A note on conjugacy degrees of a finite group

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Abstract. Let G be a group, $g_1, g_2, \ldots, g_k$ are a complete set of representatives for the conjugacy classes of G. Then

$$k(G) = \frac{1}{|G|^2}\sum_{i=1}^{k}|(g_i)^G|^2 = \sum_{i=1}^{k}\frac{1}{|C_G(g_i)|^2}$$

is called the conjugate degree of G (see [6]). In this short paper, we investigate the influence of the conjugacy degree of G on the structure of finite groups. The authors get the formulas of the conjugacy degree of the dihedral group and the generalized quaternion group and classify those groups G such that $k(G) \geq \frac{1}{3}$.

Keywords: conjugacy degree, dihedral group, generalized quaternion group.

1. Introduction

All groups considered in this paper are finite and G always denotes a group. Let $\pi(|G|)$ stand for the set of all prime divisors of the order of G. The symbol $[A]B$ denotes the semidirect product of the groups A and B, where B is an operator group of A. Let C_n be the cyclic group of order n, where n is a positive integer. The other notions and notations are standard, as in [1].

In the past few years, there has been a growing interest in the application of probability in finite group theory. One of the most important aspects that have

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been studied is the probability that two elements of a finite group G commute. In 1979, D.J. Rusin [2] introduced the commutativity degree of G denoted by $d(G)$.

$$(*)\qquad d(G) = \frac{1}{|G|^2}|\{(x,y) \in (G,G)|xy = yx\}|.$$

Evidently, G is an abelian group if and only if $d(G) = 1$, and the author also get, if $\frac{1}{2} < d(G) < 1$, then $d(G) \in \{\frac{1}{2}(1+\frac{1}{4^n})|n \in \mathbf{N}, n \geq 1\}$. Later, P. Lescot continues to research on this topic, and get the following results. If G is non abelian, then $d(G) \leq \frac{5}{8}$; if $d(G) > \frac{1}{2}$, then G is nilpotent (see [5]). If $d(G) = \frac{1}{2}$, and G is not nilpotent, then $G/Z(G) \cong \sum_3$ and $G' \cong Z_3$ (see [4]).

As we know, the elements of any group may be partitioned into conjugacy classes, so the conjugacy relation plays an important role in group theory. In [6], S. Blackburn, J. Britnell and M. Wildon replace 'community of elements' with 'conjugation of elements', introduce the concept of the probability that a pair of elements of a finite group are conjugate in 2012, we call it conjugate degree of G, denoted by

$$k(G) = \frac{1}{|G|^2}|\{(x,y) \in (G,G)|x \sim y\}| = \frac{1}{|G|^2}\sum_{i=1}^{k}|(g_i)^G|^2 = \sum_{i=1}^{k}\frac{1}{|C_G(g_i)|^2},$$

where $\sim$ is the conjugacy relation, $g_1, g_2, \ldots, g_k$ are a complete set of representatives for the conjugacy classes of G. Using this concept, the authors investigate the structure of finite groups, and show that G is abelian whenever $k(G)|G| < \frac{7}{4}$. Specializing to the symmetric group S_n, they show that $k(S_n) \leq C/n^2$ for an explicitly determined constant C. This bound leads to an elementary proof of a result of Flajolet et al. (see [3]), that $k(S_n) \sim A/n^2$ as $n \to \infty$ for some constant A. In this paper, we continue to research on this topic. We investigate the influence of the conjugacy degree of G on the structure of finite groups, and get the formulas of the conjugacy degree of the dihedral group and the generalized quaternion group and classify those groups G such that $k(G) \geq \frac{1}{3}$.

2. Preliminaries

For the same of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([6, Proposition 4.2])**.** *Suppose that G is a finite group and that $t \in G$ is a self-centralizing involution. Then $k(G) = 1/4 + 1/|G| - 1/|G|^2$ and G has a normal abelian subgroup A of odd order such that $|G : A| = 2$.*

To state our following lemma, we shall need the majorization (or dominance) order, denoted by $\succcurlyeq$, which is defined on R^k by setting

$$(x_1, x_2, \ldots, x_k) \succcurlyeq (y_1, y_2, \ldots, y_k),$$

if and only if $\sum_{i=1}^{j} x_i \geq \sum_{i=1}^{j} y_i$ for all j such that $1 \leq j \leq k$.

Lemma 2.2 ([6, Lemma 3.3]). *Let x, $y \in R^k$ be decreasing k-tuples of real numbers such that $\sum_{i=1}^k x_i = \sum_{i=1}^k y_i = 1$. Suppose that $x \succcurlyeq y$. Then $\sum_{i=1}^k x_i^2 \geq \sum_{i=1}^l y_i^2$, and equality holds if and only if $x = y$.*

3. Main results

The dihedral group D_{2n} $(n \geq 2)$ is the symmetry group of a regular polygon with n sides and it has the order $2n$. The most convenient abstract description of D_{2n} is obtained by using its generators: a rotation a of order n and a reflection b of order 2. Under these notations, we have

$$D_{2n} = \langle a, b | a^n = b^2 = 1, bab = a^{-1} \rangle = [\langle a \rangle]\langle b \rangle.$$

Proposition 3.1. *The conjugacy degree $k(D_{2n})$ of the dihedral group D_{2n} is given by the following equality:*

$$k(D_{2n}) = \begin{cases} \dfrac{n^2+4n-4}{8n^2}, & n \text{ is even;} \\ \dfrac{n^2+2n-1}{4n^2}, & n \text{ is odd.} \end{cases}$$

Proof. By the definition of the dihedral group D_{2n}, we have that $D_{2n} = \{e, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$.

Case 1. n is even. Then $\frac{n}{2}$ is integer, $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$. Observing that each element of the center $Z(D_{2n})$ forms a conjugacy class containing just itself. And $\{a^i, (a^i)^{-1}\}$ forms a conjugate class, where $i < n$ and $i \neq 0, \frac{n}{2}$, so there are $\frac{n-2}{2}$ conjugate classes of this type. Since $a^n = b^2 = 1, bab = a^{-1}$, we have that $b^a = a^{-1}ba = baa = ba^2$, $(ba^2)^a = a^{-1}(ba^2)a = ba^4$, $\ldots$, $(ba^{n-2})^a = a^{-1}(ba^{n-2})a = ba^n = b$; And $(ba)^a = a^{-1}(ba)a = baaa = ba^3$, $(ba^3)^a = a^{-1}(ba^3)a = ba^5$, $\ldots$, $(ba^{n-1})^a = a^{-1}(ba^{n-1})a = ba$. So $\{b, ba^2, \ldots, ba^{n-2}\}$ forms a conjugate class and $\{ba, ba^3, \ldots, ba^{n-1}\}$ forms a conjugate class. By the definition of conjugacy degree of G, we get that $k(D_{2n}) = \frac{1+1+\frac{n-2}{2}\times 2^2 + 2\times(\frac{n}{2})^2}{(2n)^2} = \frac{n^2+4n-4}{8n^2}$.

Case 2. n is odd. By the similar argument as in Case 1, $\{e\}$ forms a conjugate class; $\{a^i, (a^i)^{-1}\}$ forms a conjugate class, where $i < n$ and $i \neq 0$, thus there are $\frac{n-1}{2}$ conjugate classes of this type; And $\{b, ba, \ldots, ba^{n-1}\}$ forms a conjugate class by n is odd; Thus by the definition of conjugacy degree of G, we get that $k(D_{2n}) = \frac{1+\frac{n-1}{2}\times 2^2 + n^2}{(2n)^2} = \frac{n^2+2n-1}{4n^2}$.

Therefore,

$$k(D_{2n}) = \begin{cases} \frac{n^2+4n-4}{8n^2}, & \text{n is even;} \\ \frac{n^2+2n-1}{4n^2}, & \text{n is odd.} \end{cases}$$

□

It is easy to check that $k(D_{2n}) = \frac{n^2+4n-4}{8n^2}$ or $\frac{n^2+2n-1}{4n^2}$, is a monotonic decreasing function. If n is even, then $\lim_{n\to\infty} k(D_{2n}) = \frac{1}{8}$, so $\frac{1}{8} < k(D_{2n}) \leq \frac{1}{4}$.

If n is odd and $n \geq 2$, then $\lim_{n\to\infty} k(D_{2n}) = \frac{1}{4}$, so $\frac{1}{4} < k(D_{2n}) \leq \frac{7}{18}$. Thus for any integer n, we have $\frac{1}{8} < k(D_{2n}) \leq \frac{7}{18}$.

We can also calculate the conjugate degree of D_{2n} by using GAP (see [7]), the following is calculation program.

```
gap > G := DihedralGroup(2n);
gap > CC := ConjugacyClasses(G);
gap > a := 0;;
gap > i := 1;;
gap > A := 0;;
gap > L := Length(CC);
gap > while   i <= L do
> A := Size(Centralizer(CC[i]));
> a := a + 1/A^2;
> Print(a," n");
> i := i + 1;
>od;
```

Some authors define generalized quaternion group to be the same as dicyclic group.

$$\langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1}\rangle,$$

for some integer $n \geq 2$. This group is denoted Q_{4n} and has order $4n$.

Proposition 3.2. *The conjugacy degree $k(Q_{4n})$ of the generalized quaternion group Q_{4n} is given by the following equality:*

$$k(Q_{4n}) = \frac{n^2 + 2n - 1}{8n^2}.$$

Proof. By the definition of the generalized quaternion group Q_{4n}, we have that $Q_{4n} = \{e, a, \ldots, a^{2n-1}, b, ab, \ldots, a^{2n-1}b\}$. By the similar argument as in Proposition 3.1, $Z(Q_{4n}) = \{e, a^n\}$, then each element of the center $Z(Q_{4n})$ forms a conjugacy class containing just itself; $\{a^i, a^{2n-i}\}$ forms a conjugate class, where $0 < i < 2n$ and $i \neq n$, so there are $\frac{2n-2}{2}$ conjugate classes of this type; Since $ab = ba^{2n-1}$, $ba = a^{2n-1}b$, we get that $b^a = a^{-1}ba = a^{2n-2}b$, $(a^{2n-2}b)^a = a^{-1}(a^{2n-2}b)a = a^{2n-4}b$, $\ldots$, $(a^2b)^a = a^{-1}(a^2b)a = b$. Thus $\{b, a^2b, \ldots, a^{2n-2}b\}$ forms a conjugate class. And $(ab)^a = a^{-1}(ab)a = a^{2n-1}b$, $(a^{2n-1}b)^a = a^{-1}(a^{2n-1}b)a = a^{2n-3}b$, $\ldots$, $(a^3b)^a = a^{-1}(a^3b)a = ab$. So $\{ab, a^3b, \ldots, a^{2n-1}b\}$ forms a conjugate class. By the definition of conjugacy degree of G, we get that $k(Q_{4n}) = \frac{1+1+\frac{2n-2}{2}\times 2^2+2\times n^2}{(4n)^2} = \frac{n^2+2n-1}{8n^2}$. $\square$

It is easy to check that $k(Q_{4n}) = \frac{n^2+2n-1}{8n^2}$ is a monotonic decreasing function. And $\lim_{n\to\infty} k(Q_{4n}) = \frac{1}{8}$, so $\frac{1}{8} < k(Q_{4n}) \leq \frac{7}{32}$ by $n \geq 2$.

Of course, we can also calculate the conjugate degree of Q_{4n} by using GAP (see [7]), the calculation program is similar to the program for calculating D_{2n}. In the calculation program of D_{2n}, we just only replace '$G :=$

DihedralGroup$(2n)$' with '$G := QuaternionGroup(4n)$', then we can get what we want.

Now, we can get the following result by using the Propositions.

Theorem 3.1. *Let G be a non-trivial finite group. Then $k(G) \geq 1/3$ if and only if G is isomorphic to one of the following groups: C_2, C_3, D_6 and D_{10}.*

Proof. Let $c_i(G)$ be the size of the ith smallest centralizer in a finite group G, m be the number of conjugacy classes of G and let

$$r(G) = (1/c_1(G), 1/c_2(G), \ldots, 1/c_m(G)).$$

Since the size of the ith largest conjugacy class of G is $|G|/c_i(G)$, we have that $\sum_{i=1}^{m} 1/c_i(G) = 1$.

If $c_1(G) > 2$, then we have that $(1/3, 1/3, 1/3) \succcurlyeq r(G)$. Hence, either $(1/3, 1/3, 1/3) = r(G)$, that is, $|G| = 3$, $G \cong C_3$ or by Lemma 2.2, we have that

$$k(G) = \sum_{i=1}^{m} 1/c_i(G)^2 < 1/3.$$

So, $c_1(G) = 2$, that is, G contains an element t such that $|C_G(t)| = 2$. Since $\langle t \rangle \leq C_G(t)$, we get that $\langle t \rangle = C_G(t)$. Thus we have that t is a self-centralizing involution. By Lemma 2.1 and the hypothesis, we get that $k(G) = 1/4 + 1/|G| - 1/|G|^2 \geq 1/3$, that is, $|G|^2 - 12|G| + 12 \leq 0$, so $6 - 2\sqrt{6} \leq |G| \leq 6 + 2\sqrt{6}$. By Lemma 2.1, we get that $2 \in \pi(|G|)$ and $2 \notin \pi(|G|/2)$, so $|G| = 2, 6, 10$. It is clear that, if $|G| = 6$, then $G \cong C_6$ or D_6. If $|G| = 10$, then $G \cong C_{10}$ or D_{10}. By Proposition 3.1, we get that $k(D_6) = 7/18 > 1/3$, $k(D_{10}) = 17/50 > 1/3$, while, $k(C_2) = 1/2$, $k(C_6) = 1/6$, $k(C_{10}) = 1/10$. By $k(G) \geq 1/3$, we get that G is isomorphic to one of C_2, D_6 and D_{10}.

The proof of Theorem is completed. □

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