Stability aspects for linear Hamiltonian systems and its applications in the three body problem

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Abstract. This paper concerns stability and strong stability of the Lagrange equilateral solutions in the three body problem. For given three masses, there exists a family of periodic solutions for which each mass is at the vertex of an equilateral triangle and traveling along an elliptic Kepler orbit. By means of symmetry, we reduce the dimensions of our problem from 12 to 6 dimensions. After making a clever change of coordinates a 4 dimensional system is obtained. We show that this system is G-Hamiltonian where G can be found by the restriction of an invariant subspace to the linear system. We study the bifurcation of this system as parameter within the system are changed. In particular, we prove that the strong stability of this system at $e > 0$ is lost in two ways, through period doubling bifurcation, and through two $+1$ eigenvalues. We also prove that stability and instability regions occur in the circular case ($e = 0$) as the determinant of the characteristic polynomial of the system changes.
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1. Introduction

In this paper, some basic definitions and propositions from the Theory of Linear transformation finite dimensional spaces with indefinite metric are presented. An interesting example comes from Celestial Mechanics and the lagrangian equilateral solutions of the three body problem is discussed and stability type is investigated. The system we are considering is nonlinear, and the linearization of the vector field along the periodic solution gives rise to a linear system of Hamiltonian equations. In the circular case $e = 0$, this leads to an investigation of an equilibrium system of linear equations where as in the elliptic case $e > 0$, the linearized system is of time periodic type.

In this example, we focus on an analytic argument for stability and strong stability of the Lagrange equilateral triangle solutions in the three body problem. This is challenging since strong stability is not always preserved under

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parameter changes in this example. Moreover, the analytic calculation of the characteristic multipliers for the elliptic orbits requires computing the fundamental matrix solutions to the associated time-dependant linear system. This is difficult in general and typically requires the use of numerical methods. However, our techniques allow us to pinpoint the likely bifurcations which may occur in numerical simulation. Stability of the restricted three body problem was studied by Danby in [6] where numerical integration has been used throughtout. He used the mass value μ and the eccentricity e as parameters to acheive the stability type. In fact, he proved that the elliptic orbits are linearly stable while the circular solutions are not. Roberts in [3, 4] has used similar techniques to prove linear stability in the three body problem. In [9, 11], the authors studied minimization of differeent types of Linear Hamiltonian systems which can be used to study the question of stability or instability of some periodic solutions. The authors in [7, 8, 10, 12] have discussed the Instabilty of symmetric periodic family of orbits together with techniques from symplectic geometry and in particular the Maslov index along with the minimiztion properties applied to preclude the existence of eigenvalues of the monodromy matrix on the unit circle. In this paper, we show that the four dimensional system is G-Hamiltonian where the matrix G will be found by the restriction of the symplectic form to an invariant subspace of the linear system. We then analyze the behavior of the roots of the characteristic polynomial of the monodromy matrix. We show that the strong stability of this system is lost in three ways, through period doubling bifurcation, through two +1 eigenvalues and through Krein collision. Our arguments give an analytical proof of the absence of strong stability in the first two cases when $e > 0$.

We also analyze the behavior of the roots of the characteristic polynomial of the G-Hamiltonian system and how they vary with the eccentricity e and its determinant. We show that as the determinant of the characteristic polynomial increases from 0 to $\frac{1}{4}$ at $e = 0$, the eigenvalues remain on the imaginary axis leading to stability. We prove this analytically by showing these roots are purely imaginary. Instability is also occurs at $e = 0$ as the determinant of the characteristic polynomial passes through krein collision. We prove this analytically by showing the real part of these eigenvalues are non-zero.

The rest of the paper is organized as follows: In section 2, we present some propositions from the Theory of linear transformation of finite dimensional spaces with Indefinite metric. In section 3, we first reduce the dimensions of our problem to four degrees of freedomsystem and then change of coordinates will decouple our system into a 2 by 2 system and a 4 by 4 system. In section 4, a result about stability and strong stability of our decoupled system is discussed.

2. Some propositions from the theory of linear transformation of finite dimensional spaces with indefinite metric

In this section we present some basic definitions that are used frequently throughout this paper. Let G be a nonsingular Hermitian matrix (i.e $detG \neq 0, G^* = G$) which is neither positive nor negative definite. the matrix G in many applications is equal to iJ, where i is the imaginary unit and J has the form

$$\mathbf{J} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}. \tag{1}$$

Throughout this paper, we assume that the matrix G is Hermitian. An important special case occurs when $G = iJ$, which is connected with the linear Hamiltonian equation

$$\frac{dx}{dt} = JH(t)x, \tag{2}$$

where x is a $2n$ column vector, $H(t)$ is the $n \times n$ matrix, and the matrix J is given in (1). Below we will show that the number $\langle e, e\rangle$ is real but unlike (e, e) it need not be positive, hence $\langle e, d\rangle$ is called an *indefinite scalar product.* Given a matrix $X \in \mathbb{C}^{n\times n}$, the matrix X^G defined by $\langle Xe, d\rangle = \langle e, X^G d\rangle$, (e, d) are arbitrary vectors in $\mathbb{C}^{n\times n}$ is said to be G-adjoint to X. The explicit formula for X^G is given by $X^G = G^{-1}X^*G$.

From here we consider the following definition:

Definition 2.1. *A matrix $X \in \mathbb{C}^{n\times n}$ is called G-unitary if it satisfies the relation $X^G X = I$.*

Definition 2.2. *A matrix $Y \in \mathbb{C}^{n\times n}$ is called G-Hamiltonian (or G-antihermitian) if it satisfies the relation $Y^G = -Y$.*

Each G-Hamiltonian Y has the representation $Y = i^{-1}GH$ with some $H = H^*$. For a G-Unitary matrix, we have $X = G^{-1}(X^{-1})^*G$, the set of all multipliers of X is symmetric with respect to the unit circle.

The following proposition, is very important for what follows and its proof is straightforward and left to the reader:

Proposition 2.1. *The matrizant $X(t)$ of a Hamiltonian equation* (2) *is a G-unitary matrix for any t with $G = iJ$.*

3. The Lagrangian equilateral triangle solutions and elimination of dimensions

We let the mass and position of the N bodies be given by m_i and $q_i \in \mathbb{R}^2$, respectively. Interaction between the masses is determined by the Newtonian potential function

$$U(q) = \sum_{i<j} \frac{m_i m_j}{\|q_j - q_i\|}. \tag{3}$$

on the set of non-collision configurations (where $q_i \neq q_j$, $i \neq j$). The Hamiltonian for the N-body problem is given as

$$H = \sum_{i}^{N} \frac{1}{2m_i} \|p_i\|^2 - U(q). \tag{4}$$

The Hamiltonian equations of the N-body problem are

$$\dot{q}_i = \frac{1}{m_i} p_i, \qquad \dot{p}_i = \frac{\partial U(q)}{\partial q_i}, \qquad i = 1, 2, ..., N. \tag{5}$$

These equations are equivalent to the second-order differential equation system

$$m_i \ddot{q}_i = \sum_{i \neq j} \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} = \frac{\partial U}{\partial q_i}. \tag{6}$$

We are looking for solutions of the form

$$q_i(t) = \phi(t) a_i, \tag{7}$$

where a_i is a constant vector and $\phi(t)$ is a scalar function. Identify $\mathbb{R}^2$ with the complex plane by considering the $q_i, a_i, \phi(t)$ as a complex numbers. Substituting this into equation (6),we get

$$m_i \ddot{\phi} a_i = \sum_{i \neq j} \frac{m_i m_j (\phi a_j - \phi a_i)}{\|\phi a_j - \phi a_i\|^3} = \sum_{i \neq j} \frac{m_i m_j (a_j - a_i) \phi}{\|a_j - a_i\|^3 |\phi|^3},$$

where this equation is solvable up to quadrature ([2], page 100). In polar coordinates (r, θ), the solution with $\lambda = 1$ is given by

$$r(t) = \frac{\omega^2}{1 + e \cos \theta(t)}, \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0, \tag{8}$$

where ω, the angular momentum, and e the eccentricity of the ellipse, are two parameters.

Remark. A periodic solution of the planar N-body problem has 8 trivial characteristic multipliers of value +1. The solution is linearly stable if the remaining multipliers lie on the unit circle, the monodromy matrix $X(T)$, the matrix satisfying $X(t + T) = X(t)M$ (for example see [5]) restricted to the reduced space is diagonalizable, and strongly stable, if in addition, the multipliers of $X(T)$ are of definite kind.

3.1 Eliminating the standard integrals

We follow the discussion in Roberts [3] to reduce the system. We let the momenta of each body be $p_i = m_i \dot{q}_i$ and let $p = (p_1, p_2, p_3) \in \mathbb{R}^6$. We now start reducing the dimensions of our problem by using the following general fact.

For canonical (symplectic) transformations, the transformed equations of motion are Hamiltonian, with the new Hamiltonian obtained by substitution with the new coordinates.

Using Jacobi coordinates (see [2]), we can eliminate the center of mass and total linear momentum by setting

$$
\begin{aligned}
&u_1 = q_2 - q_1, \qquad v_1 = -\frac{m_2}{m_1 + m_2} p_1, \\
&u_2 = q_3 - \frac{1}{m_1 + m_2}(m_1 q_1 + m_2 q_2), \qquad v_2 = -\frac{m_3}{M}(p_1 + p_2 + p_3) + p_3, \\
&u_3 = \frac{1}{M}(m_1 q_1 + m_2 q_2 + m_3 q_3), \qquad v_3 = p_1 + p_2 + p_3.
\end{aligned}
$$

Using $u_3 = v_3 = 0$, the new Hamiltonian will has the following form

$$
H_1(u_1, u_2, v_1, v_2) = \frac{\|v_1\|^2}{2M_1} + \frac{\|v_2\|^2}{2M_2} - \frac{m_1 m_2}{\|u_1\|} - \frac{m_1 m_3}{\|u_2 + M_3 u_1\|} - \frac{m_2 m_3}{\|u_2 + M_4 u_1\|}.
$$

where

$$
\begin{aligned}
M_1 &= \frac{m_1 m_2}{m_1 + m_2}, \qquad M_2 = \frac{m_3(m_1 + m_2)}{M}, \\
M_3 &= \frac{m_2}{m_1 + m_2}, \qquad M_4 = -\frac{m_1}{m_1 + m_2}.
\end{aligned}
$$

This Hamiltonian is independent of u_3, v_3 that is $u_3 = v_3 = 0$. This reduces the dimension by 4 from 12 to 8.

Next, using polar coordinates (for example see [2]), we can eliminate the integrals due to the angular momentum and rotational symmetry. The new Hamiltonian will has the form

$$
H_2 = \frac{1}{2M_1}(R_1^2 + \frac{\omega_1^2}{r_1^2}) + \frac{1}{2M_2}(R_2^2 + \frac{\omega_2^2}{r_2^2}) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2}
$$

where

$$
\begin{aligned}
\delta_1 &= \sqrt{r_2^2 + M_3^2 r_1^2 + 2 M_3 r_1 r_2 \cos(\theta_2 - \theta_1)}, \\
\delta_2 &= \sqrt{r_2^2 + M_4^2 r_1^2 + 2 M_4 r_1 r_2 \cos(\theta_2 - \theta_1)}.
\end{aligned}
$$

Now, making a final change of coordinates by setting

$$
\omega_1 + \omega_2 = c, \qquad \theta_2 - \theta_1 = \phi, \qquad \omega_2 = \Phi.
$$

The new Hamiltonian will be independent of θ_1. Substituting this into the Hamiltonian H_2 yields

$$H = \frac{1}{2M_1}(R_1^2 + \frac{(c-\Phi)^2}{r_1^2}) + \frac{1}{2M_2}(R_2^2 + \frac{\Phi^2}{r_2^2}) - \frac{m_1 m_2}{r_1} - \frac{m_1 m_3}{\delta_1} - \frac{m_2 m_3}{\delta_2}$$

This reduces the system to six dimensions, in the variables $(r_1, r_2, \phi, R_1, R_2, \Phi)$. The equations of motion in these new variables are

$$\dot{r_1} = \frac{R_1}{M_1}, \dot{r_2} = \frac{R_2}{M_2}, \dot{\phi} = \frac{\Phi - c}{M_1 r_1^2} + \frac{\Phi}{M_2 r_2^2}, \dot{R_1} = \frac{(\Phi - c)^2}{M_1 r_1^3} - s_1,$$
$$\dot{R_2} = \frac{\Phi^2}{M_2 r_2^3} - s_2, \dot{\Phi} = m_3 r_1 r_2 \sin\phi s_3$$

where

$$s_1 = \frac{m_1 m_2}{r_1^2} + \frac{m_1 m_3 M_3 (r_1 M_3 + r_2 \cos\phi)}{\delta_1^3} + \frac{m_2 m_3 M_4 (r_1 M_4 + r_2 \cos\phi)}{\delta_2^3},$$
$$s_2 = \frac{m_1 m_3 (r_2 + r_1 M_3 \cos\phi)}{\delta_1^3} + \frac{m_2 m_3 (r_2 + r_1 M_4 \cos\phi)}{\delta_2^3}, \; s_3 = \frac{m_1 M_3}{\delta_1^3} + \frac{m_2 M_4}{\delta_2^3}.$$

The Kepler periodic solution, denoted in general by $\gamma(t)$, can be obtained using a simple calculation (see [3]) as

$$\begin{aligned} &r_1(t) = A_1 r(t), \quad R_1(t) = M_1 A_1 R(t), \quad r_2(t) = A_2 r(t), \\ &R_2(t) = M_2 A_2 R(t), \quad \phi = \bar{\theta_3} + \eta, \quad \Phi = \omega M_2 A_2^2, \end{aligned} \tag{9}$$

where $A_1 = M^{1/3}$, $A_2 = \bar{r_3} M/(m_1 + m_2)$, and η is the angle between the vectors $a_2 - a_1$ and the positive horizontal axis. The total angular momentum for the full problem has the value $c = \omega(M_1 A_1^2 + M_2 A_2^2)$.

In order to discuss linear stability of the Kepler solutions $\gamma(t)$ given by (9), we will linearize the reduced Hamiltonian equation along the Kepler solution, and then decouple the system before we discuss the fundamental solution, and its monodromy matrix.

Lineariziation of the six-dimensional system about the periodic solution $\gamma(t)$ leads to the time-dependant periodic linear Hamiltonian system which has the following form

$$\dot{X}(t) = J_3 D^2 H(\gamma(t)) X = J_3 A(t) X.$$

Using the solution described in (9), and after a simple calculation and simplification, we get

$$J_3\mathbf{A}(\mathbf{t}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{M_2} & 0 \\ \frac{2\omega}{A_1 r^3} & \frac{-2\omega}{A_2 r^3} & 0 & 0 & 0 & \frac{c}{\omega M_1 M_2 A_1^2 A_2^2 r^2} \\ a_{11} & a_{12} & a_{13} & 0 & 0 & \frac{-2\omega}{A_1 r^3} \\ a_{12} & a_{22} & \frac{-A_1}{A_2} a_{13} & 0 & 0 & \frac{2\omega}{A_2 r^3} \\ a_{13} & \frac{-A_1}{A_2} a_{13} & a_{33} & 0 & 0 & 0 \end{pmatrix}$$

where

$$a_{12} = \frac{9m_1m_2m_3}{4M\sqrt{m_1^2+m_1m_2+m_2^2}}.\frac{1}{r^3}, \qquad a_{13} = \frac{3\sqrt{3}m_1m_2m_3(m_1-m_2)}{4M^{2/3}(m_1+m_2)^2}.\frac{1}{r^2}$$

and

$$a_{33} = \frac{9m_1m_2m_3}{4M^{1/3}(m_1+m_2)}.\frac{1}{r}.$$

3.2 Decoupling the linear system

A linear time-dependant periodic Hamiltonian system is one of the form

$$\dot{X}(t) = JD^2H(\gamma(t))X, \tag{10}$$

where J is the standard symplectic matrix, and $D^2H(t+T) = D^2H(t)$.

Define the skew-inner product of two vectors $v, w \in \mathbb{C}^{4n}$ as $\Omega(v,w) = v^TJw$. A key trait of linear Hamiltonian systems is that the skew-orthogonal complement of an invariant subspace is also invariant, where the skew-orthogonal complement is defined as $W^\perp = \{v \in \mathbb{C}^{4n} : \Omega(v,w) = 0, \forall w \in W\}$.

Lemma 3.1. *Suppose W is an invariant subspace of the matrix $B = JD^2H(\gamma(t))$ in system* (10)*, then the skew-orthogonal complement $W^\perp$ of W is also an invariant subspace of the matrix B in* (10)*.*

The above lemma shows that a simple linear change of variables will decouple the system. The characteristic multipliers remain the same since the transformation is linear. We need to find an invariant subspace for $J_3A(t)$, to apply the same idea. We make use of the fact that the Kepler periodic solution $r(t)$ satisfies

$$\ddot{r}(t) = \frac{\omega^2}{r^3} - \frac{1}{r^2}, \qquad \dddot{r}(t) = (-\frac{3\omega^2}{r^4} + \frac{2}{r^3})\dot{r}. \tag{11}$$

From these we have

$$\gamma = \begin{pmatrix} A_1r(t) \\ A_2r(t) \\ \bar{\theta}_3+\eta \\ M_1A_1\dot{r}(t) \\ M_2A_2\dot{r}(t) \\ \omega M_2A_2^2 \end{pmatrix}, \qquad \dot{\gamma} = \begin{pmatrix} A_1\dot{r} \\ A_2\dot{r} \\ 0 \\ M_1A_1\ddot{r} \\ M_2A_2\ddot{r} \\ 0 \end{pmatrix}, \qquad \ddot{\gamma} = \begin{pmatrix} A_1\ddot{r} \\ A_2\ddot{r} \\ 0 \\ M_1A_1(-\frac{3\omega^2}{r^4}+\frac{2}{r^3})\dot{r} \\ M_2A_2(-\frac{3\omega^2}{r^4}+\frac{2}{r^3})\dot{r} \\ 0 \end{pmatrix}$$

as expressions for the first and second derivatives of the periodic orbit. A short calculation gives

$$J_3A(t)\dot{\gamma}(t) = \ddot{\gamma}(t), \qquad J_3A(t)\ddot{\gamma}(t) = (-\frac{3\omega^2}{r^4} + \frac{2}{r^3})\dot{\gamma}(t).$$

Then the vectors $w_1 := [A_1, A_2, 0, 0, 0, 0]$, $w_2 := [0, 0, 0, M_1A_1, M_2A_2, 0]$ will span an invariant subspace W for $J_3A(t)$. This is the basis for an application of lemma (3.1).

Consider the change of variables determined by

$$\begin{pmatrix} r_1 \\ r_2 \\ \phi \\ R_1 \\ R_2 \\ \Phi \end{pmatrix} = \begin{pmatrix} A_1 & 0 & \frac{\omega A_2 M_2}{c} & 0 & 0 & 0 \\ A_2 & 0 & -\frac{\omega A_1 M_1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & M_1A_1 & 0 & \frac{\omega A_2}{c} & 0 & 0 \\ 0 & M_2A_2 & 0 & -\frac{\omega A_1}{c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \chi \\ y \\ Y \\ z \\ Z \end{pmatrix}. \tag{12}$$

(Recall that $\frac{\omega}{c}(M_1A_1^2 + M_2A_2^2) = 1$). The linear transformation matrix is non-singular, because the determinant is non-zero. The columns

$$w_3 = [\frac{\omega A_2 M_2}{c}, \frac{-\omega A_1 M_1}{c}, 0, 0, 0, 0],$$

$w_4 = [0, 0, 0, \frac{\omega A_2}{c}, -\frac{\omega A_1}{c}, 0]$, $w_5 = [0, 0, 1, 0, 0, 0]$ and $w_6 = [0, 0, 0, 0, 0, 1]$ of the above matrix are chosen to form a basis for $W^{\perp}$. Consequently, this change of variables will decouple our linear system into a 2×2 and a 4×4 system. After computing the restriction of $J_3A(t)$ to the space spanned by four columns of the matrix in system (12), we get the new differential equation system

$$\begin{pmatrix} \dot{y} \\ \dot{Y} \\ \dot{z} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{M_1M_2} & 0 & 0 \\ M_1M_2(\frac{-3w^2}{r^4} + \frac{2}{r^3}) - \frac{a_{12}c}{wA_1A_2} & 0 & \frac{a_{13}c}{wA_2} & -\frac{2c}{A_1A_2r^3} \\ \frac{2w}{A_1A_2r^3} & 0 & 0 & d \\ \frac{a_{13}}{A_2} & 0 & a_{33} & 0 \end{pmatrix} \begin{pmatrix} y \\ Y \\ z \\ Z \end{pmatrix}. \tag{13}$$

An important observation her is that this system (13) is no longer Hamiltonian (due to the fact that the transformation (12) is not canonical). We denote the 4×4 system in equation (13) by the matrix B where all the parameters are described in section (3.1). To study the stability properties of this system, we first show that the system is G-Hamiltonian where the matrix G will be found by the restriction of W to the invariant subspace $W^{\perp}$. We proceed by considering this calculation, the skew symmetric form can be determined from the following:

$$\Omega(v, w) = v^T G w$$

where $v = (y, Y, z, Z)^T$, $w = (y, Y, z, Z)$. After a good calculations, it is clear that G has the form

$$\mathbf{G} = \begin{pmatrix} 0 & \frac{w}{c} & 0 & 0 \\ \frac{-w}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{14}$$

where iG is Hermitian, non-singular and symplectic with multiplier $\frac{w}{c}$, $\frac{w}{c} > 0$.

The following lemma states the G-Hamiltonian structure of the matrix B in equation (13). The explicit formula for B^G is given by $B^G = G^{-1}B^*G$ so that

$$(15)\, B^G = \begin{pmatrix} 0 & \frac{-1}{M_1M_2} & 0 & 0 \\ -M_1M_2(\frac{-3w^2}{r^4} + \frac{2}{r^3}) + \frac{a_{12}c}{wA_1A_2} & 0 & \frac{-a_{13}c}{wA_2} & \frac{2c}{A_1A_2r^3} \\ \frac{-2w}{A_1A_2r^3} & 0 & 0 & -d \\ \frac{-a_{13}}{A_2} & 0 & -a_{33} & 0 \end{pmatrix}.$$

Theorem 3.2. *The 4×4 system B given in equation (13) is G-Hamiltonian.*

Proof. In order to prove this theorem, we need to check only that $B^G = -B$. This is clear by comparing (15) with (13). □

4. Stability and strong stability analysis

The characterization of linear stability of the elliptic kepler orbits through system (13) was considered by Roberts in 2000 without consideration of strong stability [3].

Now, we wish to study the strong stability of the elliptic orbits through the same system. This requires that all the roots of the characteristic polynomial of the monodromy matrix must lie on the unit circle, and in addition, all roots of the characteristic polynomial must be of one kind. The monodromy matrix is denoted here by M, the eigenvalues of M are the characteristic multipliers.

The characteristic polynomial of M is reciprocal and has the form

$$P(\lambda) = \det(\lambda I - M) = \lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1 \tag{16}$$

where

$$a = -tr(M), \qquad b = \frac{1}{2}((tr(M))^2 - tr(M^2)), \qquad \det M = 1.$$

In order to have strong stability, we require that there are no G-isotropic eigenvectors. This simple remark leads us to a characterization of strong stability of the monodromy matrix M. There are three ways, in which strong stability can be lost:

- Periodic doubling bifurcation (two -1 eigenvalues), occurring when $b = 2a - 2$.
- Two +1 eigenvalues, occurring when $b = -(2a + 2)$.
- Krein collision, occurring when $b = \frac{a^2}{4} + 2$.

We now consider the first two cases in the following theorems:

Theorem 4.1. *Consider the characteristic polynomial (16) in the case where $b = 2a - 2$. There are two -1 eigenvalues occurring in this case, which are indefinite and therefore not strongly stable.*

Proof. Since $\lambda = -1$ is real, then the root subspace $\mathcal{L}_\lambda$ is also real and even dimensional. We define the Hermitian form to be $f(e) = (iGe, e)$. Note that, since the imaginary part of $f(e)$ is zero, then $(iG\bar{e}, \bar{e}) = -(iGe, e)$.

Now, we can consider in the root subspace $\mathcal{L}_\lambda$ iG-orthogonal basis of $2m$ vectors $e_1, e_2, ..., e_{2m}$, chosen and normalized by the condition

$$(iGe_i, e_j) = 0 \qquad \forall i \neq j, \quad (iGe_i, e_i) = \pm 1 \qquad i = 1, 2, ..., 2m.$$

Then the vectors $\bar{e_1}, \bar{e_2}, ..., \bar{e_{2m}}$ form iG-orthogonal basis in the corresponding root subspace $\mathcal{L}_\lambda$. Moreover, the sign of the Hermitian form has been changed, that is $(iG\bar{e_i}, \bar{e_i}) = -(iGe_i, e_i)$ which implies that the signature of $f(e)$ is zero, where the signature of $f(e)$ is defined to be the difference between the positive and negative eigenvalues of the Hermitian form. Therefore the number of positive and negative entries of the Hermitian form are the same. Hence $\lambda = -1$ is indefinite eigenvalue and thus the system is not strongly stable. □

Theorem 4.2. *Consider the characteristic polynomial* (16) *in the case where* $b = -(2a + 2)$. *There are two* $+1$ *eigenvalues occurring in this case, which are indefinite and therefore not strongly stable.*

Proof of this theorem is similar to that of theorem (4.1)

We are also interested to study the Krein collision case, which occurs when $b = a^2/4 + 2$. It is not easy in general to show these Krein collision eigenvalues are of different kind. Henceforth, we will restrict this discussion to the special case when $e = 0$. We study the stability properties of the uniformly rotating Lagrangian solutions when the eccentricity of the periodic orbit is zero. The important observation here is that the matrix B is time independent when $e = 0$. Stability then requires that all the roots of the characteristic polynomial of the G-Hamiltonian matrix B in system (13) must lie on the imaginary axis.

The characteristic polynomial of B is symmetric and has the form

$$P(\lambda) = \lambda^4 + b\lambda^2 + \det B \tag{17}$$

where

$$b = 1/2((tr(B))^2 - tr(B^2)). \tag{18}$$

It is furthermore easy to see that the trace of B is zero and the trace of B^2 is -2. Substituting this in equation (18), we get $b = 1$. The characteristic polynomial (17) can then be written as

$$\lambda^4 + \lambda^2 + \det B. \tag{19}$$

It has the roots

$$\lambda^2 = \frac{1}{2}[-1 \pm \sqrt{1 - 4\det B}.] \tag{20}$$

We follow the behavior of the eigenvalues of (19) as $\det B$ varies. In order to have stability, the roots (20) must be purely imaginary. It is easy to see that this occurs only when $0 \le \det B \le \frac{1}{4}$.

For simplicity, Let $q(\rho) = \rho^2 + \rho + \det B$ and denote the two roots of $q(\rho)$ as ρ_1 and ρ_2 where $-\pi^2 < \rho_1 \le \rho_2 < 0$. As $\det B$ increases from 0 to 1/4, ρ_1 increases from -1 to $-\frac{1}{2}$ and ρ_2 decreases from 0 to $-\frac{1}{2}$. In the interval $(0, \frac{1}{4})$, the multipliers come in the pairs $\lambda_1 = \pm\sqrt{\rho_1}$, $\lambda_2 = \pm\sqrt{\rho_2}$ where ρ_1 and ρ_2 are given in equation (20).

When the value of $\det B = \frac{1}{4}$ in equation (20), we have the double eigenvalues $\pm i\frac{\sqrt{2}}{2}$. In other words, as $\det B$ increases from 0 to $\frac{1}{4}$, two pairs of eigenvalues meet on the imaginary axis at $\pm i\frac{\sqrt{2}}{2}$. For values of $\det B > \frac{1}{4}$, the characteristic polynomial (19) has four complex roots and thus there will be a complex quartet of eigenvalues $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ leading to instability. Since the quartet of eigenvalues move off the imaginary axis and contravene the stability condition, then the system becomes unstable.

We visualize the result of the above argument in Figure 1.

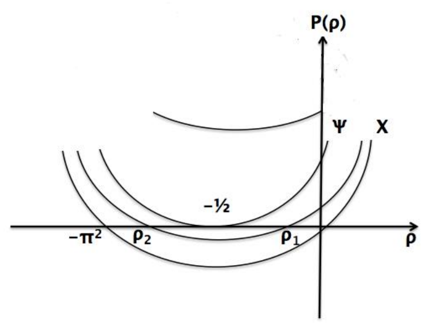

Figure 1: Stability Region for the Case $e = 0$

Beginning on the parabola curve X (the first curve from the bottom) corresponding to $e = 0$, when $\det B$ passes the point zero, the calculations in the above argument show that the curve moves up to intersect the real line at ρ_1 and ρ_2 where $-\pi^2 < \rho_1 \le \rho_2 < 0$ (the second curve from the bottom), hence this pushes us into the stability region. For $\det B = \frac{1}{4}$, the curve moves up to intersect the real line at $\rho_1 = \rho_2 = -\frac{1}{2}$, which is the point of Krein collision. In contrast as $\det B$ increases form $\frac{1}{4}$, the square root in (20) produces imaginary values, and so λ will be complex with a non-zero real part. We prove that easily by assuming $\lambda = A + iB$ in the above argument, and

$$\lambda^2 = \rho, \tag{21}$$

where $\rho = \frac{-1 \pm i\sqrt{4\det B - 1}}{2}$ the root of $q(\rho)$. Since $\det B > \frac{1}{4}$, the imaginary part of ρ never can be zero, it follows that the real part A of λ is also non-zero. This is clear by comparing both sides of equation (21). The eigenvalues of (20) lie off the imaginary axis and the system can't be stable. Thus, we have proved that for $\det B > \frac{1}{4}$, stability is immediately lost for $e = 0$ while for $0 < \det B < \frac{1}{4}$, the system is strongly stable.

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