An open problem on Tarski associative NET-groupoids and GTA-NET-groupoids

Xiaogang An[*]
School of Arts and Sciences
Shaanxi University of Science & Technology
Xi'an 710021
China
anxiaogang@sust.edu.cn

Mingming Chen
School of Arts and Sciences
Shaanxi University of Science & Technology
Xi'an 710021
China
chenmingming@sust.edu.cn

Xiaohong Zhang
School of Arts and Sciences
Shaanxi University of Science & Technology
Xi'an 710021
China
Shaanxi Joint Laboratory of Artificial Intelligence
Shaanxi University of Science & Technology
Xi'an 710021
China
zhangxiaohong@sust.edu.cn

Abstract. Tarski associative groupoid (TA-groupoid) is a kind of non-associative groupoid satisfying Tarski associative law. Tarski associative NET-groupoid (TA-NET-groupoid) is a neutrosophic extended triplet set with Tarski associative law. In this paper, we prove that every TA-NET-groupoid is a semigroup, thus successfully solving an open problem. Moreover, we propose the new notions of generalized TA-NET-groupoid (GTA-NET-groupoid) and weak commutative GTA-NET-groupoid, and investigate their properties and structural characteristics. We prove that GTA-NET-groupoid is equivalent to quasi strong regular TA-groupoid, and give the necessary and sufficient conditions for TA-groupoid to be a weak commutative GTA-NET-groupoid.

Keywords: Tarski associative groupoid (TA-groupoid), semigroup, TA-NET-groupoid, generalized TA-NET-groupoid (GTA-NET-groupoid), strong regular TA-groupoid.

*. Corresponding author

1. Introduction

Neutrosophic set (NS) was first proposed by Smarandache in [1]. In the past 15 years, NS theory has been developing rapidly. Until now, NS theory has been successfully applied to many fields such as professional selection [2], data filling [3], analysis of age group and time [4], automated skin lesion segmentation[5], clustering methods [6], etc. Moreover, further new theoretical studies on NS in [7, 8, 9, 10] have been performed, and some important results have been gotten.

As early as 1929, Suschkewitsch [11] studied the following generalized associative law which was originally called "Postulate A". In a finite group $(G, *)$, $\forall x, a, b \in G, \exists c \in G$, $(x * a) * b = x * c$, where the element c depends upon the element a and b only, and not upon x. When $c = b * a$, we can get the identity:

$$(x * a) * b = x * (b * a) \quad \text{(Tarski associative law)}.$$

Apparently, Tarski associative law is a special case of this Postulate A.

Function equations satisfying Tarski associative law were first discussed by Hosszú [12] in 1954. Continuous and strictly monotonic solutions of function equations satisfying Tarski associative law were investigated in [13]. Schölzel and Tomaschek [14] characterized the power series solutions of function equations satisfying Tarski associative law in the complex domain. A class of rings symmetric to Tarski associative law is studied in [15]. Pushkashu [16] studied properties of left (right) division (cancellative) groupoids satisfying Tarski associative law.

The concept of Tarski associative groupoid (TA-groupoid) and Tarski associative neutrosophic extended triplet groupoid (TA-NET-groupoid) were first given by Xiaohong Zhang et al. [17] in 2020. A groupoid $(G, *)$ is called a TA-groupoid, if it holds the Tarski associative law, that is, $\forall x, y, z \in G$, $(x * y) * z = x * (z * y)$.

Yingcang Ma et al. [18] put forward the concept of generalized neutrosophic extended triplet (GNET) set, which is an extension of the NS. In this paper, we study the algebraic structure with the Tarski associative law on GNET set, thus extending TA-NET-groupoid and obtaining generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid). That is, GTA-NET-groupoid is both TA-groupoid and GNET set. GTA-NET-groupoid relaxes the restrictions on the elements in TA-groupoid as compared to TA-NET-groupoid. This is also a concrete embodiment of the research method of regular semigroups to quasi-regular semigroups in TA-groupoid.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on TA-NET-groupoid are given. The relations between TA-NET-groupoid and regular TA-groupoid are discussed in Section 3. We propose the GTA-NET-groupoid and discuss its properties and structure in Section 4. Lastly, Section 5 presents the summary and the direction of future efforts.

2. Preliminaries

In this section, the related research and results of the TA-groupoid and TA-NET-groupoid are presented. Some related notions are introduced first.

A TA-groupoid $(G, *)$ is called locally associative if $\forall x \in G, (x * x) * x = x * (x * x)$. It is readily verified that every TA-groupoid is locally associative. In a TA-groupoid $(G, *)$, $\forall x \in G$, $k \in Z^+$, x^k is defined as follows: $x^1 = x, x^2 = x * x, x^3 = x^2 * x, x^4 = x^3 * x, ..., x^k = x^{k-1} * x$.

Definition 2.1 ([19])**.** *Let G be a non-empty set with a binary operation $*$. Then, G is called a neutrosophic extended triplet set if $\forall x \in G$, there exist a neutral of "x" and an opposite of "x" (denoted by $neut(x)$ and $anti(x)$ respectively), such that $neut(x), anti(x) \in G$ and: $neut(x) * x = x * neut(x) = x, anti(x) * x = x * anti(x) = neut(x)$. The triplet $(x, neut(x), anti(x))$ is called a neutrosophic extended triplet (NET).*

Definition 2.2 ([20])**.** *A NET set $(G, *)$ is called a NET-Loop, if $\forall x, y \in G$, one has $x * y \in G$.*

Definition 2.3 ([17])**.** *A TA-groupoid $(G, *)$ is called a TA-NET-groupoid if it is a NET-Loop.*

A TA-NET-groupoid $(G, *)$ is called a commutative TA-NET-groupoid if $\forall x, y \in G, x * y = y * x$. Example 2.1 illustrates that a TA-NET-groupoid is not always a commutative TA-NET-groupoid.

Example 2.1. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 1. From Definition 2.3, G is a TA-NET-groupoid. However, it is not a commutative TA-NET-groupoid since $a * c \neq c * a$.

Table 1: A TA-NET-groupoid of Example 2.1.

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$
a	a	a	a	a	a	a
b	b	b	b	b	b	b
c	c	c	c	c	c	c
d	a	a	a	d	e	e
e	a	a	a	e	e	e
f	c	c	c	f	f	f

Proposition 2.1 ([17])**.** *Let $(G, *)$ be a TA-groupoid. Then, $\forall a, b, c, d \in G$, $(a * b) * (c * d) = (a * d) * (c * b)$.*

Theorem 2.1 ([17])**.** *Let $(G, *)$ be a TA-NET-groupoid. Then, $\forall x \in G$, for any $a, b \in \{anti(x)\}$,*

(1) *$neut(x) * neut(x) = neut(x)$, and $neut(x)$ is unique.*

(2) *$neut(x)*a = a*neut(x) = neut(x)*b$, that is, $neut(x)*anti(x)$ is unique.*

(3) *$(neut(x) * a) * x = x * (neut(x) * a) = neut(x)$.*

Definition 2.4. *A TA-groupoid $(G, *)$ is called a TA-(l,l)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $c * a = b$.*

Definition 2.5. *A TA-groupoid $(G, *)$ is called a TA-(r,r)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $a * b = a$, and $a * c = b$.*

Definition 2.6. *A TA-groupoid $(G, *)$ is called a TA-(l,r)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $a * c = b$.*

Definition 2.7. *A TA-groupoid $(G, *)$ is called a TA-(r,l)-groupoid, if $\forall a \in G$, $\exists b, c \in G$, $a * b = a$, and $c * a = b$.*

From Definition 2.5 and Definition 2.6, it is readily verified that a TA-(r,r)-groupoid is equal to a TA-(l,r)-groupoid.

3. Regular TA-groupoid and TA-NET-groupoid

In this section, the relations between regular TA-groupoid and TA-NET-groupoid are established. Moreover, we prove that every TA-NET-groupoid is a semi-group, thus successfully solving an open problem.

Theorem 3.1. *Let $(G, *)$ be a finite TA-groupoid. Then, $\exists a \in G, a^2 = a$. That is, there exists an idempotent element in G.*

Proof. Assume that $(G, *)$ is a finite TA-groupoid. Then, $\forall a \in G$, $n \in Z^+$, $a^n \in G$. Since G is finite, there exist $i, j \in Z^+$ such that $a^i = a^{i+j}$. According to the value of i and j, we have three cases to discuss.

Case 1: if $j = i$, then $a^i = a^{2i}$, that is, $a^i = a^i * a^i$, a^i is an idempotent element in G.

Case 2: if $j > i$, then from $a^i = a^{i+j}$ we have
$a^j = a^i * a^{j-i} = a^{i+j} * a^{j-i} = a^{2j} = a^j * a^j$.
This means that a^j is an idempotent element in G.

Case 3: if $j < i$, then from $a^i = a^{i+j}$ we have
$a^i = a^{i+j} = a^i * a^j = a^{i+j} * a^j = a^{i+2j}$;
$a^i = a^{i+2j} = a^i * a^{2j} = a^{i+j} * a^{2j} = a^{i+3j}$;
......
$a^i = a^{i+ij}$.

Since $i, j \in Z^+$, then $ij \geq i$. For $a^i = a^{i+ij}$, Case 3 becomes Case 1 when $ij = i$, and Case 3 becomes Case 2 when $ij > i$. Thus, we know that there exists an idempotent element in G. □

Theorem 3.2. *Let $(G, *)$ be a TA-groupoid. Then, the following three statements are equivalent:*

(1) *G is a TA-(l,l)-groupoid;*

(2) *G is a TA-(l,r)-groupoid;*

(3) *G is a TA-NET-groupoid.*

Proof. (1)⇒(2). Suppose G is a TA-(l,l)-groupoid, from Definition 2.4, $\forall a \in G$, $\exists b, c \in G$, $b * a = a$, and $c * a = b$. We have $a * c = (b * a) * c = b * (c * a) = b * b = (c * a) * b = c * (b * a) = c * a = b$. By Definition 2.6, G is a TA-(l,r)-groupoid.

(2)⇒(3). Assume that G is a TA-(l,r)-groupoid, from Definition 2.6, $\forall a \in G$, $\exists b, c \in G, b * a = a$, and $a * c = b$. We have $a * b = a * (a * c) = (a * c) * a = b * a = a$. Let $d = b^2, f = b * c$, there are $d * a = b^2 * a = (b * b) * a = b * (a * b) = b * a = a$, $a * d = a * (b * b) = (a * b) * b = a * b = a$, $a * f = a * (b * c) = (a * c) * b = b^2 = d, f * a = (b * c) * a = b * (a * c) = b^2 = d$. By Definition 2.3, G is a TA-NET-groupoid.

(3)⇒(1). It is obvious that a TA-NET-groupoid is a TA-(l,l)-groupoid. □

Definition 3.1. *A TA-groupoid $(G, *)$ is called a regular TA-groupoid if $\forall a \in G, \exists x \in G, a = a * (x * a)$.*

From Definition 3.1 and Definition 2.7, it is readily verified that a TA-(r,l)-groupoid is equal to a regular TA-groupoid. Example 3.1 illustrates that a TA-groupoid is not always a regular TA-groupoid.

Example 3.1. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 2. G is a TA-groupoid. However, there is no element $x \in G$ such that $4 = 4 * (x * 4)$. From Definition 3.1, G is not a regular TA-groupoid.

Table 2: A TA-groupoid of Example 3.1.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	3	1	1	3
2	2	2	2	2	2
3	1	3	1	1	3
4	1	3	1	3	4
5	2	2	2	2	5

Definition 3.2. *A TA-groupoid $(G, *)$ is called a strong regular TA-groupoid if for any $x \in G$, there is a unary operation $x \to x^{-1}$ in G such that*

$$x = (x^{-1})^{-1}, \quad x * x^{-1} = x^{-1} * x, \quad x = x * (x^{-1} * x).$$

Example 3.2 illustrates that a regular TA-groupoid is not always a strong regular TA-groupoid.

Example 3.2. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 3. From Definition 3.1, G is a regular TA-groupoid. However, for element 3, $3 = 3 * (5 * 3)$, $5 * 3 \neq 3 * 5$. Thus, G is not a strong regular TA-groupoid.

Table 3: A regular TA-groupoid of Example 3.2.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	2	2	1	3
2	2	1	1	3	1
3	2	1	1	3	1
4	4	5	5	4	5
5	5	4	4	5	4

Proposition 3.1. *Let $(G, *)$ be a TA-NET-groupoid. Then, $\forall x \in G$, for any $a \in \{anti(x)\}$,*

$$neut(neut(x) * a) * anti(neut(x) * a) = x.$$

Proof. For any $a \in \{anti(x)\}$, we can get

$$\begin{aligned}
&(neut(x) * a) * neut(x)\\
&= neut(x) * (neut(x) * a) \quad \text{(applying the Tarski associative law)}\\
&= neut(x) * (a * neut(x)) \quad \text{(by Theorem 2.1 (2))}\\
&= (neut(x) * neut(x)) * a \quad \text{(applying the Tarski associative law)}\\
&= neut(x) * a. \quad \text{(by Theorem 2.1 (1))}.
\end{aligned}$$

From Theorem 2.1 (3) $(neut(x) * a) * x = x * (neut(x) * a) = neut(x)$, we can get

$$neut(neut(x) * a) = neut(x), x \in anti\{neut(x) * a\}.$$

From Theorem 2.1 (2) $neut(x) * anti(x)$ is unique, we have

$$neut(neut(x) * a) * anti(neut(x) * a) = neut(x) * x = x. \qquad \square$$

Example 3.3. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 4. By Definition 2.3, G is a TA-NET-groupoid. For element 1, $neut(1) = 1, \{anti(1)\} = \{1, 4, 5\}, 1 * 1 = 1 * 4 = 1 * 5 = 1$,

$$\begin{aligned}
neut(neut(1) * 1) * anti(neut(1) * 1) &= neut(neut(1) * 4) * anti(neut(1) * 4)\\
&= neut(neut(1) * 5) * anti(neut(1) * 5) = 1.
\end{aligned}$$

The other cases can be checked and verified, that is $\forall x \in G$, for any $a \in \{anti(x)\}$, $neut(neut(x) * a) * anti(neut(x) * a) = x$.

Table 4: A TA-NET-groupoid of Example 3.3.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	2	3	1	1
2	2	3	1	2	2
3	3	1	2	3	3
4	1	2	3	4	4
5	1	2	3	5	5

Theorem 3.3. *Let $(G, *)$ be a TA-groupoid. Then, G is a TA-NET-groupoid iff it is a strong regular TA-groupoid.*

Proof. Suppose G is a TA-NET-groupoid, from Definition 2.3, $\forall x \in G$, there is a NET $(x, neut(x), anti(x))$ in G.

Set $x^{-1} = neut(x) * anti(x)$, by Theorem 2.1 (2), x^{-1} is unique.

By Proposition 3.1, there are $(x^{-1})^{-1} = neut(neut(x)*anti(x))*anti(neut(x)*anti(x)) = x$,

$$\begin{aligned} x^{-1} * x &= (neut(x) * anti(x)) * x \\ &= neut(x) * (x * anti(x)) \quad (applying\ the\ Tarski\ associative\ law) \\ &= neut(x) * neut(x) \\ &= (x * anti(x)) * neut(x) \\ &= x * (neut(x) * anti(x)) \quad (applying\ the\ Tarski\ associative\ law) \\ &= x * x^{-1}, \end{aligned}$$

$x * (x^{-1} * x) = x * (neut(x) * neut(x)) = x * neut(x) = x.$

From Definition 3.2, G is a strong regular TA-groupoid.

In contrast, suppose G is a strong regular TA-groupoid and $x^{-1} \in G$, such that $x^{-1} * x = x * x^{-1}$ and $x = x * (x^{-1} * x)$. Set $neut(x) = x * x^{-1}$, then $x * x^{-1} = x^{-1} * x = neut(x)$,

$$\begin{aligned} neut(x) * x = (x * x^{-1}) * x = x * (x * x^{-1}) &= x * neut(x) \\ &= x * (x^{-1} * x) = x. \end{aligned}$$

By Definition 2.3, G is a TA-NET-groupoid and $x^{-1} \in \{anti(x)\}$. □

Example 3.4. Let $G = \{1, 2, 3, 4, 5\}$, an operation $*$ on G is defined as in Table 5. By Definition 2.3, G is a TA-NET-groupoid. For element 2 and element 3, $neut(2) = 4, anti(2) = 3$, $neut(3) = 4, anti(3) = 2$, we can get

$$\begin{aligned} &2^{-1} = neut(2) * anti(2) = 3, \quad 3^{-1} = neut(3) * anti(3) = 2, \\ &(2^{-1})^{-1} = 3^{-1} = 2, \quad 2 * (2^{-1} * 2) = 2, \quad 2 * 2^{-1} = 4 = 2^{-1} * 2, \\ &(3^{-1})^{-1} = 2^{-1} = 3, \quad 3 * (3^{-1} * 3) = 3, \quad 3 * 3^{-1} = 4 = 3^{-1} * 3. \end{aligned}$$

The other cases can be checked and verified. Thus, G is a strong regular TA-groupoid by Definition 3.2.

Table 5: A strong regular TA-groupoid of Example 3.4.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	1
2	1	3	4	2	1
3	1	4	2	3	1
4	1	2	3	4	1
5	5	5	5	5	5

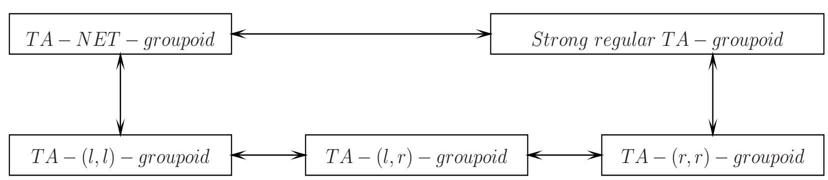

Figure 1: The relationship between TA-NET-groupoid and strong regular TA-groupoid.

Figure 1 shows that the relationship between TA-NET-groupoid and strong regular TA-groupoid. In Figure 1, a TA-NET-groupoid, a strong regular TA-groupoid, a TA-(l,l)-groupoid, a TA-(l,r)-groupoid, and a TA-(r,r)-groupoid are equivalent to each other.

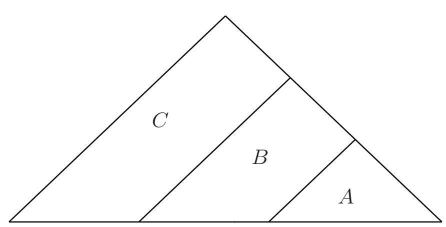

Figure 2: The relationship between regular TA-groupoid and TA-NET-groupoid.

Figure 2 can be used to express the relationship between regular TA-groupoid and TA-NET-groupoid. Here, A stands for TA-NET-groupoid; B stands for regular TA-groupoid shown in Example 3.2 rather than TA-NET-groupoid; and C stands for TA-groupoid shown in Example 3.1 rather than regular TA-groupoid. A+B stands for regular TA-groupoid, and A+B+C stands for TA-groupoid.

Xiaohong Zhang et al. [17] have proved the decomposition theorem of TA-NET-groupoid (see, Theorem 3.4) and proposed an open problem, "are there some TA-NET-groupoids which are not semigroups?". We try to look for such an example by computer programming in finite TA-NET-groupoids, which is not a semigroup. But it is not found. So we consider another possibility, a TA-NET-groupoid is a semigroup. Fortunately, this conjecture can prove to be correct.

Theorem 3.4 ([17])**.** *Let $(G,*)$ be a TA-NET-groupoid. The set of all different neutral element in G is denoted as $E(G)$, $\forall e \in E(G), G(e) = \{a \in G | neut(a) = e\}$. Then:*

(1) *$G(e)$ is a sub Abelian group of G;*

(2) *for any $e_1, e_2 \in E(G)$, if $e_1 \neq e_2$, then $G(e_1) \cap G(e_2) = \emptyset$;*

(3) *$G = \bigcup_{e \in E(G)} G_e$.*

Theorem 3.5. *A TA-NET-groupoid is a semigroup.*

Proof. Let $(G,*)$ be a TA-NET-groupoid. Denote $E(G)$ is the set of all different neutral elements in G, $\forall e \in E(G), G(e) = \{a \in G | neut(a) = e\}$. Suppose $E(G) = \{e_1, e_2, e_3, ...\}$, from Theorem 3.4 (1), $G(e_1), G(e_2)$ and $G(e_3)$ are all sub Abelian groups of G. Let $a \in G(e_1), b \in G(e_2)$ and $c \in G(e_3)$ be arbitrary elements. Set $d = (a*b)*c$ and $f = a*(b*c)$, we can get

$$
\begin{aligned}
d*d &= ((a*b)*c)*((a*b)*c) \\
&= (a*b)*((a*b)*c)*c) \quad (applying\ the\ Tarski\ associative\ law) \\
&= (a*b)*((a*b)*(c*c)) \quad (applying\ the\ Tarski\ associative\ law) \\
&= (a*b)*((a*b)*c^2) \\
&= a*(((a*b)*c^2)*b) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*((a*b)*(b*c^2)) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*((a*c^2)*(b*b)) \quad (by\ Propositoin\ 2.1) \\
&= a*((a*c^2)*b^2) \\
&= (a*b^2)*(a*c^2),
\end{aligned}
$$

$$
\begin{aligned}
f*f &= (a*(b*c))*(a*(b*c)) \\
&= a*((a*(b*c))*(b*c)) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*(a*((b*c)*(b*c))) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*(a*(b*((b*c)*c))) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*(a*(b*(b*(c*c)))) \quad (applying\ the\ Tarski\ associative\ law) \\
&= a*(a*(b*(b*c^2)))
\end{aligned}
$$

$$
\begin{aligned}
&= a*(a*((b*c^2)*b)) \quad (applying\ the\ Tarski\ associative\ law)\\
&= a*((a*b)*(b*c^2)) \quad (applying\ the\ Tarski\ associative\ law)\\
&= a*((a*c^2)*(b*b))\\
&= a*((a*c^2)*b^2)\\
&= (a*b^2)*(a*c^2),
\end{aligned}
$$

$$
\begin{aligned}
d*f &= ((a*b)*c))*(a*(b*c))\\
&= (a*b)*((a*(b*c))*c) \quad (applying\ the\ Tarski\ associative\ law)\\
&= (a*b)*(a*(c*(b*c))) \quad (applying\ the\ Tarski\ associative\ law)\\
&= (a*b)*(a*((c*c)*b)) \quad (applying\ the\ Tarski\ associative\ law)\\
&= (a*b)*(a*(c^2*b))\\
&= a*((a*(c^2*b))*b) \quad (applying\ the\ Tarski\ associative\ law)\\
&= a*(a*(b*(c^2*b))) \quad (applying\ the\ Tarski\ associative\ law)\\
&= a*(a*((b*b)*c^2)) \quad (applying\ the\ Tarski\ associative\ law)\\
&= a*(a*(b^2*c^2))\\
&= a*((a*c^2)*b^2) \quad (applying\ the\ Tarski\ associative\ law)\\
&= (a*b^2)*(a*c^2).
\end{aligned}
$$

Since $d*d = f*f$, from Theorem 3.4, we have d and f belong to the same subgroup $G(e)$. Since $d*d = d*f$ and the subgroup $G(e)$ satisfies the cancellation law, we have $d = f$. Thus, $(a*b)*c = a*(b*c)$, that is, G is a semigroup. □

Example 3.5. Let $G = \{1,2,3,4,5,6,7,8,9\}$, the definition of operation $*$ on G is shown in Table 6. From Definition 2.3, G is a TA-NET-groupoid. $E(G) = \{1,4,7\}$, $G(e_1) = \{1,2,3\}$, $G(e_2) = \{4,5,6\}$, $G(e_3) = \{7,8,9\}$. $G(e_1)$, $G(e_2)$ and $G(e_3)$ are all sub Abelian groups of G. It is easy to verify that $G(e_1)\cap G(e_2) = \emptyset$, $G(e_2)\cap G(e_3) = \emptyset$, $G(e_1)\cap G(e_3) = \emptyset$, and $G = G(e_1)\cup G(e_2)\cup G(e_3)$. For any $a \in G(e_1), b \in G(e_2)$ and $c \in G(e_3)$, without losing generality, let $a = 2, b = 5$ and $c = 7$. So, we can get

$$d = (2*5)*7 = 3, \quad f = 2*(5*7) = 3, \quad d*d = f*f = d*f = 2, \quad d, f \in G(e_1).$$

The other cases can be verified, thus G is a semigroup.

Example 3.6 illustrates that a semigroup is not always a TA-groupoid.

Example 3.6. Let $G = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, the definition of operation $*$ on G is shown in Table 7. G is a semigroup. However, it is not a TA-groupoid since $(x_3*x_2)*x_3 \neq x_3*(x_3*x_2)$.

Table 6: A TA-NET-groupoid of Example 3.5.

$*$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	1	2	3	1	2	3	1	2	3
2	2	3	1	2	3	1	2	3	1
3	3	1	2	3	1	2	3	1	2
4	4	5	6	4	5	6	4	5	6
5	5	6	4	5	6	4	5	6	4
6	6	4	5	6	4	5	6	4	5
7	7	8	9	7	8	9	7	8	9
8	8	9	7	8	9	7	8	9	7
9	9	7	8	9	7	8	9	7	8

Table 7: A semigroup of Example 3.6.

$*$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$
x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_1	x_1	x_2	x_1
x_3	x_1	x_2	x_3	x_3	x_2	x_3
x_4	x_1	x_2	x_4	x_4	x_2	x_4
x_5	x_1	x_1	x_1	x_1	x_5	x_1
x_6	x_1	x_2	x_3	x_3	x_2	x_6

Example 3.7 illustrates that a groupoid which is both a semigroup and a TA-groupoid is not always a TA-NET-groupoid.

Example 3.7. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 8. G is both a semigroup and a TA-groupoid. However, there is no element $x \in G$ such that $x * b = b$. From Definition 2.3 G is not a TA-NET-groupoid.

Table 8: A semigroup of Example 3.7.

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$
a	a	a	a	a	a	a
b	a	a	b	b	b	a
c	a	a	c	e	e	a
d	a	a	e	d	e	f
e	a	a	e	e	e	a
f	a	a	a	f	a	f

Figure 3 can be used to express the relationship between TA-groupoid and semigroup. Here, A stands for TA-NET-groupoid; B stands for both TA-

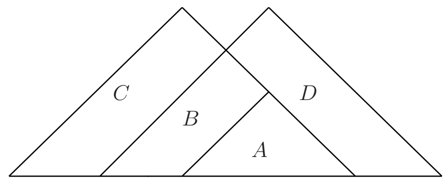

Figure 3: The relationship between TA-groupoid and semigroup.

groupoid and semigroup shown in Example 3.7 rather than TA-NET-groupoid; C stands for semigroup shown in Example 3.6 rather than TA-groupoid; and D stands for TA-groupoid rather than semigroup. A+B+C stands for semigroup, and A+B+D stands for TA-groupoid.

4. Generalized Tarski associative neutrosophic extended triplet groupoid

In this section, the new notion of generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid) is proposed, and some properties and structural characteristics are discussed.

Definition 4.1 ([18, 21])**.** *Let G be a non-empty set with a binary operation $*$. Then, G is called a generalized neutrosophic extended triplet set if for any $a \in G$, there is at least $n \in Z^+$, such that a^n has neutral element and opposite element (denoted by $neut(a^n)$ and $anti(a^n)$ respectively). The triplet $(a, neut(a^n), anti(a^n))$ is called a generalized neutrosophic extended triplet (GNET) with degree n.*

Definition 4.2. *Let $(G, *)$ be a GNET set. Then, G is called a generalized Tarski associative neutrosophic extended triplet groupoid (GTA-NET-groupoid), if $\forall a, b, c \in G$, $a * (c * b) = (a * b) * c$.*

A GTA-NET-groupoid $(G, *)$ is called a commutative GTA-NET-groupoid if $\forall x, y \in G, y * x = x * y$.

Example 4.1. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 9. For element 4, there is no element $x \in G$ such that $x * 4 = 4$. Thus, G is not a TA-NET-groupoid. Moreover, there is a GNET $(4, neut(4^2), anti(4^2))$ with degree 2 in G, thus G is a GTA-NET-groupoid. Since $5 * 4 \neq 4 * 5$ it is not a commutative GTA-NET-groupoid.

Proposition 4.1. *Let $(G, *)$ be a GTA-NET-groupoid, $x \in G, n \in Z^+$, and $(x, neut(x^n), anti(x^n))$ is a GNET with degree n. Then,*

(1) $neut(x^n)$ is unique.

(2) $neut(x^n) * neut(x^n) = neut(x^n)$, $neut(neut(x^n)) = neut(x^n)$.

Table 9: A GTA-NET-groupoid of Example 4.1.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	1	3	1	1
2	1	2	3	2	2
3	3	3	1	3	3
4	1	2	3	2	4
5	1	2	3	2	5

(3) $neut(x^n) * a = a * neut(x^n) = neut(x^n) * b$, *for any* $a, b \in \{anti(x^n)\}$.

(4) $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$, *for any* $a \in \{anti(x^n)\}$.

(5) $(neut(x^n) * a) * neut(x^n) = neut(x^n) * (neut(x^n) * a) = neut(x^n) * a$, *for any* $a \in \{anti(x^n)\}$.

(6) $neut(neut(x^n) * a) * anti(neut(x^n) * a) = x^n$, *for any* $a \in \{anti(x^n)\}$.

Proof.

(1) Assume $c, d \in \{neut(x^n)\}$, so $x^n * c = c * x^n = x^n, x^n * d = d * x^n = x^n$, and there exist $a, b \in G$ such that $x^n * a = a * x^n = c, x^n * b = b * x^n = d$. We have $c * d = (a * x^n) * d = a * (d * x^n) = a * x^n = c$, $c * d = c * (b * x^n) = (c * x^n) * b = x^n * b = d$. Thus, $c = d$, $neut(x^n)$ is unique.

(2)

$$\begin{aligned} neut(x^n) * neut(x^n) &= (anti(x^n) * x^n) * neut(x^n) \\ &= anti(x^n) * (neut(x^n) * x^n) \\ &\quad (applying\ the\ Tarski\ associative\ law) \\ &= anti(x^n) * x^n \\ &= neut(x^n). \end{aligned}$$

Thus, $neut(neut(x^n)) = neut(x^n)$.

(3) For any $a, b \in \{anti(x^n)\}$, $neut(x^n) * a = (a * x^n) * a = a * (a * x^n) = a * neut(x^n) = a * (b * x^n) = (a * x^n) * b = neut(x^n) * b$.

(4) For any $a \in \{anti(x^n)\}$, we have $x^n * (neut(x^n) * a) = (x^n * a) * neut(x^n) = neut(x^n) * neut(x^n) = neut(x^n)$, $(neut(x^n) * a) * x^n = neut(x^n) * (x^n * a) = neut(x^n) * neut(x^n) = neut(x^n)$. Thus, $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$.

(5) For any $a \in \{anti(x^n)\}$, we have

$$\begin{aligned} &(neut(x^n) * a) * neut(x^n) \\ &= neut(x^n) * (neut(x^n) * a) \quad (applying\ the\ Tarski\ associative\ law) \\ &= neut(x^n) * (a * neut(x^n)) \quad (applying\ (3)) \\ &= (neut(x^n) * neut(x^n)) * a \quad (applying\ the\ Tarski\ associative\ law) \\ &= neut(x^n) * a. \quad (applying\ (2)) \end{aligned}$$

Thus, $(neut(x^n) * a) * neut(x^n) = neut(x^n) * (neut(x^n) * a) = neut(x^n) * a$.

(6) For any $a \in \{anti(x^n)\}$, from (4) and (5), we have $neut(neut(x^n) * a) = neut(x^n)$, $x^n \in anti\{neut(x^n) * a\}$. From (3), $neut(x^n) * anti(x^n)$ is unique, we get $neut(neut(x^n) * a) * anti(neut(x^n) * a) = neut(neut(x^n) * a) * x^n = neut(x^n) * x^n = x^n$. □

Example 4.2. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 10. For element 1, $neut(1) = 1, \{anti(1)\} = \{1, 3\}$; for element 2, $neut(2) = 2, \{anti(2)\} = \{2, 4\}$; and for element 5, $neut(5) = anti(5) = 5$. There is no element $x \in G$ such that $x * 3 = 3, x * 4 = 4$. However, $3^2 = 1, 4^2 = 2$. Thus, G is a GTA-NET-groupoid. We can get the results that correspond to Proposition 4.1:

Table 10: A GTA-NET-groupoid of Example 4.2.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	3	1	3	3
2	4	2	4	2	4
3	1	3	1	3	3
4	4	2	4	2	4
5	5	5	5	5	5

(1) It is easily to verify that $neut(x^n)$ is unique.

(2) Being $neut(1^1) * neut(1^1) = neut(1^1), neut(2^1) * neut(2^1) = neut(2^1)$,

$neut(3^2) * neut(3^2) = neut(3^2), neut(4^2) * neut(4^2) = neut(4^2), neut(5^1) * neut(5^1) = neut(5^1)$,

that is, $neut(x^n) * neut(x^n) = neut(x^n)$ and $neut(neut(x^n)) = neut(x^n)$.

(3) For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1, 3\}$, we have

$neut(3^2) * 1 = 1 * neut(3^2) = neut(3^2) * 3 = 3 * neut(3^2) = 1$.

The other cases can be checked and verified. That is $neut(x^n) * a = a * neut(x^n) = neut(x^n) * b$, for any $a, b \in \{anti(x^n)\}$.

(4) For element 4, $neut(4^2) = 2, \{anti(4^2)\} = \{2, 4\}$, we have

$4^2 * (neut(4^2) * 2) = (neut(4^2) * 2) * 4^2 = 4^2 * (neut(4^2) * 4) = (neut(4^2) * 4) * 4^2 = neut(4^2)$.

The other cases can be checked and verified. That is, $x^n * (neut(x^n) * a) = (neut(x^n) * a) * x^n = neut(x^n)$, for any $a \in \{anti(x^n)\}$.

(5) For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1, 3\}$, we have

$(neut(3^2) * 1) * neut(3^2) = neut(3^2) * (neut(3^2) * 1) = neut(3^2) * 1$,

$(neut(3^2) * 3) * neut(3^2) = neut(3^2) * (neut(3^2) * 3) = neut(3^2) * 3$.

The other cases can be checked and verified. That is, $(neut(x^n) * a) * neut(x^n) = neut(x^n) * (neut(x^n) * a) = neut(x^n) * a$, for any $a \in \{anti(x^n)\}$.

(6) For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1, 3\}$,

$neut(neut(3^2) * 1) * anti(neut(3^2) * 1) = 3^2$,

$neut(neut(3^2) * 3) * anti(neut(3^2) * 3) = 3^2$.

For element 4, $neut(4^2) = 2, \{anti(4^2)\} = \{2, 4\}$,

$neut(neut(4^2) * 2) * anti(neut(4^2) * 2) = 4^2$,

$neut(neut(4^2) * 4) * anti(neut(4^2) * 4) = 4^2$.

The other cases can be checked and verified. That is, $neut(neut(x^n) * a) * anti(neut(x^n) * a) = x^n$, for any $a \in \{anti(x^n)\}$.

Definition 4.3. *A TA-groupoid $(G, *)$ is called a quasi strong regular TA-groupoid, if $\forall x \in G$, there is a positive integer n, $x^n \in G$, and a unary operation $x^n \to (x^n)^{-1}$ on G such that $((x^n)^{-1})^{-1} = x^n$, $(x^n)^{-1} * x^n = x^n * (x^n)^{-1}$, $x^n = x^n * ((x^n)^{-1} * x^n)$.*

Theorem 4.1. *Let $(G, *)$ be a groupoid. Then, G is a GTA-NET-groupoid iff it is a quasi strong regular TA-groupoid.*

Proof. Suppose G is a GTA-NET-groupoid, from Definition 4.2, for each $x \in G$, there exists a GNET $(x, neut(x^n), anti(x^n))$ with degree n. Set $(x^n)^{-1} = neut(x^n) * anti(x^n)$. From Proposition 4.1 (6), we get $((x^n)^{-1})^{-1} = neut(neut(x^n) * anti(x^n)) * anti(neut(x^n) * anti(x^n)) = x^n$. Being

$$
\begin{aligned}
(x^n)^{-1} * x^n &= (neut(x^n) * anti(x^n)) * x^n \\
&= neut(x^n) * (x^n * anti(x^n)) \ \ (applying\ the\ Tarski\ associative\ law) \\
&= neut(x^n) * neut(x^n) \\
&= (x^n * anti(x^n)) * neut(x^n) \\
&= x^n * (neut(x^n) * anti(x^n)) \ \ (applying\ the\ Tarski\ associative\ law) \\
&= x^n * (x^n)^{-1},
\end{aligned}
$$

$x^n * ((x^n)^{-1} * x^n) = x^n * (neut(x^n) * neut(x^n)) = x^n * neut(x^n) = x^n$, from Definition 4.3, G is a quasi strong regular TA-groupoid.

In contrast, if G is a quasi strong regular TA-groupoid and $(x^n)^{-1} \in G$, such that $x^n * (x^n)^{-1} = (x^n)^{-1} * x^n$ and $x^n * ((x^n)^{-1} * x^n) = x^n$. Set $neut(x^n) = x^n * (x^n)^{-1}$, then $x^n * (x^n)^{-1} = (x^n)^{-1} * x^n = neut(x^n)$. Being $neut(x^n) * x^n = (x^n * (x^n)^{-1}) * x^n = x^n * (x^n * (x^n)^{-1}) = x^n * neut(x^n) = x^n * ((x^n)^{-1} * x^n) = x^n$, from Definition 4.2, G is a GTA-NET-groupoid and $(x^n)^{-1} \in \{anti(x^n)\}$. □

Example 4.3. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 11. G is a GTA-NET-groupoid. For element 3, $neut(3^2) = 1, \{anti(3^2)\} = \{1,3\}$, $((3^2)^{-1})^{-1} = 3^2$,$3^2 * ((3^2)^{-1} * 3^2) = 3^2$, $3^2 * (3^2)^{-1} = (3^2)^{-1} * 3^2$. For element 4, $neut(4^2) = 2, \{anti(4^2)\} = \{2, 4\}$, $((4^2)^{-1})^{-1} = 4^2$, $4^2 * ((4^2)^{-1} * 4^2) = 4^2$, $4^2 * (4^2)^{-1} = (4^2)^{-1} * 4^2$. The other cases can be checked and verified. Thus G is a quasi strong regular TA-groupoid by Definition 4.3.

Table 11: A GTA-NET-groupoid of Example 4.3.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	3	1	3	1
2	4	2	4	2	4
3	1	3	1	3	1
4	4	2	4	2	4
5	5	5	5	5	5

Definition 4.4. *Let $(G, *)$ be a GTA-NET-groupoid. G is called a weak commutative GTA-NET-groupoid (WC-GTA-NET-groupoid) if $\forall x, y \in G$, there exist a GNET $(x, neut(x^n), anti(x^n))$ with degree n and a GNET $(y, neut(y^m), anti(y^m))$ with degree m $(n, m \in Z^+)$, $x^n * neut(y^m) = neut(y^m) * x^n$.*

Example 4.4. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 12. From Definition 4.2 G is a GTA-NET-groupoid. It is not a commutative GTA-NET-groupoid since $5 * 4 \neq 4 * 5$. We can show that it is a WC-GTA-NET-groupoid. For element 1 and element 2, $neut(1) = neut(2) = 1$, we have $1 * neut(2) = neut(2) * 1$, $2 * neut(1) = neut(1) * 2$. For element 3 and element 5, $neut(3^2) = 1, neut(5) = 5$, we have $3^2 * neut(5) = neut(5) * 3^2$, $5 * neut(3^2) = neut(3^2) * 5$. The other cases can be checked and verified. Thus, G is a WC-GTA-NET-groupoid.

Example 4.5 illustrates that a commutative GTA-NET-groupoid may not be a commutative TA-NET-groupoid.

Example 4.5. Let $G = \{a, b, c, d, e, f\}$, the definition of operation $*$ on G is shown in Table 13. G is a commutative GTA-NET-groupoid. Since there is no element x in G such that $(x * b) * b = b$, it is not a commutative TA-NET-groupoid.

Table 12: A WC-GTA-NET-groupoid of Example 4.4.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	2	2	2	1
2	2	1	1	1	2
3	2	1	1	1	3
4	2	1	1	1	4
5	1	2	3	3	5

Table 13: A commutative GTA-NET-groupoid of Example 4.5.

$*$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$	$\boldsymbol{f}$
a	a	a	a	a	a	a
b	a	a	b	a	b	b
c	a	b	c	a	c	c
d	a	a	a	d	d	a
e	a	b	c	d	e	c
f	a	b	c	a	c	f

Example 4.6 illustrates that a GTA-NET-groupoid may not be either a WC-GTA-NET-groupoid or a TA-NET-groupoid.

Example 4.6. Let $G = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, the definition of operation $*$ on G is shown in Table 14. From Definition 4.2, G is a GTA-NET-groupoid. There is no element a in G such that $a * x_2 = x_2$, and $x_1 * neut(x_3) \neq neut(x_3) * x_1$. Thus, it is neither a TA-NET-groupoid nor a WC-GTA-NET-groupoid.

Table 14: A GTA-NET-groupoid of Example 4.6.

$*$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	$\boldsymbol{x_6}$
x_1	x_1	x_1	x_2	x_2	x_1	x_1
x_2	x_1	x_1	x_2	x_2	x_1	x_1
x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_4	x_3	x_3	x_3	x_4	x_3	x_3
x_5	x_1	x_1	x_2	x_2	x_5	x_6
x_6	x_1	x_1	x_2	x_2	x_6	x_6

Figure 4 shows the relationships among WC-GTA-NET-groupoid and TA-NET-groupoid. Here, A stands for commutative TA-NET-groupoid; B stands for commutative GTA-NET-groupoid shown in Example 4.5 rather than commutative TA-NET-groupoid; C stands for WC-GTA-NET-groupoid shown in Example 4.4 rather than commutative GTA-NET-groupoid; D stands for non-

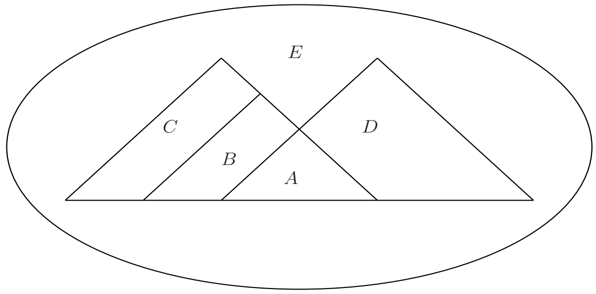

Figure 4: The relationships among WC-GTA-NET-groupoid and TA-NET-groupoid.

commutative TA-NET-groupoid shown in Example 2.1; and E stands for GTA-NET-groupoid shown in Example 4.6, it is, however, not either TA-NET-groupoid or WC-GTA-NET-groupoid. A+B stands for commutative GTA-NET-groupoid, A+B+C stands for WC-GTA-NET-groupoid, A+D stands for TA-NET-groupoid, and A+B+C+D+E stands for GTA-NET-groupoid.

Proposition 4.2. *Let* $(G,*)$ *be a GTA-NET-groupoid. Then,* G *is a WC-GTA-NET-groupoid iff* $\forall x,y \in G$*, there exist a GNET* $(x, neut(x^n), anti(x^n))$ *with degree* n *and a GNET* $(y, neut(y^m), anti(y^m))$ *with degree* m *(*$n,m \in Z^+$*),* $x^n * y^m = y^m * x^n$.

Proof. If $(G,*)$ is a GTA-NET-groupoid, then $\forall x,y \in G$, there exist a GNET $(x, neut(x^n), anti(x^n))$ with degree n and a GNET $(y, neut(y^m), anti(y^m))$ with degree m ($n,m \in Z^+$). Suppose G is a WC-GTA-NET-groupoid, $x^n * neut(y^m) = neut(y^m) * x^n$, we have

$$\begin{aligned} x^n * y^m &= (neut(x^n) * x^n) * (neut(y^m) * y^m) \\ &= (neut(x^n) * y^m) * (neut(y^m) * x^n) \quad (by\ Propositoin\ 2.1) \end{aligned}$$

$$\begin{aligned} &= (y^m * neut(x^n)) * (x^n * neut(y^m)) \\ &= (y^m * neut(y^m)) * (x^n * neut(x^n)) \quad (by\ Propositoin\ 2.1) \\ &= y^m * x^n. \end{aligned}$$

In contrast, $(x, neut(x^n), anti(x^n))$ and $(y, neut(y^m), anti(y^m))$ are two GNETs in G. From Proposition 4.1 (2), we can get $(neut(y^m), neut(y^m), neut(y^m))$ is a GNET with degree 1. Suppose that G satisfies the condition $x^n * y^m = y^m * x^n$, we get $x^n * neut(y^m) = neut(y^m) * x^n$. From Definition 4.4, G is a WC-GTA-NET-groupoid. □

Definition 4.5. *A quasi strong regular TA-groupoid* $(G,*)$ *is called a quasi Clifford TA-groupoid, if* $\forall x,y \in G$*, there exist* $n,m \in Z^+$*,* $(x^n)^{-1}, (y^m)^{-1} \in G$*, and* $x^n * (y^m * (y^m)^{-1}) = (y^m * (y^m)^{-1}) * x^n$.

Theorem 4.2. *Let $(G, *)$ be a groupoid. Then G is a WC-GTA-NET-groupoid iff it is a quasi Clifford TA-groupoid.*

Proof. Suppose that G is a WC-GTA-NET-groupoid, by Theorem 4.1, G is a quasi strong regular TA-groupoid. $\forall x, y \in G$, there exist $n, m \in Z^+$, $(x^n)^{-1}$, $(y^m)^{-1} \in G$. Since $(y, neut(y^m), anti(y^m))$ is a GNET with degree m, we have $(y^m, neut(y^m), anti(y^m))$ is a GNET with degree 1.

Set $(y^m)^{-1} = neut(y^m) * anti(y^m)$, so, we have

$$x^n * (y^m * (y^m)^{-1}) = x^n * neut(y^m) = neut(y^m) * x^n = (y^m * (y^m)^{-1}) * x^n.$$

Thus, G is a quasi Clifford TA-groupoid.

In contrast, assume that G is a quasi Clifford TA-groupoid, then $\forall x, y \in G$, there exist $n, m \in Z^+$, $(x^n)^{-1}, (y^m)^{-1} \in G$. By Theorem 4.1, G is a GTA-NET-groupoid, so $(x^n, neut(x^n), anti(x^n))$ and $(y^m, neut(y^m), anti(y^m))$ are two GNETs with degree 1.

Set $neut(x^n) = x^n * (x^n)^{-1}$, $neut(y^m) = y^m * (y^m)^{-1}$. From Definition 4.5, being $x^n * (y^m * (y^m)^{-1}) = (y^m * (y^m)^{-1}) * x^n$, we have $x^n * neut(y^m) = neut(y^m) * x^n$. Thus, G is a WC-GTA-NET-groupoid by Definition 4.4. □

Example 4.7. Let $G = \{1, 2, 3, 4, 5\}$, the definition of operation $*$ on G is shown in Table 15. It is a WC-GTA-NET-groupoid. From Theorem 4.1, G is a quasi strong regular TA-groupoid. For element 1 and element 2, $1^{-1} = 1, 2^{-1} = 2$, $1 * (2 * 2^{-1}) = (2 * 2^{-1}) * 1$, $2 * (1 * 1^{-1}) = (1 * 1^{-1}) * 2$. For element 3 and element 5, $(3^2)^{-1} = 1, 5^{-1} = 5$, $3^2 * (5 * 5^{-1}) = (5 * 5^{-1}) * 3^2$, $5 * (3^2 * (3^2)^{-1}) = (3^2 * (3^2)^{-1}) * 5$.

The other cases can be checked and verified, thus G is a quasi Clifford TA-groupoid.

Table 15: A WC-GTA-NET-groupoid of Example 4.7.

$*$	**1**	**2**	**3**	**4**	**5**
1	1	2	2	2	1
2	2	1	1	1	2
3	2	1	1	1	3
4	2	1	1	1	4
5	1	2	2	4	5

5. Conclusion

In this paper, the GTA-NET-groupoid has been introduced, the structures of TA-groupoid and GTA-NET-groupoid have been studied in detail, and some important results have been obtained. We prove that TA-NET-groupoid, TA-(l,l)-groupoid, TA-(r,r)-groupoid, TA-(l,r)-groupoid, and strong regular TA-groupoid are equivalent (see, Theorem 3.2 and Theorem 3.3); quasi strong regular TA-groupoid and GTA-NET-groupoid are equivalent (see, Theorem 4.1); quasi Clifford TA-groupoid and WC-GTA-NET-groupoid are equivalent (see, Theorem 4.2).

Furthermore, using the research results of Theorem 3.4, we successfully solve an open problem in [17]. We prove that a TA-NET-groupoid is a semigroup (see, Theorem 3.5), thus illuminating the structure of TA-NET-groupoid. Figure 5 shows the main results of this paper. Future efforts will be directed towards discussing the relationships between TA-groupoid and related logic algebras (see, [22, 23]).

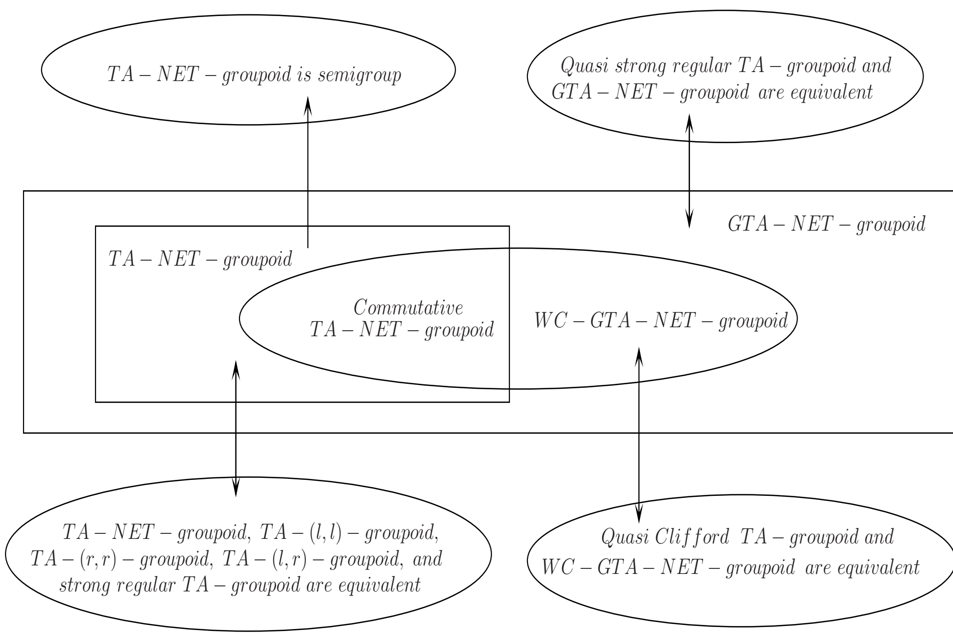

Figure 5: The main results of this paper.

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