppf* being finite. Hence $\phi_{u_n,0}(f) \longrightarrow \phi_{\bar{r},0}(f)$ $(f \in l^1(\mathbb{Z}_+^2, \omega))$ and $\phi_{\bar{r},0} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$.

Case (iii). $\bar{r} = 0$ and $\bar{s} > 0$. By similar arguments as given in Case (ii), we have $\phi_{0,\bar{s}} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$.

Case (iv). $\bar{r} > 0$ and $\bar{s} > 0$. Choose $k_0 \in \mathbb{N}$ sufficiently large enough so that $0 < \frac{1}{k_0} < \bar{r}$ and $0 < \frac{1}{k_0} < \bar{s}$. For any $k \geq k_0$, there exists $(u_k, v_k) \in \mathcal{C}$ such that $(\bar{r} - \frac{1}{k}, \bar{s} - \frac{1}{k}) < (u_k, v_k) \leq (\bar{r}, \bar{s})$. This implies that $\phi_{\bar{r},\bar{s}}(\delta_{(m,n)}) = \bar{r}^m \bar{s}^n = \lim_{k\to\infty} u_k^m v_k^n = \lim_{k\to\infty} \phi_{u_k,v_k}(\delta_{(m,n)})$. Thus $\phi_{u_k,v_k}(\delta_{(m,n)}) \longrightarrow \phi_{\bar{r},\bar{s}}(\delta_{(m,n)})$. This implies that $\phi_{u_k,v_k}(f) \longrightarrow \phi_{\bar{r},\bar{s}}(f)$ $(f \in l^1(\mathbb{Z}_+^2, \omega))$. Hence$(\bar{r}, \bar{s}) \in \mathcal{D}$.

Therefore, by Zorn's Lemma, $\mathcal{D}$ has a maximal element. Hence $M \neq \emptyset$.

Finally, we claim that $\mathbb{G}(\omega) = \cup_{i\in\Lambda}(\mathbb{D}_{r_i} \times \mathbb{D}_{s_i})$. Let $(z, w) \in \mathbb{G}(\omega)$. Then $(z, w) \in \mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$ for some $i \in \Lambda$. This implies that $|z| \leq r_i$ and $|w| \leq s_i$. Therefore $\phi_{|z|,|w|} \in \Delta(l^1(\mathbb{Z}_+^2, \omega))$ because $(r_i, s_i) \in \mathcal{D}$. Hence $(z, w) \in \mathbb{G}(\omega)$.

Conversely, let $(z, w) \in \mathbb{G}(\omega)$. Then $(|z|, |w|) \in \mathcal{D}$. Since $\mathcal{D}$ has maximal elements, there exists $i \in \Lambda$ such that $(|z|, |w|) \leq (r_i, s_i)$. So that $(z, w) \in \mathbb{D}_{r_i} \times \mathbb{D}_{s_i}$. Hence $\mathbb{G}(\omega) \subseteq \cup_{i\in\Lambda}(\mathbb{D}_{r_i} \times \mathbb{D}_{s_i})$. This completes the proof. □

Now we identify the Gel'fand space of $l^1(\mathbb{Z}^2, \omega)$ with a closed subset of $\mathbb{C}^{\bullet^2}$. Note that Dabhi identified the Gel'fand space $\Delta(l^1(\mathbb{Z}_+^2, \omega))$ with $\mathbb{T}(\omega)$ for those weights satisfying $\omega(m, n) \geq 1$ $(m, n \in \mathbb{Z})$ [2, Corollary 2]. Here we do not put any condition on ω. In order to this, we need to introduce some notations.

Notations. Let $r, s > 0$ and ω be a weight on $\mathbb{Z}^2$. Then

(i) $\Gamma(r) = \{z \in \mathbb{C} : |z| = r\}$;

(ii) $\Gamma(r, s) = \{z \in \mathbb{C} : r \leq |z| \leq s\}$;

(iii) $\mathbb{T}(\omega) = \{(z, w) \in \mathbb{C}^2 : |z|^m |w|^n \leq \omega(m, n) \ \ (m, n \in \mathbb{Z})\}$.

It is clear that $\Gamma(r, s) \neq \emptyset$ iff $r \leq s$. Moreover, $\mathbb{T}(\omega) \subset \mathbb{C}^{\bullet^2}$.

Theorem 3.4. *The Gel'fand space* $\Delta(l^1(\mathbb{Z}^2, \omega))$ *is homeomorphic to* $\mathbb{T}(\omega)$.

Proof. Define $\Lambda : \Delta(l^1(\mathbb{Z}^2, \omega)) \longrightarrow \mathbb{C}^2$ as $\Lambda(\varphi) = (\varphi(\delta_{e_1}), \varphi(\delta_{e_2}))$. It is enough to prove that $\varphi(\delta_{e_1}) \neq 0$ and $\varphi(\delta_{e_2}) \neq 0$. The rest of the proof is similar to the proof of Theorem 3.1. Since $l^1(\mathbb{Z}^2, \omega)$ is a unital, commutative, Banach algebra with identity $\delta_{(0,0)}$, we have $\varphi(\delta_{(0,0)}) = 1$, for all $\varphi \in \Delta(l^1(\mathbb{Z}^2, \omega))$. Now, $\delta_{e_1} * \delta_{-e_1} = \delta_{(0,0)}$. This implies $\varphi(\delta_{e_1})\varphi(\delta_{-e_1}) = \varphi(\delta_{(0,0)}) = 1$. Hence $\varphi(\delta_{e_1}) \neq 0$. Similarly, we can show that $\varphi(\delta_{e_2}) \neq 0$. □

Theorem 3.5. *Let* $\omega \geq 1$ *be a weight on* $\mathbb{Z}^2$. *Then*

(i) $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subseteq \mathbb{T}(\omega)$;

(ii) $\mathbb{T}(\omega) \subseteq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$;

(iii) $(\Gamma(\alpha_1) \times \Gamma(1)) \cup (\Gamma(1) \times \Gamma(\alpha_2)) \cup (\Gamma(\beta_1) \times \Gamma(1)) \cup (\Gamma(1) \times \Gamma(\beta_2)) \subseteq \mathbb{T}(\omega)$.

Proof. (i) Let $(z, w) \in \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2})$. Then

$$
\begin{aligned}
&\sqrt{\alpha_1} \leq |z| \leq \sqrt{\beta_1} \\
&\Rightarrow |z|^2 \leq \omega(m, n)^{1/m} (m \in \mathbb{N}; n \in \mathbb{Z}) \text{ and } \omega(-m, n)^{-1/m} \leq |z|^2 (m \in \mathbb{N}; n \in \mathbb{Z}) \\
&\Rightarrow |z|^{2m} \leq \omega(m, n)(m \in \mathbb{N}; n \in \mathbb{Z}) \text{ and } |z|^{-2m} \leq \omega(-m, n)(m \in \mathbb{N}; n \in \mathbb{Z}) \\
&\Rightarrow |z|^{2m} \leq \omega(m, n), (m, n \in \mathbb{Z}).
\end{aligned}
$$

Similarly, we get, $|w|^{2n} \leq \omega(m,n), m \in \mathbb{Z}; n \in \mathbb{Z}$. It follows that $|z|^m|w|^n \leq \omega(m,n)\ \ m,n \in \mathbb{Z}$. Therefore, $(z,w) \in \Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2})$.

(ii) Let $(z,w) \in \mathbb{T}(\omega)$. Define $z = \varphi(\delta_{e_1})$ and $w = \varphi(\delta_{e_2})$. Then $\varphi = \phi_{z,w}$. Since $\phi_{z,w}$ is continuous, $|z|^m = |\varphi(\delta_{(m,0)})| \leq \|\delta_{(m,0)}\| = \omega(m,0)$, for all $m \in \mathbb{Z}$. Therefore $\sup_{m\in\mathbb{N}} \omega(-m,0)^{\frac{-1}{m}} \leq |z| \leq \inf_{m\in\mathbb{N}} \omega(m,0)^{\frac{1}{m}}$ and similarly, $\sup_{n\in\mathbb{N}} \omega(0,-n)^{\frac{-1}{n}} \leq |w| \leq \inf_{n\in\mathbb{N}} \omega(0,n)^{\frac{1}{n}}$. Hence $(z,w) \in \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

(iii) Let $|z| = \alpha_1$ and $|w| = 1$. Then, for $m \in \mathbb{Z}_+, n \in \mathbb{Z}$, we have, $|z|^m \leq \omega(m,n)$ as $\omega \geq 1$. Now, for $m \in \mathbb{N}, n \in \mathbb{Z}$, we have $|z|^{-m} \leq \omega(-m,n)$. Hence $\Gamma(\alpha_1) \times \Gamma(1) \subseteq \mathbb{T}(\omega)$. Similarly, $\Gamma(1) \times \Gamma(\alpha_2) \subseteq \mathbb{T}(\omega)$. Now let $|z| = \beta_1$ and $|w| = 1$. Then for $m \in \mathbb{N}, n \in \mathbb{Z}$, we have $|z|^m \leq \omega(m,n)$. For $m < 0, n \in \mathbb{Z}$, $|z|^m \leq \omega(m,n)$ as $\omega \geq 1$. Therefore, $\Gamma(\beta_1) \times \Gamma(1) \subseteq \mathbb{T}(\omega)$. Similarly, $\Gamma(1) \times \Gamma(\beta_2) \subseteq \mathbb{T}(\omega)$. □

Theorem 3.6. *Let ω be any weight on $\mathbb{Z}^2$. Then there exist a subset Λ of $\mathbb{R}_+^{\bullet}$ and a subset $\{(s_r, t_r) : r \in \Lambda\}$ of $\mathbb{R}_+^{\bullet 2}$ such that*

$$\mathbb{T}(\omega) = \cup_{r\in\Lambda}(\Gamma(r) \times \Gamma(s_r, t_r)).$$

Proof. Let $\mathcal{D} = \{(r,s) \in \mathbb{R}_+^2 : r^m s^n \leq \omega(m,n)\ (m,n \in \mathbb{Z})\}$. Then $\mathcal{D}$ is a partially ordered set with respect to the product order. Next we define $\Lambda = \{r \in \mathbb{R}_+^{\bullet} : (r,s) \in \mathcal{D}$ for some $s > 0\}$. Then, for each $r \in \Lambda$, the set $\mathcal{C}_r = \{(r,s) : (r,s) \in \mathcal{D}\}$ is a chain in $\mathcal{D}$, and so it has minimum and maximum elements, say (r, s_r) and (r, t_r), respectively.

Now, we *claim* that $\mathbb{T}(\omega) = \cup_{r\in\Lambda}(\Gamma(r) \times \Gamma(s_r, t_r))$. Let $(z,w) \in \mathbb{T}(\omega)$ and set $|z| = r$. Then $(r, |w|) \in \mathcal{D}$ and so $r \in \Lambda$. This implies that $(z,w) \in \Gamma(r) \times \Gamma(s_r, t_r)$. Therefore $\mathbb{T}(\omega) \subset \cup_{r\in\Lambda}(\Gamma(r) \times \Gamma(s_r, t_r))$. Conversely, let $r \in \Lambda$ be arbitrary. Let $(z,w) \in \Gamma(r) \times \Gamma(s_r, t_r)$. Then $|z| = r$ and $s_r \leq |w| \leq t_r$. Let $m, n \in \mathbb{Z}$.

(1) If $m \geq 0$ and $n \geq 0$, then $|z|^m|w|^n \leq r^m t_r^n \leq \omega(m,n)$;

(2) If $m \geq 0$ and $n \leq 0$, then $|z|^m|w|^n \leq r^m s_r^n \leq \omega(m,n)$;

(3) If $m \leq 0$ and $n \geq 0$, then $|z|^m|w|^n \leq r^m t_r^n \leq \omega(m,n)$;

(4) If $m \leq 0$ and $n \leq 0$, then $|z|^m|w|^n \leq r^m s_r^n \leq \omega(m,n)$.

Therefore, $(z,w) \in \mathbb{T}(\omega)$. Hence $\cup_{r\in\Lambda}(\Gamma(r) \times \Gamma(s_r, t_r)) \subset \mathbb{T}(\omega)$. □

Theorem 3.7. *Let ω_1 and ω_2 be any two weights either on $\mathbb{Z}_+$ or on $\mathbb{Z}$. Let $\omega(m,n) = \omega_1(m)\omega_2(n)$.*

(i) If ω is defined on $\mathbb{Z}_+^2$, then $\mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;

(ii) If ω is defined on $\mathbb{Z}^2$, then $\mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Proof. (i) By Theorem 3.2, it is clear that $\mathbb{G}(\omega) \subseteq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. Conversely, let $(z, w) \in \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$. For $m, n \in \mathbb{Z}_+$, we have $|z|^m \leq \omega(m, 0) = \omega_1(m)\omega_2(0)$ and $|w|^n \leq \omega(0, n) = \omega_1(0)\omega_2(n)$. This implies that

$$|z|^m |w|^n \leq \omega_1(m)\omega_2(n)\omega_1(0)\omega_2(0) = c\omega(m, n),$$

where $c = \omega_1(0)\omega_2(0)$. Hence $(z, w) \in \mathbb{G}(\omega)$.

(ii) By Theorem 3.5, it is clear that $\mathbb{T}(\omega) \subseteq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$. Conversely, let $(z, w) \in \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$. Then, for $m, n \in \mathbb{Z}$, we have, $\omega_1(-m)^{-1/m}\omega_2(0)^{-1/m} \leq |z| \leq \omega_1(m)^{1/m}\omega_2(0)^{1/m}$ and $\omega_1(0)^{-1/n}\omega_2(-n)^{-1/n} \leq |w| \leq \omega_1(0)^{1/n}\omega_2(n)^{1/n}$. Therefore, $|z|^m|w|^n \leq \omega_1(m)\omega_2(n)\omega_1(0)\omega_2(0) = c\omega(m, n)$, for all $m, n \in \mathbb{Z}$, where $c = \omega_1(0)\omega_2(0)$. Hence $(z, w) \in \mathbb{T}(\omega)$ by Lemma 3.1. □

4. Examples of weights

Here, we discuss some examples of weights on $\mathbb{Z}_+^2$ and $\mathbb{Z}^2$ and then find the Gel'fand spaces of the corresponding Beurling algebras $l^1(\mathbb{Z}_+^2, \omega)$ and $l^1(\mathbb{Z}^2, \omega)$, i.e, $\mathbb{G}(\omega)$ and $\mathbb{T}(\omega)$. Note that a weight on $\mathbb{Z}^2$ is also a weight on $\mathbb{Z}_+^2$. So we shall take examples of weights on $\mathbb{Z}^2$.

Example 1. Let $\omega(m, n) = 1$ be a weight on $\mathbb{Z}^2$. Then $\mathbb{G}(\omega) = \mathbb{D}_1 \times \mathbb{D}_1$ and $\mathbb{T}(\omega) = \Gamma(1) \times \Gamma(1)$. Because $\mu_2 = \beta_{1,0} = \beta_{2,0} = 1$. Also, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha_{1,0} = \alpha_{2,0} = \beta_{1,0} = \beta_{2,0} = 1$.

Example 2. Let $\omega(m, n) = e^{|m|} + e^{|n|}$ $(m, n \in \mathbb{Z})$ be a weight on $\mathbb{Z}^2$. Then

(i) $\alpha_1 = \alpha_2 = \alpha_{1,0} = \alpha_{2,0} = \rho_1 = e^{-1}$;

(ii) $\beta_1 = \beta_2 = \beta_{1,0} = \beta_{2,0} = \rho_2 = e$;

(iii) $\mu_2 = \sqrt{e}$;

(iv) $\mathbb{G}(\omega) = (\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}}) \cup (\mathbb{U}_1 \times \Gamma(\sqrt{e}, e)) \cup (\Gamma(\sqrt{e}, e) \times \mathbb{U}_1) \cup (\mathbb{S}_4 \cup \mathbb{S}_4^{op})$,
where $\mathbb{S}_4 = \{(z, w) : 1 \leq |z| \leq \sqrt{e} \leq |w| \leq e \text{ and } |z||w| \leq e\}$
and $\mathbb{S}_4^{op} = \{(w, z) : (z, w) \in \mathbb{S}_4\}$
i.e. $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) \subsetneq \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$;

(v) $\mathbb{T}(\omega) = \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \cup [\cup_{i=1}^4 (\mathbb{T}_i \cup \mathbb{T}_i^{op})]$
where $\mathbb{T}_1 = \{(z, w) : 1 \leq |z| \leq \sqrt{e} \leq |w| \leq e \text{ and } |z||w| \leq e\}$,
$\mathbb{T}_2 = \{(z, w) : \frac{1}{\sqrt{e}} \leq |z| \leq 1, \sqrt{e} \leq |w| \leq e \text{ and } |z|^{-1}|w| \leq e\}$,
$\mathbb{T}_3 = \{(z, w) : \frac{1}{e} \leq |z| \leq \frac{1}{\sqrt{e}}, 1 \leq |w| \leq \sqrt{e} \text{ and } |z|^{-1}|w| \leq e\}$,
$\mathbb{T}_4 = \{(z, w) : \frac{1}{e} \leq |z| \leq \frac{1}{\sqrt{e}} \leq |w| \leq 1 \text{ and } |z||w| \geq \frac{1}{e}\}$, and
$\mathbb{T}_i^{op} = \{(w, z) : (z, w) \in \mathbb{T}_i\}$ $(i = 1, 2, 3, 4)$
i.e. $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) \subsetneq \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Solution. Let $\eta(m, n) = \log \omega(m, n)$ $(m, n \in \mathbb{Z})$.

(i) $\alpha_1 = \alpha_2 = \alpha_{1,0} = \alpha_{2,0} = \rho_1 = e^{-1}$.
It is enough to prove that $\sup\{\frac{\eta(-m,n)}{-m} : m \in \mathbb{N}, n \in \mathbb{Z}\} = -1$. Note that, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{\eta(-m,n)}{-m} &= \frac{\log \omega(-m,n)}{-m} = \frac{\log(e^m + e^{|n|})}{-m} > \frac{\log(e^m + e^{|n|+1})}{-m} \\ &> \frac{\log(e^m e^{|n|+1})}{-m} = \frac{\log e^{m+|n|+1}}{-m} = \frac{m+|n|+1}{-m}. \end{aligned}$$

Hence, $\sup\{\frac{\eta(-m,n)}{-m}: m\in\mathbb{N}, n\in\mathbb{Z}\} \geq \limsup_{m\to\infty} \frac{\eta(-m,n)}{-m} \geq \limsup_{m\to\infty} \frac{m+|n|+1}{-m} = -1$. On the other hand, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have

$$\frac{\eta(-m,n)}{-m} = \frac{\log \omega(-m,n)}{-m} = \frac{\log(e^m + e^{|n|})}{-m} < \frac{\log e^m}{-m} = -1.$$

Therefore, $\alpha_1 = e^{-1}$.

Similarly, $\alpha_2 = e^{-1}$, because $\omega(m,n) = \omega(n,m)$.

Since $\alpha_{1,0} = \lim_{m\to\infty} \omega(-m,0)^{-1/m} = \lim_{m\to\infty}(e^m + 1)^{-1/m} = e^{-1}$. Similarly, we have, $\alpha_{2,0} = e^{-1}$.

We know that $\rho_1 = \lim_{n\to\infty} \omega(-n,-n)^{-1/n} = \lim_{n\to\infty}(e^n + e^n)^{-1/n} = e^{-1}$.

(ii) This can be proved by similar arguments given in (i) above, we get, $\beta_1 = \beta_2 = \beta_{1,0} = \beta_{2,0} = \rho_2 = e$.

(iii) $\mu_2 = \sqrt{e}$.

It is enough to show that $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} = \frac{1}{2}$. For $m = n$, note that,

$$\frac{\eta(n,n)}{n+n} = \frac{\log(e^n + e^n)}{2n} = \frac{\log 2 + n}{2n}.$$

Hence, $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} \leq \lim_{n\to\infty} \frac{\eta(n,n)}{2n} \leq \lim_{n\to\infty} \frac{\log 2+n}{2n} = \frac{1}{2}$. Thus $\mu_2 \leq \frac{1}{2}$ On the other hand, we have

$$\frac{\eta(m,n)}{m+n} = \frac{\log \omega(m,n)}{m+n} = \frac{\log(e^m + e^n)}{m+n} > \frac{1}{2}.$$

Hence $\mu_2 = e^{\frac{1}{2}}$.

(iv) $\mathbb{G}(\omega) = (\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}}) \cup (\mathbb{U}_1 \times \Gamma(\sqrt{e}, e)) \cup (\Gamma(\sqrt{e}, e) \times \mathbb{U}_1) \cup (\mathbb{S}_4 \cup \mathbb{S}_4^{op})$.

Note that $\mu_2 = \sqrt{e}$. So, by Theorem 3.2(i), we have $\mathbb{U}_{\sqrt{e}} \times \mathbb{U}_{\sqrt{e}} \subset \mathbb{G}(\omega)$.

Let $(z,w) \in \mathbb{U}_1 \times \Gamma(\sqrt{e}, e)$. Then $|z| \leq 1$ and $|w| \leq e$. This implies that

$$|z|^m |w|^n \leq |w|^n \leq e^n < e^m + e^n = \omega(m,n).$$

Thus $\mathbb{U}_1 \times \Gamma(\sqrt{e}, e) \subset \mathbb{G}(\omega)$. Similarly $\Gamma(\sqrt{e}, e) \times \mathbb{U}_1 \subset \mathbb{G}(\omega)$.

Let $(z,w)\in\mathbb{S}_4$ and let $m,k\in\mathbb{Z}_+$ and $n=m+k$. Then

$$\begin{aligned}|z|^m|w|^n &= |z|^m|w|^m|w|^k\le e^m|w|^k\\ &\le e^me^k<e^m(1+e^k)=e^m+e^{m+k}\\ &= e^m+e^n=\omega(m,n).\end{aligned}$$

Similarly if $(z,w)\in\mathbb{S}_4^{op}$. Then $|z|^m|w|^n\le\omega(m,n)$.
Conversely, let $(z,w)\in\mathbb{G}(\omega)$. Then $|z|^m|w|^m\le\omega(m,m)=2e^m$ $(m\in\mathbb{Z}_+)$. This implies that $|z||w|\le e$. Now, we have following two cases:
Case (i). $0<|z|<\sqrt{e}$. Then we have either $0<|w|<\sqrt{e}$ or $\sqrt{e}\le|w|\le e$. Thus (z,w) belongs to $\mathbb{U}_{\sqrt{e}}\times\mathbb{U}_{\sqrt{e}}$ or $\mathbb{S}_4(\omega)$ or $\mathbb{U}_1\times\Gamma(\sqrt{e},e)$.
Case (ii). $\sqrt{e}\le|z|\le e$. Then $\sqrt{e}|w|\le|z||w|\le e$. This implies $|w|\le\sqrt{e}$. Thus (z,w) belongs to $\Gamma(\sqrt{e},e)\times\mathbb{U}_1$ or $(z,w)\in\mathbb{S}_4^{op}$.

(v) $\mathbb{T}(\omega)=\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})\times\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})\cup[\cup_{i=1}^4(\mathbb{T}_i\cup\mathbb{T}_i^{op})]$;

Note that, $\alpha_1=\alpha_2=e^{-1}$ and $\beta_1=\beta_2=e$. By Theorem 3.5(i), we have $\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})\times\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})\subset\mathbb{T}(\omega)$. Now let $(z,w)\in\mathbb{T}_1$.

Let $m,k\in\mathbb{Z}_+$ or $\mathbb{Z}_-$ and $n=m+k$. Then

$$\begin{aligned}|z|^m|w|^n &= |z|^m|w|^m|w|^k\le|z|^{|m|}|w|^{|m|}|w|^{|k|}\\ &\le e^{|m|}|w|^{|k|}<e^{|m|}(1+e^{|k|})=e^{|m|}+e^{|m|+|k|}\\ &= e^{|m|}+e^{|n|}=\omega(m,n).\end{aligned}$$

Let $m\in\mathbb{Z}_+$, $k\in\mathbb{Z}_-$, and $n=m+k$. Then, it is clear that $|w|^k\le 1+e^k$, for all $k\in\mathbb{Z}$. By repeating above arguments, we get

$$|z|^m|w|^n\le e^m|w|^k\le e^m(1+e^k)=e^m+e^{|n|}=\omega(m,n).$$

If $m\in\mathbb{Z}_-$ and $k\in\mathbb{Z}_+$, then by reversing the role of m and k in above argument, we can prove that $|z|^m|w|^n\le\omega(m,n)$. Hence $\mathbb{T}_1\subset\mathbb{T}(\omega)$.

By similar argument we can show that $\mathbb{T}_1^{op}\cup\mathbb{T}_2\cup\mathbb{T}_2^{op}\cup\mathbb{T}_3\cup\mathbb{T}_3^{op}\cup\mathbb{T}_4\cup\mathbb{T}_4^{op}\subset\mathbb{T}(\omega)$.

Conversely, let $(z,w)\in\mathbb{T}(\omega)$. Then $|z|^m|w|^m\le\omega(m,m)=2e^m$, $|z|^{-m}|w|^{-m}\le 2e^m$, $|z|^m|w|^{-m}\le 2e^m$, and $|z|^{-m}|w|^m\le 2e^m$, for all $m\in\mathbb{Z}_+$. This implies that $|z||w|\le e, |z|^{-1}|w|\le e, |z||w|^{-1}\le e,$ and $|z|^{-1}|w|^{-1}\le e$. Since $\alpha_1=\alpha_{1,0}=\alpha_2=\alpha_{2,0}=e^{-1}$ and $\beta_1=\beta_{1,0}=\beta_2=\beta_{2,0}=e$, we have the following cases for z and w:
Case (i). Let $\frac{1}{e}\le|z|\le\frac{1}{\sqrt{e}}$. Then $|z|^{-1}|w|^{-1}\le e$ implies that $\frac{1}{\sqrt{e}}\le|w|\le 1$ and $|z|^{-1}|w|\le e$ implies that $1\le|w|\le\sqrt{e}$. Therefore$(z,w)\in\mathbb{T}_3\cup\mathbb{T}_4$.

Case (ii). Let $\frac{1}{\sqrt{e}}\le|z|\le\sqrt{e}$. Then $\frac{1}{e}\le|w|\le e$.

If $\frac{1}{\sqrt{e}}\le|z|\le 1$ and $\frac{1}{e}\le|w|\le\frac{1}{\sqrt{e}}$. Then $(z,w)\in\mathbb{T}_4^{op}$.

If $\frac{1}{\sqrt{e}}\le|z|\le 1$ and $\frac{1}{\sqrt{e}}\le|w|\le\sqrt{e}$. Then $(z,w)\in\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})\times\Gamma(\frac{1}{\sqrt{e}},\sqrt{e})$.

If $\frac{1}{\sqrt{e}}\le|z|\le 1$ and $\sqrt{e}\le|w|\le e$. Then $(z,w)\in\mathbb{T}_2$.

If $1\le|z|\le\sqrt{e}$ and $\frac{1}{e}\le|w|\le\frac{1}{\sqrt{e}}$. Then $(z,w)\in\mathbb{T}_3^{op}$.

If $1 \leq |z| \leq \sqrt{e}$ and $\frac{1}{\sqrt{e}} \leq |w| \leq \sqrt{e}$. Then $(z,w) \in \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e})$.

If $1 \leq |z| \leq \sqrt{e}$ and $\sqrt{e} \leq |w| \leq e$. Then $(z,w) \in \mathbb{T}_1$.

Case (iii). Let $\sqrt{e} \leq |z| \leq e$. Then $|z||w| \leq e$ implies that $1 \leq |w| \leq \sqrt{e}$ and $|z||w|^{-1} \leq e$ implies that $\frac{1}{\sqrt{e}} \leq |w| \leq 1$. Therefore, $(z,w) \in \mathbb{T}_1^{op} \cup \mathbb{T}_2^{op}$.

Example 3. Let $\omega(m,n) = e^{|m|} + e^{-n}$ $(m,n \in \mathbb{Z})$. Then

(i) $\alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}$; $\alpha_2 = \alpha_{2,0} = 1$;

(ii) $\beta_1 = \beta_{1,0} = \rho_2 = e$; $\beta_2 = \beta_{2,0} = 1$;

(iii) $\mu_2 = 1$;

(iv) $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$. i.e $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$.

(v) $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$.
i.e. $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Solution. Define $\eta(m,n) = \log \omega(m,n)$ $(m,n \in \mathbb{Z})$.

(i) $\alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}$; $\alpha_2 = \alpha_{2,0} = 1$ can be proved as per the argument given in (ii) below.

(ii) $\beta_1 = \beta_{1,0} = \rho_2 = e$; $\beta_2 = \beta_{2,0} = 1$;

It is enough to prove that $\inf\{\frac{\eta(m,n)}{m} : m \in \mathbb{N}, n \in \mathbb{Z}\} = 1$. Note that, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \frac{\eta(m,n)}{m} &= \frac{\log \omega(m,n)}{m} = \frac{\log(e^m + e^{-n})}{m} \\ &\leq \frac{\log(e^m + e^m)}{m} \leq \frac{\log 2 + m}{m}. \end{aligned}$$

Hence, $\inf\{\frac{\eta(m,n)}{n} : m \in \mathbb{N}, n \in \mathbb{Z}\} \leq \liminf_{m\to\infty} \frac{\eta(m,n)}{m} \leq \liminf_{m\to\infty} \frac{\log 2 + m}{m} = 1$. On the other hand, for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have

$$\frac{\eta(m,n)}{m} = \frac{\log(e^m + e^{-n})}{m} \geq \frac{\log e^m}{m} = 1.$$

Thus $\beta_1 = e$.

It is enough to prove that $\inf\{\frac{\eta(m,n)}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} = 0$. Note that, for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \frac{\eta(m,n)}{n} &= \frac{\log \omega(m,n)}{n} = \frac{\log(e^{|m|} + e^{-n})}{n} \\ &\leq \frac{\log(e^{|m|} + e^{|m|})}{n} \leq \frac{\log 2 + |m|}{n} \end{aligned}$$

Hence, $0 \leq \inf\{\frac{\eta(m,n)}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \leq \liminf_{n\to\infty} \frac{\eta(m,n)}{n} \leq \liminf_{n\to\infty} \frac{\log 2 + |m|}{n}$ $= 0$. Therefore, $\beta_2 = 1$.

We know that $\beta_{1,0} = \lim_{m\to\infty} \omega(m,0)^{1/m} = \lim_{m\to\infty} (e^{|m|} + 1)^{1/m} = e$.

Also, $\beta_{2,0} = \lim_{n\to\infty} \omega(0,n)^{1/n} = \lim_{n\to\infty}(1+e^{-n})^{1/n} = 1$ and $\rho_2 = \lim_{n\to\infty} \omega(n,n)^{1/n} = \lim_{n\to\infty}(e^{|n|}+e^{-n})^{1/n} = e$.

(iii) $\mu_2 = 1$.

It is enough to prove that $\inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} = 0$. Note that, for $(m,n) \neq (0,0)$, we have

$$\frac{\eta(m,n)}{m+n} \quad = \quad \frac{\log(e^m + e^{-n})}{m+n} \leq \frac{\log(e^m+e^m)}{m+n} \leq \frac{\log 2 + m}{m+n}$$

Hence, $0 \leq \inf\{\frac{\eta(m,n)}{m+n} : (m,n) \neq (0,0)\} \leq \liminf_{n\to\infty} \frac{\log 2+m}{m+n} = 0$. Thus $\mu_2 = 1$.

(iv) $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

Here, $\beta_{1,0} = e$ and $\beta_{2,0} = 1$. Now let $(z,w) \in \mathbb{D}_e \times \mathbb{D}_1$ and $f \in l^1(\mathbb{Z}_+^2, \omega)$, $m,n \in \mathbb{Z}_+$. Then

$$|z|^m |w|^n \leq e^m \leq (e^m + e^{-n}) = \omega(m,n).$$

This implies that $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

(v) $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$;

Since $\alpha_1 = \alpha_{1,0} = e^{-1}, \beta_1 = \beta_{1,0} = e$ and $\alpha_2 = \alpha_{2,0} = \beta_2 = \beta_{2,0} = 1$. By Theorem 3.5(i), we have $\Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(1) \subset \mathbb{T}(\omega)$.

Let $\frac{1}{e} \leq |z| \leq e$ and $|w| = 1$. Then for $m,n \in \mathbb{Z}$, note that

$$|z|^m |w|^n \leq |z|^m \leq e^{|m|} \leq e^{|m|} + e^{-n} = \omega(m,n).$$

Hence $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$.

Example 4. Let $\omega(m,n) = e^{|m|} + |n|$ be a weight on $\mathbb{Z}^2$. Then

(i) $\alpha_1 = \alpha_{1,0} = \rho_1 = e^{-1}, \alpha_2 = \alpha_{2,0} = 1$;

(ii) $\beta_1 = \beta_{1,0} = \rho_2 = e, \beta_2 = \beta_{2,0} = 1$;

(iii) $\mu_2 = 1$;

(iv) The Gel'fand set $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$. i.e $\mathbb{D}_{\mu_2} \times \mathbb{D}_{\mu_2} \subsetneq \mathbb{G}(\omega) = \mathbb{D}_{\beta_{1,0}} \times \mathbb{D}_{\beta_{2,0}}$.

(v) The Gel'fand set $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$. i.e. $\Gamma(\sqrt{\alpha_1}, \sqrt{\beta_1}) \times \Gamma(\sqrt{\alpha_2}, \sqrt{\beta_2}) \subsetneq \mathbb{T}(\omega) = \Gamma(\alpha_{1,0}, \beta_{1,0}) \times \Gamma(\alpha_{2,0}, \beta_{2,0})$.

Solution. By the arguments given in Example 3 (i), (ii), and (iii), the first three statements can be proved easily.

(iv) Let $(z,w) \in \mathbb{D}_e \times \mathbb{D}_1$ and $m,n \in \mathbb{Z}_+$. Then

$$|z|^m |w|^n \leq e^m \leq (e^m + n) = \omega(m,n).$$

This implies that $\mathbb{G}(\omega) = \mathbb{D}_e \times \mathbb{D}_1$.

(v) Let $(z,w) \in \Gamma(\frac{1}{e}, e) \times \Gamma(1)$ and $m,n \in \mathbb{Z}$. Then

$$|z|^m |w|^n \leq e^{|m|} \leq (e^{|m|} + |n|) = \omega(m,n).$$

Hence $\mathbb{T}(\omega) = \Gamma(\frac{1}{e}, e) \times \Gamma(1)$.

Acknowledgement

The second author is thankful to CSIR-HRDG for providing Junior Research Fellowship.

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Accepted: October 19, 2020