A new secant type method for solving one variable functions

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Abstract. For solving one variable functions, Newton method is an important and basic method which convergences quadratically. In this paper, we deriving a new secant type methods based on the approximating the second derivative information for solving one minimization problem. The new secant type iterative method has convergence of order two. Application examples are given for demonstrated that the proposed method has better numerical characteristics compared to Newton method.
Keywords: Newton's methods, secant method, test functions.

1. Introduction

The majority useful optimization problems have many variables, know the study of single variable minimization may seem academic. For function of one variable, the classical approach finds the values of σ at the turning points of $f(\sigma)$ as the solutions of:

$$f'(\sigma) = 0. \quad (1)$$

For more details can be found in [3].

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Newton's classic method is the best broadly use of these types of methods, which is given as follows:

$$\sigma_{\ell+1} = \sigma_\ell - \frac{f'(\sigma_\ell)}{f''(\sigma_\ell)}. \tag{2}$$

For details see [14]. Secant method [4,5] is derived from the Newton-Raphson Method. Sometimes evaluating the second derivative of the function could be very tedious and cumbersome. To overcome this, the derivative is approximated and the next estimate is given as:

$$\sigma_{\ell+1} = \sigma_\ell - \frac{f'(\sigma_\ell)}{f'(\sigma_\ell) - f'(\sigma_{\ell-1})}(\sigma_\ell - \sigma_{\ell-1}). \tag{3}$$

Lately, some authors have derived new iterative methods which are more efficient than of Newton's [4,6-10,13 and 15].

The new update deriving for trying to approximate the secant update by approximating the second derivative $f''(\sigma_\ell)$ information.

2. A new modification into Newton's method

Theoretically, the Taylor's theorem has an important role in iterative methods. Taking Three-order Taylor expansion around the point σ_ℓ, as follows [12]:

$$f(\sigma_\ell) \cong f(\sigma_\ell) + (\sigma - \sigma_\ell)f'(\sigma_\ell) + \frac{1}{2}(\sigma - \sigma_\ell)^2 f''(\sigma_\ell) + \frac{1}{6}(\sigma - \sigma_\ell)^3 f'''(\sigma_\ell) \tag{4}$$

finding the minimum point where:

$$f(\sigma_\ell)' \cong f''(\sigma_\ell) + (\sigma - \sigma_\ell)f''(\sigma_\ell) + \frac{1}{2}(\sigma - \sigma_\ell)^2 f'''(\sigma_\ell) = 0. \tag{5}$$

It reduces to:

$$(\sigma - \sigma_\ell)^2 f'''(\sigma_\ell) = -2(f'(\sigma_\ell) + (\sigma - \sigma_\ell)f''(\sigma_\ell)). \tag{6}$$

Putting Eq. 6 in Eq. 4 we get:

$$(\sigma - \sigma_\ell)^2 f''(\sigma_\ell) = 6((f(\sigma) - f(\sigma_\ell)) - 4(\sigma - \sigma_\ell)f'(\sigma_\ell)).$$

This yields:

$$f''(\sigma_\ell) = \frac{6((f(\sigma) - f(\sigma_\ell)) - 4(\sigma - \sigma_\ell)f'(\sigma_\ell))}{(\sigma - \sigma_\ell)^2}. \tag{7}$$

Substituting σ with $\sigma_{\ell-1}$ in the above equation, we get:

$$f''(\sigma_\ell) = \frac{6((f(\sigma_{\ell-1}) - f(\sigma_\ell)) - 4(\sigma_{\ell-1} - \sigma_\ell)f'(\sigma_\ell))}{(\sigma_{\ell-1} - \sigma_\ell)^2}. \tag{8}$$

Putting Eq. 8 into Eq. 1, we get:

$$\sigma_{\ell+1} = \sigma_\ell - \frac{(\sigma_{\ell-1} - \sigma_\ell)^2 f'(\sigma_\ell)}{6(f(\sigma_{\ell-1}) - f(\sigma_\ell)) - 4((\sigma_{\ell-1} - \sigma_\ell) f'(\sigma_\ell))}, \tag{9}$$

which is new modification into Newton's method.

Now, we are prepared to state the new algorithm based on the above preparation.

Stage 1. Let σ_0 be the estimate of a minimum of $f(\sigma)$.

Stage 2. Set $\ell = \ell + 1$.

Stage 3. Set $\sigma_\ell = \sigma_{\ell-1} - \dfrac{f'(\sigma_{\ell-1})}{f''(\sigma_{\ell-1})}$.

Stage 4. Compute $\sigma_{\ell+1} = \sigma_\ell - \dfrac{(\sigma_{\ell-1} - \sigma_\ell)^2 f'(\sigma_\ell)}{6(f(\sigma_{\ell-1}) - f(\sigma_\ell)) - 4((\sigma_{\ell-1} - \sigma_\ell) f'(\sigma_\ell))}$.

Stage 5. If stopping criteria satisfies, then stop, otherwise go to 2.

3. Convergence analysis

In the next theorem, we will study the new method's convergence.

Theorem 3.1. *Let $f : I \longrightarrow R$ be a sufficiently differentiable function where $I$is an open interval, assume $\sigma^* \in R$ be a zero of $f(\sigma)$. If $\sigma_0$is sufficiently close to this zero, then the convergence of the new method is quadratic.*

Proof. The modification into Newton method is as:

$$\sigma_{\ell+1} = \sigma_\ell - \frac{(\sigma_{\ell-1} - \sigma_\ell)^2 f'(\sigma_\ell)}{6(f(\sigma_{\ell-1}) - f(\sigma_\ell)) - 4((\sigma_{\ell-1} - \sigma_\ell) f'(\sigma_\ell))}. \tag{10}$$

By subtracting σ^* from both sides of equation 10, we obtain:

$$e_{\ell+1} = e_\ell - \frac{f'(\sigma_\ell)(e_{\ell-1} - e_\ell)^2}{6(f(\sigma_{\ell-1}) - f(\sigma_\ell)) - 4((e_{\ell-1} - e_\ell) f'(\sigma_\ell))}, \tag{11}$$

where $e_\ell = \sigma_{\ell+1} - \sigma^*$. Appling expanding Taylor's series, we get:

$$f(\sigma^*) = f(\sigma_\ell) + (\sigma^* - \sigma_\ell) f'(\sigma_\ell) + \frac{1}{2!}(\sigma^* - \sigma_\ell)^2 f^{(2)}(\sigma_\ell) + \frac{1}{3!}(\sigma^* - \sigma_\ell)^3 f^{(3)}(\sigma_\ell)$$
$$+ \frac{1}{4!}(\sigma^* - \sigma_\ell)^4 f^{(4)}(z), \tag{12}$$

where z lies between σ^* and σ_ℓ, the derivative of equation (12), equal to zero, we obtained:

$$f'(\sigma^*) = f'(\sigma_\ell) + (\sigma^* - \sigma_\ell) f^{(2)}(\sigma_\ell) + \frac{1}{2}(\sigma^* - \sigma_\ell)^2 f^{(3)} + \frac{1}{6}(\sigma^* - \sigma_\ell)^3 f^{(4)}(z) \tag{13}$$

and using:

$$-f^{(1)}(\sigma^*) = -e_\ell f^{(2)}(\sigma_\ell) + \frac{e_\ell^2}{2} f^{(3)}(\sigma_\ell) - \frac{e_\ell^3}{6} f^{(4)}(z) \tag{14}$$

from equation 11 we get:

$$e_{\ell+1} = \frac{e_\ell^2 f^{(3)}(\sigma_\ell)}{2f^{(2)}(\sigma_\ell)} - \frac{e_\ell^3 f^{(4)}(z)}{6f^{(2)}(\sigma_\ell)}. \tag{15}$$

This means the order of convergence is quadratically. □

4. Computational results

Applying the iterative methods to solve seven unimodal function, in Matlab, it is implemented. In addition, there are other test functions for multidimensional case as in [1,2]. Using the execution time and number of iteration for the various functions for comparison between the iterative methods. We use accuracy is $\epsilon = 10^{-10}$, for computer programs.

Problem 1. Function $f(\sigma) = cos(\sigma) + (\sigma - 2)^2$ **starting point** $\sigma_0 = 2$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	4	2.3542	0.5790
New method	3	2.3684	0.1410

Problem 2. Function $f(\sigma) = e^\sigma - 3\sigma^2$ **starting point** $\sigma_0 = 0.25$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	4	0.2045	0.2660
New method	3	0.2043	0.1880

Problem 3. Function $f(\sigma) = e^\sigma + \sigma^2$ **starting point** $\sigma_0 = 1$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	5	0.3517	0.3130
New method	3	0.3257	0.1570

Problem 4. Function $f(\sigma) = -\sigma e^{-\sigma}$ **starting point** $\sigma_0 = 0$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	7	1	0.3900
New method	4	0.5630	0.2030

Problem 5. Function $f(\sigma) = 0.65 - 0.75/(1 + \sigma^2) - 0.65\sigma tan^{-1}(1/\sigma)$ **starting point** $\sigma_0 = 0.1$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	4	0.4809	0.7970
New method	4	0.3971	0.2810

Problem 6. Function $f(\sigma) = 0.5\sigma^2 - sin(\sigma)$ **starting point** $\sigma_0 = 2$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	5	0.7391	0.4210
New method	3	0.7359	0.1560

Problem 7. Function $f(\sigma) = \sigma^4 + 2\sigma^2 - \sigma - 3$ **starting point** $\sigma_0 = 1$.

Methods	Number of iterations	σ_ℓ	Execution time
Newton method	7	0.2367	0.6870
New method	4	0.5124	0.1880

5. Conclusion

A new secant type method for solving one minimization problem has been derived and analyzed. In certain examples, both strategies which fail to converge to the minimum and may not actually converge to the global minimum, either.

Based on our results, we now believe that the most powerful of the Newton methods we studied in this analysis is the new method. At each step, we need to determine the first derivative and it is likely that we will have trouble gaining the global minimum. It's not much different than its affinity, in the worst situation. The new iterative method is performing very well in comparison to Newton method as noted in the tables.

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References

[1] A.H. Basim, M.S. Hameed, *A modified class of conjugate gradient algorithms based on quadratic model for nonlinear unconstrained optimization*, Raf. J. of Comp. & Math's., 1 (2014), 25-37.

[2] A.H. Basim, M.E. Omer, *A new sufficient descent conjugate gradient method for nonlinear optimization*, Iraqi Journal of Statistical Sciences, 26 (2014), 13-24.

[3] B. Bundy, *Basic optimization method,* Edward Arnold, Bedford Square, London, U.K., 1984.

[4] K. Emin, C. Jinhai, *A modified secant method for unconstrained optimization*, Applied Mathematics and Computation, 176 (2007), 123-127.

[5] K.P. Edwin, H.Z. Stanislaw, *An introduction to optimization*, A John WILEY & Sons, Inc., Publication, Fourth Edition, 1980.

[6] K. Emin, *Modified secant-type methods for unconstrained optimization,* Applied Mathematics and Computation, 181 (2006), 1349-1356.

[7] M. Frontini, E. Sormani, *Some variants of Newton's method with third-order convergence*, Appl. Math. Comput., 140 (2003), 419–426.

[8] M. Frontini, E. Sormani, *Modified Newton's method with third-order convergence and multiple roots*, J. Comput. Appl. Math., 156 (2003), 345–354.

[9] H.H. Homeier, *A modified Newton method with cubic convergence: the multivariate case*, J. Comput. Appl. Math., 169 (2004), 161–169.

[10] H.H. Homeier, *On Newton-type methods with cubic convergence*, J. Comput. Appl. Math., 176 (2005), 425–432.

[11] S.S. Rao, *Engineering optimization theory and practice*, 4th edition, John Wiley & Sons Inc., New Jersey, Canada, 2009.

[12] R.L. Rardin, *Optimization in operations research*, Prentice-Hall, Inc., NJ, 1998.

[13] A.Y. Ozban, *Some new variants of Newton's method*, Applied Mathematics Letters, 17 (2004), 677-682.

[14] S. Weerakoom, T.G.I. Fernando, *A variant of Newton's method with accelerated third-order convergence*, Appl. Math. Lett., 13 (2000), 87–93.

[15] Y. Yuan, *A modified BFGS algorithm for unconstrained optimization,* IMA Journal Numerical Analysis, 11 (1991), 325-332.

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