# Finslerian hypersurfaces of a Finsler space with special $(\alpha, \beta)$-metric

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**Abstract.** In the present paper we have studied the Finslerian hypersurfaces of a Finsler space with the special $(\alpha, \beta)$ metric. We have examined the hypersurfaces of this special metric as a hyperplane of first, second and third kinds. Further, we the condition under which this hypersurface will be C2-like Finsler space also obtained the value of Main Scalar $I$ in two-Dimensional frame of Finsler Space.

**Keywords:** Finslerian hypersurface, $(\alpha, \beta)$ metric, Douglas metric, infinite series metric.

## 1. Introduction

In 1992 Matsumoto [5] summarised all the result of a Finsler space based on a Finsler metric $L(\alpha, \beta)$ in a differentiable manifold $M^n$ by considering L is a positively homogeneous function of degree one in two variable $\alpha$ which was a Riemannian metric and $\beta$ one-form metric on $M^n$. Further in 1998 he [7] introduced a special $(\alpha, \beta)$- metric which was known as Douglas metric and studied the conditions for some special Finsler spaces. Since the Douglas space is the generalization of Berwald space form the viewpoint of geodesic equation, so this metric is very important in the development of the Finsler geometry. Further in 2004 Lee and Park [1] introduced a r-th series $(\alpha, \beta)$-metric and by considering $r = \infty$ he defined the above metric as infinite series metric.

The concept of Finslerian hypersurface was first introduced by Matsumoto in 1985 and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further many authors studied

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these hyperplanes in different changes of the Finsler metric [2, 3, 8, 9, 10, 11, 12] and obtained different important results.

In the present paper we combine the metric Douglas metric and Infinite series metric in a special manner and introdue a generalized metric and examined the hypersurfaces of this metric as a hyperplane of first, second and third kinds. Further we obtained the condition under which this hypersurface will be C2-like Finsler space and the value of Main Scalar $I$ in two-Dimensional frame of Finsler Space.

## 2. Preliminaries

In 1998 Matsumoto introdued the concept of Douglas type metric which was defined as

$$L = \alpha + \frac{\beta^2}{\alpha}, \tag{1}$$

where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ one-form metric.

In 2004 Lee and Park [1] introduced a r-th series $(\alpha,\beta)$-metric

$$L(\alpha,\beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k, \tag{2}$$

where they assume $\alpha < \beta$. If we consider the above infinite series metric in the case of $\beta < \alpha$ and $r = \infty$ then the above metric becomes

$$L(\alpha,\beta) = \frac{\beta^2}{\alpha - \beta} \tag{3}$$

and the metric given by (3) named as special infinite series $(\alpha,\beta)$-metric.

In the present paper we combine the metric (1) and (3) in a special manner and introdue a new combination of metric in following way

$$L(\alpha,\beta) = A_1(\alpha + \frac{\beta^2}{\alpha}) + A_2(\frac{\beta^2}{\alpha - \beta}), \tag{4}$$

where $A_1$ and $A_2$ are any constant. If $A_1 = 0$ then the above metric becomes special infinite series metric also if $A_2 = 0$ the above metric becomes Doglas type.

Now, differentiating equation (4) partially w.r.t $\alpha$ and $\beta$ are given by

$$L_\alpha = A_1(1 - \frac{\beta^2}{\alpha}) + A_2(-\frac{\beta^2}{(\alpha-\beta)^2}), \qquad L_\beta = A_1(\frac{2\beta}{\alpha}) + A_2(\frac{2\alpha\beta - \beta^2}{(\alpha-\beta)^2}),$$
$$L_{\alpha\alpha} = A_1(\frac{2\beta^2}{\alpha^3}) + A_2(\frac{2\beta^2}{(\alpha-\beta)^3}), \qquad L_{\beta\beta} = A_1(\frac{2}{\alpha}) + A_2(\frac{2\alpha^2}{(\alpha-\beta)^3})$$
$$L_{\alpha\beta} = A_1(\frac{-2\beta}{\alpha^2}) + A_2(\frac{-2\alpha\beta}{(\alpha-\beta)^3}),$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$.

In the Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor $h_{ij}$ are given by [5]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,$$
$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j,$$

where $Y_i = a_{ij} y^j$.

For the fundamental function (4) above constants are

$$\begin{aligned}
p &= L L_\alpha \alpha^{-1} = A_1^2 \frac{(\alpha^4 - \beta^4)}{\alpha^4} - A_1 A_2 \frac{(\alpha+\beta)\beta^3}{\alpha^2(\alpha-\beta)^2} - A_2^2 \frac{\beta^4}{\alpha(\alpha-\beta)^3},\\
q_0 &= L L_{\beta\beta} = A_1^2 \frac{(2\alpha^2 + 2\beta^2)}{\alpha^2} + A_1 A_2 \frac{(\alpha^2+\beta^2)^2 - 4\alpha\beta^3}{\alpha(\alpha-\beta)^3} + A_2^2 \frac{2\alpha^2\beta^2}{(\alpha-\beta)^4},\\
q_{-1} &= L L_{\alpha\beta} \alpha^{-1} = -A_1^2 \frac{(2\alpha^2 + 2\beta^2)\beta}{\alpha^4} - A_2^2 \frac{2\beta^3}{(\alpha-\beta)^4}\\
&\quad - A_1 A_2 \frac{(2\alpha^4\beta + 4\alpha^2\beta^3 + 2\beta^5 - 4\alpha\beta^4}{\alpha^3(\alpha-\beta)^3},\\
q_{-2} &= L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = A_1^2 \frac{(2\alpha^2\beta^2 + 3\beta^4 - \alpha^4)}{\alpha^6} + +A_2^2 \frac{(3\alpha-\beta)\beta^4}{\alpha^3(\alpha-\beta)^4}\\
&\quad - A_1 A_2 \frac{(2\alpha^4 - 7\alpha\beta^3 + 2\alpha^3\beta + 3\beta^4)\beta^2}{\alpha^5(\alpha-\beta)^3}
\end{aligned} \tag{5}$$

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2$ and its reciprocal tensor $g^{ij}$ for $L = L(\alpha,\beta)$ are given by [4, 5]

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j, \tag{6}$$

where

$$p_0 = q_0 + L_\beta^2, \;\; p_{-1} = q_{-1} + L^{-1} p L_\beta, \;\; p_{-2} = q_{-2} + p^2 L^{-2}. \tag{7}$$

The reciprocal tensor $g^{ij}$ of $g_{ij}$ is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{8}$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

$$\begin{aligned}
s_0 &= \frac{1}{\tau p}\{p p_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\},\\
s_{-1} &= \frac{1}{\tau p}\{p p_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\},\\
s_{-2} &= \frac{1}{\tau p}\{p p_{-2} + (p_0 p_{-2} - p_{-1}^2) b^2\},\\
\tau &= p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2).
\end{aligned} \tag{9}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [9]

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{10}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{11}$$

Here $m_i$ is a non-vanishing covariant vector orthogonal to the element of support $y^i$.

Let $\{^{\,i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space $R^n$ and $\bigtriangledown_k$ be the covariant derivative with respect to $x^k$ relative to this christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji}, \tag{12}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the cartan connection of $F^n$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space $F^n$ is given by

$$\begin{aligned} D^i_{jk} &= B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ &\quad - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} \\ &\quad + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{13}$$

where

$$\begin{aligned} B_k &= p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \\ B_{ij} &= \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B^k_i = g^{kj} B_{ji}, \\ A^m_k &= B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ \lambda^m &= B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i, \end{aligned} \tag{14}$$

where 0 denote contraction with $y^i$ except for the quantities $p_0, q_0$ and $s_0$.

## 3. Induced Cartan connection

Let $F^{n-1}$ be a hypersurface of $F^n$ given by the equation $x^i = x^i(u^\alpha)$ (where $\alpha = 1, 2, 3...(n-1)$). The element of support $y^i$ of $F^n$ is to be taken tangential to $F^{n-1}$, that is [6],

$$y^i = B^i_\alpha(u) v^\alpha \tag{15}$$

the metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of $F^{n-1}$ are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma$$

and at each point $(u^\alpha)$ of $F^{n-1}$, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}\{x(u,v),y(u,v)\}B^i_\alpha N^j = 0, \quad g_{ij}\{x(u,v),y(u,v)\}N^i N^j = 1.$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^i N^j = 1, \tag{16}$$

$(B^\alpha_i, N_i)$ inverse of $(B^i_\alpha, N^i)$ is given by

$$\begin{aligned} &B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0,\\ &N_i = g_{ij}N^j, \quad B^k_i = g^{kj}B_{ji}, \quad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \end{aligned}$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of $F^{n-1}$ from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by [6].

$$\begin{aligned} \Gamma^{*\alpha}_{\beta\gamma} &= B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,\\ G^\alpha_\beta &= B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma, \end{aligned}$$

where

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{oj}B^j_\beta)$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \qquad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

The quantities $M_{\beta\gamma}$ and $H_\beta$ are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{17}$$

where

$$M_\beta = N_i C^i_{jk}B^j_\beta N^k. \tag{18}$$

The relative h and v-covariant derivatives of projection factor $B^i_\alpha$ with respect to $IC\Gamma$ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta}N^i.$$

It is obvious form the equation (17) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{19}$$

The above equation yield

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{20}$$

We shall use following lemmas which are due to Matsumoto [6] in the coming section

**Lemma 3.1.** *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.*

**Lemma 3.2.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$.*

**Lemma 3.3.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

**Lemma 3.4.** *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

## 4. Hypersurface $F^{(n-1)}(c)$ of a special Finsler space

Let us consider a Finsler space with the metric $L(\alpha, \beta) = A_1(\alpha + \frac{\beta^2}{\alpha}) + A_2(\frac{\beta^2}{\alpha - \beta})$, where, vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now, we consider a hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [9].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0, \quad \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0, \quad b_i B^i_\alpha = 0.$$

Above shows that $b_i(x)$ are covariant component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0 \quad i.e \quad \beta = 0 \tag{21}$$

and induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{22}$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (6), (7) and (9) we get

$$\begin{aligned} p &= A_1^2, \quad q_0 = 2(A_1^2 + A_1A_2), \quad q_{-1} = 0 \quad q_{-2} = -A_1^2\alpha^{-2}, \\ p_0 &= 2(A_1^2 + A_1A_2) \quad p_{-1} = 0 \quad p_{-2} = 0 \quad \tau = A_1^3(3A_1 + 2A_2)b^2, \\ s_0 &= \frac{2(A_1 + A_2)}{A_1^2(3A_1 + 2A_2)b^2} \quad s_{-1} = 0 \quad s_{-2} = 0, \end{aligned} \tag{23}$$

from (8) we get,

$$g^{ij} = \frac{1}{A_1^2} a^{ij} - \frac{2(A_1 + A_2)}{A_1^2(3A_1 + 2A_2)b^2} b^i b^j \tag{24}$$

thus, along $F^{n-1}(c)$, (24) and (21) leads to

$$g^{ij} b_i b_j = \frac{b^2}{A_1(3A_1 + 2A_2)}.$$

So, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{A_1(3A_1+2A_2)}}N_i, \qquad b^2 = a^{ij}b_ib_j, \tag{25}$$

where $b$ is the length of the vector $b^i$.

Again from (24) and (25), we get

$$b^i = a^{ij}b_j = A_1(3A_1+2A_2)N^i \tag{26}$$

thus, we have

**Theorem 4.1.** *In a special Finsler hypersurface $F^{(n-1)}(c)$, the Induced Riemannian metric is given by* (22) *and the scalar function b(x)is given by* (25) *and* (26)*.*

Now, the angular metric tensor $h_{ij}$ and metric tensor $g_{ij}$ of $F^n$ are given by

$$h_{ij} = A_1^2 a_{ij} + 2A_1(A_1+A_2)b_ib_j - \frac{A_1^2}{\alpha^2}Y_iY_j \tag{27}$$

and

$$g_{ij} = A_1^2 a_{ij} + 2A_1(A_1+A_2)b_ib_j.$$

From equation (21), (27) and (16) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}_{(c)}$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

Thus, along $F^{n-1}_{(c)}$, $\frac{\partial p_0}{\partial \beta} = \frac{6A_1A_2}{\alpha}$ from equation (12) we get $r_1 = \frac{6A_1^3A_2}{\alpha}$, $m_i = b_i$ then hv-torsion tensor becomes

$$C_{ijk} = \frac{3A_1A_2}{\alpha}b_ib_jb_k \tag{28}$$

in the special Finsler hypersurface $F^{(n-1)}_{(c)}$. Due to fact from (16), (17), (19), (21) and (28) we have

$$M_{\alpha\beta} = 0, \quad M_\alpha = 0. \tag{29}$$

Therefore, from equation (20) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

**Theorem 4.2.** *The second fundamental v-tensor of the special Finsler hypersurface $F^{(n-1)}_{(c)}$ is given by* (29) *and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now, from (21) we have $b_iB^i_\alpha = 0$. Then we have $b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0$. Therefore, from (19) and using $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_jN^jH_\beta$, we have

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0 \tag{30}$$

since $b_i|_j = -b_h C^h_{ij}$, we get $b_{i|j} B^i_\alpha N^j = 0$. Therefore, from equation (30), we have

$$\sqrt{\frac{b^2}{A_1(3A_1+2A_2)}} H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0, \tag{31}$$

because $b_{i|j}$ is symmetric. Now, contracting (31) with $v^\beta$ and using (15) we get

$$\sqrt{\frac{b^2}{A_1(3A_1+2A_2)}} H_\alpha + b_{i|j} B^i_\alpha y^j = 0. \tag{32}$$

Again, contracting by $v^\alpha$ equation (32) and using (15), we have

$$\sqrt{\frac{b^2}{A_1(3A_1+2A_2)}} H_0 + b_{i|j} y^i y^j = 0. \tag{33}$$

From Lemma 3.1 and 3.2, it is clear that the hypersuface $F^{(n-1)}_{(c)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus from (33) it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j} y^i y^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of $F^n$ defined on $y^i$, but $b_{ij} = \bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence $b_{ij}$ does not depend on $y^i$. We shall consider the difference $b_{i|j} - b_{ij}$ where $b_{ij} = \bigtriangledown_j b_i$ in the following. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by (14). Since $b_i$ is a gradient vector, then from (13) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F^i_j = 0$. Thus, (14) reduces to

$$\begin{aligned} D^i_{jk} = {} & B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k \\ & - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{34}$$

where

$$\begin{aligned} & B_i = 2A_1(A_1+A_2) b_i, \quad B^i = \frac{2(A_1+A_2)}{(3A_1+2A_2)} b^i, \\ & \lambda^m = B^m b_{00}, \quad B_{ij} = \frac{3A_1A_2}{\alpha} b_i b_j, \\ & B^i_j = \frac{3A_2}{\alpha(3A_1+A_2)} b^i b_j, \\ & A^m_k = B^m_k b_{00} + B^m b_{k0}. \end{aligned} \tag{35}$$

In view of (23) and (24), the relation in (15) becomes to by virtute of (35) we have $B^i_0 = 0, B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now, contracting (34) by $y^k$ we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again, contracting the above equation with respect to $y^j$ we have

$$D^i_{00} = B^i b_{00} = \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{00}.$$

Paying attention to (21), along $F^{(n-1)}_{(c)}$, we get

$$\begin{aligned} b_i D^i_{j0} &= \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{j0} + \frac{3A_2 b^2}{\alpha(3A_1 + 2A_2)} b_j b_{00} \\ &- \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b_i b^m C^i_{jm} b_{00}. \end{aligned} \tag{36}$$

Now, we contract (36) by $y^j$ we have

$$b_i D^i_{00} = \frac{2(A_1 + A_2)}{(3A_1 + 2A_2)} b^2 b_{00}. \tag{37}$$

From (17), (25), (26), (29) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ the equation (36) and (37) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \{\frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)}\} b_{00}.$$

Consequently (32) and (33) may be written as

$$\begin{aligned} &\sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_\alpha + \{\frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)}\} b_{i0} B^i_\alpha = 0, \\ &\sqrt{\frac{b^2}{A_1(3A_1 + 2A_2)}} H_0 + \{\frac{A_1(3 - 2b^2) + 2A_2(1 - b^2)}{(3A_1 + 2A_2)}\} b_{00} = 0. \end{aligned} \tag{38}$$

Thus, the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus, we can write

$$2b_{ij} = b_i c_j + b_j c_i. \tag{39}$$

Now, from (21) and (39) we get $b_{00} = 0$, $b_{ij} B^i_\alpha B^j_\beta = 0$, $b_{ij} B^i_\alpha y^j = 0$. Hence, from (38) we get $H_\alpha = 0$, again from (39) and (35) we get $b_{i0} b^i = \frac{c_0 b^2}{2}, \lambda^m = 0, A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = 0$.

Now, we use equation (17), (24), (25), (26), (29) and (34) then, we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = 0. \tag{40}$$

Thus, the equation (31) reduces to

$$\sqrt{\frac{b^2}{A_1(3A_1+2A_2)}}H_{\alpha\beta}=0. \tag{41}$$

Hence, the hypersurface $F^{n-1}_{(c)}$ is umbilic.

**Theorem 4.3.** *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space $F^n$ equipped with a special $(\alpha,\beta)$-metric defined in equation* (4) *to be a hyperplane of first kind is* (39).

Now, from Lemma 3.3, $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha=0$ and $H_{\alpha\beta}=0$. Thus from (40), we get $c_0=c_i(x)y^i=0$. Therefore, there exist a function $\psi(x)$ such that $c_i(x)=\psi(x)b_i(x)$. Therefore (39) we get $2b_{ij}=b_i(x)\psi(x)b_j(x)+b_j(x)\psi(x)b_i(x)$. This can also be written as $b_{ij}=\psi(x)b_ib_j$.

**Theorem 4.4.** *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space $F^n$ equipped with a special $(\alpha,\beta)$-metric defined in equation* (4) *to be a hyperplane of second kind is* (41).

Again Lemma 3.4, together with (29) and $M_\alpha=0$ shows that $F^{n-1}_{(c)}$ become a hyperplane of third kind.

**Theorem 4.5.** *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space $F^n$ equipped with a special $(\alpha,\beta)$-metric defined in equation* (4) *is a hyperplane of the third kind.*

## 5. Some important result of hypersurface $F^{(n-1)}(c)$ of a Finsler space $F^n(c)$ with metric $A_1(\alpha+\frac{\beta^2}{\alpha})+A_2(\frac{\beta^2}{\alpha-\beta})$

The hv-torsion tensor $C_{ijk}$ of $F^{(n-1)}(c)$ with deformed infinite series metric written in equation (28) as

$$C_{ijk}=\frac{3A_1A_2}{\alpha}b_ib_jb_k.$$

Contracting by $g^{jk}$, we have

$$C_i=C_{ijk}g^{jk}=\frac{3A_1A_2}{\alpha}b^2b_i.$$

This implies that $b_i=\frac{\alpha}{3A_1A_2b^2}C_i$. Therefore, equation (28) becomes

$$C_{ijk}=\frac{\alpha^2}{9A_1^2A_2^2b^6}C_iC_jC_k. \tag{42}$$

**Definition 5.1.** *A Finsler space $F^n$ is called C2-like,if the (h) hv-tortion tensor $C_{ijk}$ is written in the form $C_{ijk}=\frac{1}{C^2}C_iC_jC_k$.*

Thus, using the above definition and equation (42) we have

$$C = \frac{3A_1A_2b^3}{\alpha}. \tag{43}$$

Thus

**Proposition 5.1.** *The Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space $F^n$ equipped with a special $(\alpha, \beta)$-metric defined in equation (4) is always be a C2-like Finsler space if equation (43) is satisfied.*

Since the main scalar of two dimensional Finsler space is defined as $LC_{ijk} = Im_im_jm_k$. Since $m_i = b_i$ then we have $LC_{ijk} = Ib_ib_jb_k$. Contracting $g^{jk}$ we have $LC_i = Ib^2b_i$ which gives $b_i = \frac{L}{b^2}C_i$.

Now, the main scalar of two dimensional Finsler

$$LC_{ijk} = \frac{IL^3}{b^6}C_iC_jC_k. \tag{44}$$

From equation (5.1) and (5.3), we have

$$I = \frac{\alpha^2}{9A_1A_2L^2}. \tag{45}$$

**Proposition 5.2.** *The main scalar $I$ of a Finslerian hypersurface $F^{(n-1)}(c)$ for the Finsler space $F^n$ equipped with a special $(\alpha, \beta)$-metric defined in equation (4) in a two dimensional case is given by (45).*

## 6. Conclusion

In the present paper we have combined Matsumoto douglas type Finsler metric and special Infinite series metric and introduced a special Finsler metric (4) with certain scalar $A_1$ and $A_2$. Further we obtained the necessary and sufficient condition for a Finslerian Hypersurface $F^{(n-1)}(c)$ of a Finsler space $F^n$ equiped with a special $(\alpha, \beta)$ metric will be hyperplane of first, second and third kind in the Theorem 4.3, 4.4 and 4.5 respectively. Further as application point of view for the above deformed metric we obtain a Proposition 5.1 in which it stated that this hypersurface will be C2-like Finsler space and in propostion (5.2) the value of main scalar $I$ in two-dimensional frame of Finsler Space is obtained.

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Accepted: January 21, 2021