A new characterization of sporadic groups

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Abstract. Let G be a finite group, n a positive integer. $\pi(n)$ denotes the set of all prime divisors of n and $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of G, defined by Grenberg and Kegel, is a graph whose vertex set is $\pi(G)$, two vertices p, q in $\pi(G)$ joined by an edge if and only if G contains an element of order pq. In this article, a new characterization of sporadic simple groups is obtained, that is, if G is a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and $\Gamma(G)$ is disconnected. This characterization unifies the several characterizations that can conclude the group has disconnected prime graphs, hence several known characterizations of sporadic simple groups become the corollaries of this new characterization.
Keywords: sporadic groups, order, prime graph, characterization.

1. Introduction

In the past three decades, as a very interesting topic, quantitative characterization of a group, especially a simple group, has been being an active topic

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in the theory of finite simple group since classification of finite simple groups completed in the early of 1980s. When Wujie Shi began to investigate the topic whether a finite simple group can be uniquely determined by its order and the set of its element orders, he proposed a famous conjecture in 1987, which was recorded as Problem 12.39 in [1].

Shi's Conjecture. Let G be a finite group, S a finite simple group, then $G \cong S$ if and only if $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$, where $\pi_e(G)$ denotes the set of element orders in G.

Research on Shi's conjecture began an era of quantitative characterization of finite simple groups. At last, this conjecture was completely proved in 2009. In the series of papers to prove Shi's conjecture, an important concept the prime graph of a finite group was frequently used for dealing with those simple groups with disconnected prime graph, which was defined by Gruenberg and Kegel in [2] as following:

Let G be a finite group, n a positive integer. $\pi(n)$ denotes the set of all prime divisors of n and $\pi(G) = \pi(|G|)$. The prime graph $\Gamma(G)$ of G, defined by Grenberg and Kegel, is a graph whose vertex set is $\pi(G)$, two vertices p, q in $\pi(G)$ joined by an edge if and only if G contains an element of order pq. We denote the number of connected components of $\Gamma(G)$ as $t(G)$, the connected components of $\Gamma(G)$ as $\{\pi_i(G),\ i = 1, \cdots,\ t(G)\}$, and we always assume 2 is in $\pi_1(G)$ if $2\big||G|$. The components of prime graphs of all finite simple groups were given by J. S. William, A. S. Kondratév, M. Suzuki, N. Iiyora and H. Yamaki etc. (see [2-5]).

The prime graph once be used by the second author to study the famous Thompson's conjecture:

Thompson's Conjecture. Let G be a finite group with $Z(G) = 1$, $N(G) = \{n \in \mathbf{N} | G$ has a conjugacy class of length $n\}$. If M is a finite simple group such that $N(G) = N(M)$, then $G \cong M$.

During the second author studying Thompson's conjecture, he proved that $|G| = |M|$ if G and M satisfy conditions of Thompson's Conjecture and the prime graph $\Gamma(M)$ is disconnected. For a finite group with disconnected prime graph, he divided its order into co-prime divisors, each of them exactly corresponding to each of components of the prime graph, and called these divisors the order components of the finite group and found that many finite simple groups can be uniquely determined by their order components. Actually, many simple groups with disconnected prime graphs have been proved to be uniquely determined by their order components, for example, in a series of articles , for example [6-11], etc, it is shown many simple groups with disconnected prime graphs are characterized by order components of their prime graphs. There are some other topics on characterization of a finite simple group by its order and some other quantitative properties related to disconnected prime graph, for example, in [12-16], characterization of a finite simple group by its order and maximal element order (the largest element order or the second largest element order, or both of them), characterization of a finite simple group by its order and the set of orders of maximal abelian subgroups, etc. In these topics, the discussed finite group are usually ascribed to a finite group having disconnected prime graph, whose order is the same as some finite simple group. Therefore, it is a meaningful topic to study the finite group having its prime graph disconnected and its order being equal to a finite simple group. In this article, we shall discuss this topic and specially focus on a finite group having its prime graph disconnected and its order being equal to a sporadic group. we shall prove the following theorem:

Main theorem. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and the prime graph of G is disconnected.*

By above theorem, the following known characterizations of sporadic simple groups including Shi' Conjecture and Thompson Conjecture, become its corollaries since under the hypothesis the prime graphs of the groups are disconnected, for example:

Corollary 1.1. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and $\pi_e(G) = \pi_e(S)$.*

Corollary 1.2. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $Z(G) = 1$ and $N(G) = N(S)$.*

Corollary 1.3. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if G and S have the same order components.*

Corollary 1.4. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if $|G| = |S|$ and the sets of orders of maximal abelian subgroups of G and S are equal.*

Throughout the paper, the actions unspecified always means conjugate action. Let G be a finite group, for a prime $p \in \pi(G)$, G_p denotes the *Sylow* p-subgroup of G. In addition, $G = U \ltimes V$ denotes G is the semidirect product of U and V, especially, $V \trianglelefteq G$.

2. The Proof of the Main Theorem

In order to prove the Main Theorem, we first present some lemmas which are useful in our proof.

Lemma 2.1 ([2], Theorem *A*)**.** *If G is a finite group whose prime graph has more than one components, then one of the following holds:*

(1) *G is a Frobenius group;*

(2) *G is a 2-Frobenius group;*

(3) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are $\pi_1(G)$-groups, K/H is a non-abelian simple group, H is a nilpotent group, where $2 \in \pi_1(G)$. And $|G/K| \big| |\mathrm{Out}(K/H)|$.*

Lemma 2.2 ([18], Lemma 2.6(1)). *Let G be a Frobenius group with Frobenius kernel K and Frobenius complement H. Then K is nilpotent and $|H|\big||K|-1$. Moreover, $t(G)=2$, and $\{\pi_1(G), \pi_2(G)\} = \{\pi(H),\ \pi(K)\}$.*

Lemma 2.3 ([18], Lemma 2.6(2)). *Let G be a 2-Frobenius group, then $G = DEF$, where D and DE are normal subgroups in G, DE and EF are Frobenius groups with kernels D and E, respectively. Moreover $t(G)=2$, $\pi_1(G)=\pi(D)\cup\pi(F)$ and $\pi_2(G)=\pi(E)$.*

Lemma 2.4 ([4], Lemma 14). *Any odd number from $\pi(\mathrm{Out(L)})$, where L is a non-abelian simple group, either lies in $\pi(L)$ or does not exceed $\frac{p}{2}$, where $p=\max(\pi(L))$.*

Lemma 2.5. *Let G be a finite group, L a non-abelian simple group such that $|L|\big||G|$, $|G|\big||\mathrm{Aut(L)}|$. Then any prime divisor r of $|G|$ with $\frac{p}{2}\le r\le p$, where $p=\max_{t\in\pi(G)} t$, lies in $\pi(L)$.*

Proof. It follows straight forward from $t\in\pi(L)$ by Lemma 2.4. □

Lemma 2.6 ([19], 8.2.3). *Let G and A be two groups. Let p be a prime divisor of $|G|$. Suppose that the action of A on G is coprime. Then there exists an A-invariant Sylow p-subgroup of G.*

Lemma 2.7 ([18], Lemma 2.4). *Let G be a p-group of order p^n, and G act on a q-group H of order q^m, where $p,\ q$ are two distinct primes. If $|G|\nmid|\mathrm{GL(m,q)}|$, then $pq\in\pi_e(G\ltimes H)$.*

By the above Lemma, we have the following corollary.

Corollary 2.1. *Let G be a p-group of order p^n, and G act on a q-group H of order q^m. If $p^r\nmid\prod_{i=1}^{m}(q^i-1)$, where $1\le r\le n$, then $pq\in\pi_e(G\ltimes H)$.*

Remark. Let G be a finite group satisfying the hypothesis of the Main Theorem, then by Lemma 2.1, G may be a Frobenius group or a 2-Frobenius group. In order to prove the Main Theorem, we first show that G cannot be a Frobenius group or a 2-Frobenius group, the proof will be separated into six lemmas. In the proofs of next six lemmas, when we mention a Frobenius group or a $2-Frobenius$ group, we are referring groups and notations in Lemma 2.2 and Lemma 2.3 without explanations. Moreover, we get that K is a $Hall$ subgroup of $G=HK$, and E is a $Hall$ subgroup of $G=DEF$, therefore, the following property holds:

(*) G **has no normal Sylow subgroup of the smallest order.**

2.1 To prove that G is neither a Frobenius group nor a 2-Frobenius group

Now we start to show that G is neither a Frobenius group nor a 2-Frobenius group.

Lemma 2.8. *Let G be a group and S a Mathieu group. If $|G| = |S|$, and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Mathieu groups are the following:

$$|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11,\ |M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11,\ |M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11,$$
$$|M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23,\ |M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23.$$

(1) Since $|G| = |S|$, it follows that $11 \in \pi(G)$. Now, we prove that G has no nilpotent normal subgroup with order divided by 11. Otherwise, let T be such a group. Then $T_{11} \trianglelefteq G$, so that $\pi(G) \setminus 5 \subseteq \pi(C_G(T_{11}))$ by $|G/C_G(T_{11})| \big| 10$. In view of the fact that $t(G) > 1$, we conclude that $5 \notin \pi(C_G(T_{11}))$. Note that $3 \in \pi(C_G(T_{11}))$, considering the action on $C_G(T_{11})$ by an element of order 5 and using Lemmas 2.6 and 2.7, we come to that $15 \in \pi_e(G)$. Hence, $t(G) = 1$, a contradiction. Similarly, we get that if $23 \in \pi(G)$, then G has no nilpotent normal subgroup with order divided by 23.

(2) Assume first that G is a Frobenius group, then $G = KH$ and $11 \in \pi(H)$ by (1). Applying Lemma 2.2, we obtain that, for any $r \in \pi(K)$, $t \in \pi(H)$, $|H_t| \big| (|K_r| - 1)$. Therefore, we get that $|K_2| = 2^{10}$ or $|K_2| = 1$ after checking calculation. Moreover, $|K_3| = 1$. Since $|G_2| > |G_r|$ for any $r \neq 2$, we conclude that $|K_2| = 2^{10}$. But $3^2 \big| |H|$, which contradicts Lemma 2.2 as $2^{10} - 1 = 3 \times 11 \times 31$. Therefore, G is not a Frobenius group.

(3) Now suppose that $G = DEF$ is a 2-Frobenius group. Using (1), we get that $11 \in \pi(E)$ or $11 \in \pi(F)$. If the former holds, then $|F| \big| 10$. Further, by calculating and using the fact $23 \notin \pi(D)$ by (1), we come to that $D_r = 1$ for $r \neq 2$. Hence $|D| = 2^{10}$ and $|G| = |M_{24}|$. Therefore, $7 \in \pi(E)$ for $|F| \big| 10$, contradicting $|E| \big| (|D| - 1)$ as $2^{10} - 1 = 3 \times 11 \times 31$. Thus, $11 \notin \pi(E)$. So that $11 \in \pi(F)$, forcing that $|E| = 23$ and $|G| = |M_{24}|$, which indicates that $|F| \big| 22$, yielding that $|D_5| = 5 < |E|$, contradicting Lemma 2.2. The lemma holds. □

Lemma 2.9. *Let G be a group and S a Janko group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. We write the proof in several steps.
Step 1. To show the lemma follows while $|G| = |J_1|$ or $|J_3|$.

Noticing $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, similarly to Lemma 2.8(1), we can prove that G has no nilpotent normal subgroup with order divided by 19.

Suppose that $G = KH$ is a Frobenius group, then $19 \in \pi(H)$. By calculating, we have, $|G_r| \not\equiv 1 (\text{mod } 19)$ for each $r \in \pi(G)$, a contradiction to Lemma 2.2. Hence, G can not be a Frobenius group. Now Assume that G is a 2-Frobenius group, referring to above and using the fact that E is a *Hall* subgroup of G, we conclude that $19 \notin \pi(F)$. Clearly, $19 \notin \pi(D)$, therefore, $19 \in \pi(E)$, which is impossible as $19 \nmid |\mathrm{GL}(7, 2)|$, $19 \nmid |\mathrm{GL}(5, 3)|$ and $5, 7, 11 < 19$. Consequently, G is not a 2-Frobenius group.

Step 2. To show the lemma follows while $|G| = |J_2|$ or $|J_4|$.

(1) Assume that $|G| = |J_2| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. If $G = KH$ is Frobenius group, then $7 \in \pi(H)$ by (*). But $|G_r| \not\equiv 1 (\text{mod } 7)$ for $r = 2, 3, 5$, a contradiction to Lemma 2.2. Now, assume that $G = DEF$ is a 2-Frobenius group, similarly, $7 \notin \pi(F)$, so $7 \in \pi(E)$ by (*), which implies $|D| = 2^3 < 3^3$, also a contradiction. As a result, G is neither a Frobenius group nor a 2-Frobenius group.

(2) Assume that $|G| = |J_4| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. If $G = KH$ is a Frobenius group, using (*), we see that $5 \in \pi(H)$, so that $43 \in \pi(H)$ for $5 \nmid 42$. Which is impossible, since $|G_r| \not\equiv 1 (\text{mod } 43)$ for any $r \in \pi(G)$. If $G = DEF$ is a 2-Frobenius group, similarly, we get that $43 \notin \pi(F)$ for E is a *Hall* subgroup of G. Consequently, $43 \in \pi(D)$ or $43 \in \pi(E)$. If the former holds, then $|E| = 7$ for $|E|$ is odd and E is a *Hall* subgroup of G. Using (*), we have, $5 \notin \pi(D)$, so that $5 \in \pi(F)$, this is impossible since EF cannot be a Frobenius group. Thus, $43 \in \pi(E)$, which forces $|D| = 2^{14}$ by calculating, so that $|F_2| = 2^7 > |E_{43}|$, a contradiction to Lemma 2.2. Hence, G is neither a Frobenius group nor a 2-Frobenius group.The lemma follows from Steps 1-2. □

Lemma 2.10. *Let G be a group and S a Conway group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Conway groups are the following:

$$|Co_1| = 2^{21}\cdot3^9\cdot5^4\cdot7^2\cdot11\cdot13\cdot23, |Co_2| = 2^{18}\cdot3^6\cdot5^3\cdot7\cdot11\cdot23, |Co_3| = 2^{10}\cdot3^7\cdot5^3\cdot7\cdot11\cdot23.$$

In the same way as Lemma 2.8(1), we get that G has no nilpotent normal subgroup with order divided by 23. Therefore, if $G = KH$ is a Frobenius group, then $23 \in \pi(H)$, this is impossible since $|G_r| \not\equiv 1 (\text{mod } 23)$ for any $r \in \pi(G)$. Now assume $G = DEF$ is a 2-Frobenius group, then $23 \in \pi(E)$ or $23 \in \pi(F)$. If $23 \in \pi(F)$, then we get a contradiction by $|E| = 2^{11}$. If $23 \in \pi(E)$, then $|D| = 2^{11}$, yielding that $|G| \neq |Co_3|$ and $2^7 \big| |F|$, contradicting $|F| \big| 22$. To summarize, G is neither a Frobenius group nor a 2-Frobenius group. □

Lemma 2.11. *Let G be a group and S a Fischer group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], we get the orders of Fischer groups are the following:

$$|Fi_{22}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13,\ |Fi_{23}| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23,$$
$$|Fi_{24}'| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29.$$

Since $t(G) > 1$, we deduce that G has no nilpotent normal subgroup with order divided by 11 or 13. Suppose that $G = KH$ is a Frobenius group, then $11, 13 \in \pi(H)$, which is impossible since $|G_r| \not\equiv 1(\mathrm{mod}\ 11 \times 13)$ for any $r \in \pi(G)$. Now assume that $G = DEF$ is a 2-Frobenius group. In this case, $11, 13 \notin \pi(D)$. If $13 \in \pi(E)$, then $|D_3| = 3^9$. Since $3^9 - 1 = 2 \times 13 \times 757$, we conclude that $11 \notin \pi(E)$, yielding that $11 \in \pi(F)$, contradicting the fact that $|F|\big||E_{13}| - 1$. Therefore, $13 \in \pi(F)$, implying that $|E| = 3^9$ by calculating. Hence, $|G| = |Fi_{22}|$ and $|D_5| = 5^2 < |E|$, a contradiction. □

Lemma 2.12. *Let G be a group and S the Monster group or the Baby group. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. From [20], the orders of the Monster group and the Baby group are the following:

$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, $|B| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$.

(1) It can be easily shown that $t(G) = 1$ if G_p is normal in G, where $p = 17, 19, 31$, hence, G has no normal subgroup with order p.

(2) Assume that $G = KH$ is a Frobenius group, then $17, 19, 31 \in \pi(H)$ by (1). By calculating, we get that $|G_r| \not\equiv 1(\mathrm{mod}\ 17 \times 19 \times 31)$ for any $r \in \pi(G)$, which contradicts Lemma 2.2. Now assume that G is a 2-Frobenius group. Then $17, 19, 31 \notin \pi(D)$ by (1). If $47 \in \pi(D)$, then $|E| = 23$, hence, $|F|\big|2 \cdot 11$, from which it follows that $17 \in \pi(D)$, a contradiction to (1). If $47 \in \pi(E)$, then $|F|$ divides $2 \cdot 23$, and D can only be a 2-group by calculating. Consequently, for any $p \in \pi(G) \setminus \{2, 23\}$, all p-elements are contained in E, so $|E| \geq 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47 > 2^{46}$, contradicting $|E|\big|(|D| - 1)$. Hence, $47 \in \pi(F)$, then, similarly, E can only be a 2-group since EF_{47} is a Frobenius group, a contradiction to $|E|$ is odd. □

Lemma 2.13. *Let G be a group and S one of Suz, HS, M^cL, He, HN, Th, $O'N$, Ly and Ru. If $|G| = |S|$ and the prime graph of G is disconnected, then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. We write the proof in three steps.

Step 1. To show the lemma follows while $|G|$ equals one of $|Suz|$, $|HS|$, $|M^cL|$, $|HN|$, $|O'N|$, $|Ly|$.

From [20], we get that: $|Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$; $|HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$; $|M^cL| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$; $|HN| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$; $|O'N| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$; $|Ly| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$.

(1) Since $t(G) > 1$, we deduce that G has no nilpotent subgroup with order divided by 7 or 11.

(2) Suppose that $G = KH$ is a Frobenius group, then (1) implies that $11 \in \pi(H)$. By calculating, we conclude that $11 \nmid |G_r| - 1$ for any $r \in \pi(G)$, which contradicts the facts that K is a $Hall$ subgroup and $|H|\big||K| - 1$. Now assume that $G = DEF$ is a 2-Frobenius group. Then $7, 11 \notin \pi(D)$ by (1). Referring to above and using the fact that $|E|$ is a $Hall$ subgroup of G, we see that $11 \notin \pi(F)$. Therefore, $11 \in \pi(E)$, then $|F|\big|2 \cdot 5$. Moreover, by calculating, we come to that $|D_2| = 2^{10}$ or $D_3 = 3^5$, $D_r = 1$ while $r \neq 2, 3$. This indicates that $7 \in \pi(E)$. But $7 \nmid 2^{10} - 1$ and $7 \nmid 3^5 - 1$, a contradiction.

Step 2. To show the lemma follows while $|G| = |He|$

By $|He| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, one has that $|G_{17}| = 17$, hence, G has no nilpotent normal subgroup with order divided by 17 for $t(G) > 1$. Assume that G is a Frobenius group. Then $17 \in \pi(H)$, but $17 \nmid |G_r| - 1|$ for every $r \in \pi(G)$, a contradiction to Lemma 2.2. Now Assume that G is a $2 - Frobenius$ group. Note that E is a $Hall$ subgroup of G and D is a nilpotent normal subgroup of G, for the same reason as above, we conclude that $17 \in \pi(E)$. Which forces that $|D| = 2^8$, so that $|F| = 2^2$, thereby, $|E_5| = 5^2$, if follows by Lemma 2.2 that $5^2\big|2^8 - 1$, contradicting $2^8 - 1 = 3 \times 5 \times 17$.

Step 3. To show the lemma follows while $|G|$ equals $|Th|$ or $|Ru|$.

(1) By [20], $|Th| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ and $|Ru| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, we see that, $|G_{13}| = 13$, and G_{13} is not normal in G, otherwise, $t(G) = 1$.

(2) Assume that G is a Frobenius group, then (1) follows that $13\big||H|$, so that $|K| = 3^3$, which yields $|K| < |H|$, contradicting $|H|\big|(|K| - 1)$. Therefore, G is not a Frobenius group.

(3) Assume that G is a $2 - Frobenius$ group. Then $13 \in \pi(E)$ or $13 \in \pi(F)$ by (1). Suppose that $13 \in \pi(E)$, and if $|G| = |Th|$, then, we get that $19 \in \pi(E)$. This is impossible since 19 does not divide $|\mathrm{GL}(14, 2)|$, $|\mathrm{GL}(10, 3)|$, $|\mathrm{GL}(3, 5)|$ and $|\mathrm{GL}(2, 7)|$. If $|G| = |Ru|$, then $29 \in \pi(E)$. However, $29 \nmid |\mathrm{GL}(14, 2)|$, we deduce that $D_2 = 1$, so that $|F_2| = |G_2| > |E_{13}|$, a contradiction. Hence, $13 \in \pi(F)$. Then, $|E| = 3^3$ by calculating, so $|G| = |Ru|$. Noticing that $3^3 - 1 = 2 \cdot 13$, we deduce that $|D_{29}| = 29$, which is impossible for ED_{29} cannot be a Frobenius group. which implies G is not a 2-Frobenius group.

This Lemma follows from Steps 1-3. □

Lemmas 2.1 and Lemmas 2.8−2.13 imply the following.

Corollary 2.2. *Let G be a finite group and S one of 26 sporadic groups. If $|G| = |S|$ and the prime graph of G is disconnected, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H and G/K are $\pi_1(G)$-groups, K/H is a non-abelian simple group with $t(K/H) \geq t(G)$, H is a nilpotent group, and $G/H \leq \mathrm{Aut}(K/H)$.*

2.2 To prove $G \cong S$

The necessity of the Main Theorem is obvious. Now we start to prove the sufficiency of The Main Theorem. In the following discussion, K and H always means subgroups of G in Corollary 2.2. We shall check the possibilities of K/H based on $|K/H|\big||S|$ by using the list of simple groups in [20], which means we need to find out all simple groups T such that $|T|\big||S|$. In [20] there is a list of non-abelian simple groups of orders$< 10^{25}$, except that $L_2(q), L_3(q), U_3(q)$, $L_4(q), U_4(q), S_4(q)$ and $G_2(q)$, which are stopped at orders $10^6(q \leq 125), 10^{12}(q \leq 31), 10^{12}(q \leq 32)$, $10^{16}(q \leq 11), 10^{16}(q \leq 11), 10^{16}(q \leq 41), 10^{20}(q \leq 25)$, respectively. For a sporadic simple group $S \ncong B$ and M, it follows that $|S| < 10^{25}$, hence we need to find out a simple group T such that $|T|\big||S|$ from the mentioned series of simple groups of Lie type. For $S = B$ or M, since $|B|\big||M|$ and $10^{53} < |M| < 10^{54}$, so it is enough for us to find out simple group T such that $|T|\big||M|$. Therefore, we set up the following lemma:

Lemma 2.14. *Let S be a sporadic group, and T a non-abelian simple group such that $|T|\big||S|$, then*

(I) If $S = M$, then T is either contained in the list of [20] or one of $A_n(26 \leq n \leq 32)$, $L_2(2^{10})$ and $L_2(13^2)$.

(II) If $S \neq M$, then T is contained in the list of [20].

Proof. (I) For a simple group T satisfying the hypothesis of (I), since $10^{53} < |M| < 10^{54}$, we have that $|T| \leq 10^{54}$, hence we have to check if there exists T of order $\leq 10^{25}$ among the series of $L_2(q)(q > 125), L_3(q)(q > 31), U_3(q)(q > 32)$, $L_4(q)(q > 11), U_4(q)(q > 11), S_4(q)(q > 41)$ and $G_2(q)(q > 25)$, and to check if there exists T of order greater than 10^{25} and smaller than 10^{54} among all non-abelian simple groups except sporadic simple groups. In order to make the proof easy to read, we check all series of non-abelian simple groups except sporadic simple groups. By Classification Theorem of Simple Groups, we have the following steps.

(1) Assume that $T \cong A_n$, an alternating group. Note that $|A_{25}| < 10^{25} < |A_{26}|$, we have $n \geq 26$. On the other hand, in view of the fact that $11^2 \| |M|$, we come to that $26 \leq n \leq 32$. By calculating the order of $A_n(26 \leq n \leq 32)$, we get that $T \cong A_n(26 \leq n \leq 32)$.

(2) Assume that $T \cong A_n(q)$. Then $|T| = q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^{i+1}-1)/(n+1, q-1)$. It follows from $|T|\big||M|$ and the largest exponent of prime power of $|M|$ is 46 that $n \leq 9$. Therefore, T can only be one of the Lie type simple groups: $A_1(q)(q > 125)$, $A_2(q)(q > 31)$, $A_3(q)(q > 11)$, $A_4(q)(q > 11)$, $A_5(q)(q > 5)$, $A_6(q)(q > 3)$, $A_7(q)(q > 2)$, $A_8(q)(q > 2)$, $A_9(q)(q < 4)$.

If $T = A_1(q)(q > 125)$, since $|T|$ divides $|M|$, we deduce that one of the cases holds: $q = 2^t(7 \leq t \leq 46)$, $q = 3^t(5 \leq t \leq 20)$, $q = 5^t(4 \leq t \leq 9)$, $q = 7^t(3 \leq t \leq 6)$, $q = 13^t(2 \leq t \leq 3)$. Now calculating the order of these groups by using Maple, we come to that q is either 2^{10} or 13^2.

If $T = A_2(q)(q > 31)$, then q^3 divides one of 2^{45}, 3^{18}, 5^9 and 7^6. Hence, q is one of: $2^t(5 \leq t \leq 15)$, $q = 3^t(4 \leq t \leq 6)$, $q = 5^3$ and $q = 7^2$. Now calculating the order of these groups, for example, by using Maple, we have $|T| \nmid |M|$, a contradiction.

By the same approach above, we can get contradictions if $T = A_n(q)$, where $3 \leq n \leq 9$.

Suppose that T is one of the remaining cases, we can get contradictions by $|T| \nmid |M|$ in the same way as above. The checking procedures is trivial, we omit them here. We come to that (I) follows.

(II) For a sporadic group S, we have that if $S \neq B$ and M, then $|S| < 10^{25}$; and if $S \ncong Ly$ or J_4, then $|S|||M|$. Hence if $S \ncong Ly$ and J_4, then by (I) T is contained in the list of [20], or T is one of $A_n (26 \leq n \leq 32)$, $L_2(2^{10})$ and $L_2(13^2)$. But $|A_n| > 10^{25}$ $(26 \leq n \leq 32)$, $|L_2(2^{10})|$ and $|L_2(13^2)|$ do not divide the order of any sporadic group except M. Therefore (II) follows for $S \ncong Ly$ and J_4. If $S \cong Ly$ or J_4, since both of $|Ly|$ and $|J_4| < 10^{25}$, we have that if T is not in the list of [20], then T is contained in the series of $L_2(q)(q > 125), L_3(q)(q > 31), U_3(q)(q > 32)$, $L_4(q)(q > 11), U_4(q)(q > 11), S_4(q)(q > 41)$ and $G_2(q)(q > 25)$. By using Maple to calculate the orders of groups with orders $< 10^{25}$ among these series, we can show that $|T|$ does not divide $|S|$. □

Now, we begin to prove $G \cong S$, in the following proofs, we use (2) in Corollary 2.2 and Lemma 2.5 frequently.

Lemma 2.15. *Let G be a group and S one of groups: M_{11}, M_{12}, M_{22}, M_{23} and M_{24}. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. By assumptions, $|G|$ is one of $|M_{11}|$, $|M_{12}|$, $|M_{22}|$, $|M_{23}|$ and $|M_{24}|$, also, $\Gamma(G)$ is disconnected. Applying Corollary 2.2, G has a unique simple section K/H such that K and H satisfying (1) in Corollary 2.2. By Lemma 2.15, K/H can only be simple groups in the list of [20]. By comparing the order of G with the orders of the simple groups in [20], we can obtain all possibilities of K/H. We discuss step by step following what S is.

Step 1. If $S = M_{11}$ or M_{12}, then $G \cong S$.

Observe that $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $|M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $|G| = |S|$. By $t(G) > 1$, we assert that $11 \notin \pi(H)$. Otherwise, $|H_{11}| = 11$ and $H_{11} \trianglelefteq G$. Since $G/C_G(H_{11}) \leq \mathrm{Aut}(H_{11})$, it follows that $\pi(G) \setminus \{5\} \in \pi(C_G(H_{11}))$. As $t(G) > 1$, we have $5 \notin \pi(C_G(H_{11}))$. Note that $3 \in \pi(C_G(H_{11}))$, considering the action of an element of order 5 on $C_G(H_{11})$. Apply Lemmas 2.6 and 2.7, we get that $15 \in \pi_e(G)$, thus $t(G) = 1$, a contradiction. Hence $11 \in \pi(G/H)$ as desired. By Corollary 2.2 and Lemma 2.5, we get that $11 \in \pi(K/H)$. Checking simple groups in the list of [20], we get that K/H has the following cases:

(1) Suppose that $S \cong M_{11}$, then $K/H \cong L_2(11)$ or M_{11}. As $|L_2(11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$, $\mathrm{Out(L_2(11))} = 2$, we see that $|H_3| = 3$, thus $t(G) = 1$, a contradiction. So $K/H \cong M_{11}$, hence, $G \cong S \cong M_{11}$ by $|G| = |M_{11}|$.

(2) Suppose that $S \cong M_{12}$, then $K/H \cong L_2(11)$, M_{11} or M_{12}. If $K/H \cong M_{12}$, then $G \cong S \cong M_{12}$, we are done. Note that $|\mathrm{Out(M_{11})}| = 1$ and refer to (1), we get $|H_3| = 3$ or 3^2 if $K/H \cong L_2(11)$ or M_{11}. This implies that $t(G) = 1$, a contradiction. Step (1) follows.

Step 2. If $S = M_{22}$, then $G \cong M_{22}$.

Since $|G| = |M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, it follows by $t(G) > 1$ that $7, 11 \in \pi(G/H)$. Using Corollary 2.2 and Lemma 2.5, $7, 11 \in \pi(K/H)$, so K/H is uniquely isomorphic to M_{22} by [20]. Therefore, $G \cong S$, and Step (2) follows.

Step 3. If $S = M_{23}$ or M_{24}, then $G \cong S$.

In these cases, either $|G| = |M_{23}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ or $|G| = |M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. For the same reason as above, $23 \in \pi(K/H)$. By [20], the following hold:

(1) Suppose that $S \cong M_{23}$, then $K/H \cong M_{23}$ or $L_2(23)$. If $K/H \cong M_{23}$, then $G \cong S$, we are done. If $K/H \cong L_2(23)$, then by $G/H \leq \mathrm{Aut}(K/H)$, we get that $|H_7| = 7$, one has that $t(G) = 1$, a contradiction.

(2) Suppose that $S \cong M_{24}$, then K/H is one of M_{24}, M_{23} and $L_2(23)$. Note that if $K/H \cong M_{23}$ or $L_2(23)$, then $|H_3| = 3$ or $|H_7| = 7$. But in either case, one has that $t(G) = 1$, this is impossible. Therefore $K/H \cong M_{24}$, thus $G \cong M_{24}$. □

Lemma 2.16. *Let G be a group and S one of groups: J_1, J_2, J_3 and J_4. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. Similar to Lemma 2.15, we need consider the possibilities of the unique simple section K/H for each S, and have the following steps.

Step 1. If S is one of J_1, J_3 and J_4, then $G \cong S$.

Since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $|J_3| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, and $|J_4| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$, it follows by $t(G) > 1$ and Corollary 2.2 and Lemma 2.5 that: if $S = J_1$, then $11, 19 \in \pi(K/H)$; if $S = J_3$, then $17, 19 \in \pi(K/H)$; if $S = J_4$, then $23, 29, 31, 37, 43 \in \pi(K/H)$. Hence, by [20], K/H can only be isomorphic to S. Therefore, $G \cong S$ by $|G| = |K/H|$.

Step 2. If $S = J_2$, then $G \cong S$.

In such case, $|G| = |J_2| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. similarly, $5, 7 \in \pi(K/H)$. We check the list of simple groups in [20] and come to that K/H is one of the following simple groups: $A_7(|A_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7, |\mathrm{Out}(\mathrm{A}_7)| = 2)$, $A_8(|A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7, |\mathrm{Out}(\mathrm{A}_8)| = 2)$, $L_3(4)(|L_3(4)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7, |\mathrm{Out}(\mathrm{L}_3(4))| = 2^2 \cdot 3)$, J_2. If $K/H \ncong J_2$, then $|H_5| = 5$, which yields $t(G) = 1$, a contradiction. Hence, $K/H \cong J_2$. Therefore, $G \cong S$, as desired. □

Lemma 2.17. *Let G be a group and S one of Co_1, Co_2 and Co_3. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. We write the proof step by step upon what S is .

Step 1. If $S = Co_1$, then $G \cong Co_1$.

By assumption, $|G| = |Co_1| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. So $13, 23 \in \pi(K/H)$ by $t(G) > 1$ and Corollary 2.2 and Lemma 2.5. Applying Lemma 2.14 and checking simple groups of orders dividing $|Co_1|$ in [20], we get that $K/H \cong S = Co_1$.

Step 2. If S is one of Co_2 and Co_3 then $G \cong S$.

Since $|G|$ equals $|Co_2| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ or $|Co_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, we get that $23 \in \pi(K/H)$. By the same reason, we get that K/H is one of the simple groups: S, $L_2(23)(|L_2(23)| = 2^3 \cdot 3 \cdot 11 \cdot 23, |\mathrm{Out}(\mathrm{L}_2(23))| = 2$, $M_{23}(|M_{23}| =$

$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23, \mathrm{Out}(\mathrm{M}_{23}) = 1)$, $M_{24}(|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23, \mathrm{Out}(\mathrm{M}_{24}) = 1)$. Comparing $|G|$ with $|\mathrm{Aut(K/H)}|$, we conclude that if $K/H \ncong S$, then $|H_5| = 5^r$, where $1 \le r \le 3$. Note that $|\mathrm{GL}(3,5)| = 2^7 \times 3 \times 31$, by $G/C_G(H_5) \le \mathrm{Aut(H_5)}$ and using Lemma 2.7, we get that $t(G) = 1$, a contradiction. Therefore, $K/H \cong S$, hence $G \cong S$, we are done. □

Lemma 2.18. *Let G be a group and $S = Fi_{22}$, Fi_{23} or Fi'_{24}. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. The proof is divided into two steps.
Step 1. If $S = Fi_{22}$, then $G \cong Fi_{22}$.

In this case, $|G| = |Fi_{22}| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, so $7, 11, 13 \in \pi(K/H)$. By the same reason, checking the list in [20], we get that K/H is isomorphic to one of the groups: Fi_{22}, $A_{13}(|A_{13}| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\mathrm{Out}(\mathrm{A}_{13})| = 2)$, $Suz(|Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\mathrm{Out(Suz)}| = 2)$. Checking the orders of Aut(K/H) and using the fact that $|G/H| \big| |\mathrm{Aut(K/H)}|$, we get that if $K/H \ncong S$, then $|H_3| = 3^r$ and $2 \le r \le 4$. Observe that $|\mathrm{GL}(4,3)| = 2^9 \times 3^6 \times 5 \times 13$, by $G/C_G(H_3) \le \mathrm{Aut}(H_3)$ and $t(G) > 1$, we have $\pi(C_G(H_3)) = \pi(G) \setminus 13$. In particular, $7 \in \pi(C_G(H_3))$. Consider the action of an element of order 13 on $C_G(H_3)$. Using Lemmas 2.6 and 2.7, $91 \in \pi_e(G)$, which implies that $t(G) = 1$, a contradiction. Therefore, $K/H \cong S$, hence $G \cong S$.
Step 2. If $S = Fi_{23}$ or Fi'_{24}, then $G \cong S$.

Note that $|Fi_{23}| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $|Fi'_{24}| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, similarly we conclude $13, 17, 23 \in \pi(K/H)$ if $S = Fi_{23}$, and $17, 23, 29 \in \pi(K/H)$ if $S = Fi'_{24}$. By the same reason, checking the list in [20], we get that K/H can only be isomorphic to S, so that $G \cong S$. □

Lemma 2.19. *Let G be a group and S a Monster group or Baby Monster group. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. Because $|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ and $|B| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, by assumptions, $t(G) > 1$, apply Corollary 2.2 and Lemma 2.5, we come to that, if $S = M$, then $41, 47, 59, 71 \in \pi(K/H)$; if $S = B$, then $31, 47 \in \pi(K/H)$. By Lemma 2.14 and [20], we get that K/H can only be isomorphic to S. Consequently, $G \cong S$. □

Lemma 2.20. *Let G be a group and S one of the groups: Ly, $O'N$, M^cL, Th, HN, He, Ru, HS, and Suz. If $|G| = |S|$ and the prime graph of G is disconnected, then $G \cong S$.*

Proof. We have the following steps.
Step 1. If $S = Ly$, $O'N$, or Th, then $G \cong Ly$.

In this case, $|G|$ is one of $|Ly| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, $|O'N| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ and $|Th| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. Similarly, we see that: If $S = Ly$, then $37, 67 \in \pi(K/H)$; if $S = O'N$ or Th, then

$19, 31 \in \pi(K/H)$. In either case, we get that $K/H \cong S$ by [20] and Lemma 2.14. Hence, $G \cong S$.

Step 2. If $S = M^cL$, then $G \cong M^cL$.

$|G| = |M^cL| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ indicates $7, 11 \in \pi(K/H)$. Applying Lemma 2.14 and by [20], we see that K/H is one of: M^cL, $M_{22}(|M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, |\mathrm{Out(M_{22})}| = 2)$; $A_{11}(|A_{11}| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11, |\mathrm{Out(A_{11})}| = 2)$. If $K/H \cong M^cL$, then $G \cong S$, we are done. If $K/H \cong A_{11}$ or M_{22}, by calculating, we get that $|H_5| = 5$ or 5^2. But in either case, we can get that $t(G) = 1$ by $G/C_G(H_5) \leq \mathrm{Aut}(H_5)$, a contradiction to our hypothesis.

Step 3. If $S = HN$, then $G \cong HN$.

Since $|G| = |HN| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$, we have $11, 19 \in \pi(K/H)$. Then apply Lemma 2.14 and [20], we get that, $K/H \cong HN$ or J_1. If $K/H \cong HN$, then $G \cong S$ as desired. If $K/H \cong J_1$, since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $|\mathrm{Out(J_1)}| = 2$, it follows by Corollary 2.2 that $|H|_3 = 3^5$. Note that $|\mathrm{GL}(5, 3)| = 2^9 \cdot 3^{10} \cdot 5 \cdot 13$, use the same method, we get that $t(G) = 1$, a contradiction. Hence $K/H \cong S$, so $G \cong S$.

Step 4. If $S = He$, then $G \cong He$.

In this case, $|G| = |He| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$, so that $17 \in \pi(K/H)$. By [20] and Lemma 2.14, we get that K/H is one of the groups: He; $L_2(16)(|L_2(16)| = 2^4 \cdot 3 \cdot 5 \cdot 17, |\mathrm{Out(L_2(16))}| = 4)$; $L_2(17)(|L_2(17)| = 2^4 \cdot 3^2 \cdot 17, |\mathrm{Out(L_2(17))}| = 2)$; $S_4(4)(|S_4(4)| = 2^8 \cdot 3^2 \cdot 5 \cdot 17, |\mathrm{Out(S_4(4))}| = 2)$.

Suppose that K/H is isomorphic to one of the groups except He, then $|H_3| = 3$ or 3^2, this yields $t(G) = 1$, a contradiction. Now we have $K/H \cong He$, hence $G \cong S$.

Step 5. If $S = Ru$, then $S \cong Ru$.

By $|G| = |Ru| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$, we have $29 \in \pi(K/H)$. Lemma 2.14 and [20] show that $K/H \cong Ru$ or $L_2(29)(|L_2(29)| = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29, |\mathrm{Out(L_2(29))}| = 2)$. If $K/H \cong L_2(29)$, then $|H_{13}| = 13$, hence $t(G) = 1$, a contradiction. Thus $K/H \cong Ru$, so $G \cong S$, as desired.

Step 6. If $S = HS$, then $G \cong HS$.

In this case, $|G| = |HS| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, thus, $7, 11 \in \pi(K/H)$. Applying Lemma 2.14 and [20], we get that $K/H \cong HS$ or $M_{22}(|M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, |\mathrm{Out(M_{22})}| = 2)$. If $K/H \cong M_{22}$, then $G/H \leq \mathrm{Aut}(K/H)$ imply $|H_5| = 5^2$. By $G/C_G(H_5) \leq \mathrm{Aut}(H_5)$, we get that $t(G) = 1$, a contradiction. Thus $K/H \cong HS$, so $G \cong S$.

Step 7. If $S = Suz$, then $G \cong Suz$.

In this case, $|G| = |Suz| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, it follows by $t(G) = 1$ and Corollary 2.2 and Lemma 2.5 that $7, 11, 13 \in \pi(K/H)$. By the same reason, we get that $K/H \cong Suz$ or $A_{13}(|A_{13}| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, |\mathrm{Out(A_{13})}| = 2)$. Again by $t(G) = 1$, we get that $K/H \cong Suz$, so that $G \cong S$, we are done.

The lemma holds by Step 1-7. □

2.3 Conclusion

Proof of the Main Theorem. The Main Theorem follows from Lemma 2.8-2.13 and 2.15−2.20.

Acknowledgements

Supported by the National Natural Science Foundation of China (Grant No. 12071376, 11971391), Fundamental Research Funds for the Central Universities (No. XDJK2019B030).

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Accepted: December 24, 2020