Two equations in unequal powers of primes and powers of 2

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Abstract. In this paper, we obtained when $k = 176$, every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of two prime squares, two prime cubes, two prime fourth powers and k powers of 2.
Keywords: Hardy-Littlewood method, Goldbach-Linnik problem, powers of 2.

1. Introduction

As an approximation to Goldbach's problem Linnik proved in 1951 [8] under the assumption of the Generalized Riemann Hypothesis (GRH), and later in 1953 [9] unconditionally, that each large even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2, namely

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}.$$

In 2002, Heath-Brown and Puchta [1] applied a rather different approach to this problem and showed that $k = 13$ and, on the GRH, $k = 7$. In 2003, Pintz and Ruzsa [16] established this latter result and announced that $k = 8$ is acceptable unconditionally. This paper is yet to appear in print. Elsholtz, in an unpublished manuscript, showed that $k = 12$; this was proved independently by Liu and Lü [15].

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In 1999, Liu, Liu and Zhan [11] proved that every large even integer N can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}.$$

Subsequently Liu and Liu [10] got that $k = 8330$ suffices. Later Liu and Lü [12] improved the value of k of (1.2) to 165, Li [7] improved it to 151 and Zhao [18] improved it to 46. Finally Platt and Trudgian [17] revised it to 45. In 2011, Liu and Lü [14] considered the problem on the representation of large even integers N in the form

$$N = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}.$$

They showed that $k = 161$ is acceptable and Zhao [19] improved it to 18. Also in 2017, Liu [13] construct that every large even integers N can be represented in the form

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}. \tag{1.1}$$

He showed that $k = 41$ is acceptable.

As a comparison, Kong [5] first considered the result on pairs of linear equations in four prime variables and powers of 2, in the form

$$\begin{cases} N_1 = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_3 + p_4 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

where k is a positive integer. She proved that the simultaneous equations are solvable for $k = 63$. Then Platt and Trudgian [17] and Kong and Liu [6] revised it to 62 and 34. Later, Hu and Liu [2], Hu and Yang [3][4] considered other equations

$$\begin{cases} N_1 = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5^2 + p_6^2 + p_7^2 + p_8^2 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

$$\begin{cases} N_1 = p_1 + p_2^2 + p_3^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_4 + p_5^2 + p_6^2 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

$$\begin{cases} N_1 = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5 + p_6^2 + p_7^3 + p_8^3 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases}$$

and proved that these equations are solvable for $k = 142, 332, 455$.

In this paper, comparing to (1.1), we shall consider the simultaneous representation of pairs of positive even integers $N_2 \gg N_1 > N_2$, in the form

$$\begin{cases} N_1 = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_7^2 + p_8^2 + p_9^3 + p_{10}^3 + p_{11}^4 + p_{12}^4 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases} \tag{1.2}$$

where k is a positive integer. Our result is stated as follows.

Theorem 1.1. *For $k = 176$, the equations* (1.2) *are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

Our proof of Theorem 1.1 uses the Hardy-Littlewood circle method and draws on the strategies adopted in the works of Hu and Yang [4] and Liu [13].

Throughout this paper, the letter ϵ denotes a positive constant which is arbitrarily small but may not the same at different occurrences. And p and v denote a prime number and a positive integer, respectively.

2. Outline of the method

Here we give an outline for the proof of Theorem 1.1.

Throughout, we assume that $N_i, i = 1, 2$ are sufficient large even integers. Then we set

$$U_i = \sqrt{(1-\eta)N_i}, \quad V_i = (\eta N/2)^{1/3}, \quad W_i = (\eta N/2)^{1/4}$$

for $i = 1, 2$, where η is a small positive constant. We set

$$\begin{aligned} f(\alpha_i, U_i) &= \sum_{U_i/2 < p \leqslant U_i} (\log p) e(p^2 \alpha_i), \\ g(\alpha_i, V_i) &= \sum_{V_i/2 < p \leqslant V_i} (\log p) e(p^3 \alpha_i), \\ h(\alpha_i, W_i) &= \sum_{W_i/2 < p \leqslant W_i} (\log p) e(p^4 \alpha_i), \end{aligned} \tag{2.1}$$

$$G(\alpha_i) = \sum_{v \leqslant L} e(2^v \alpha_i), \quad \mathcal{E}_\lambda := \{\alpha_i \in [0,1] : \ |G(\alpha_i)| \geqslant \lambda L\}. \tag{2.2}$$

where $i = 1, 2$, $e(x) := \exp(2\pi i x)$ and $L = \log_2 N_1$.

In order to apply the circle method, we set

$$P_i = N_i^{3/20 - 2\epsilon}, \quad Q_i = N_i^{17/20+\epsilon}, \tag{2.3}$$

for $i = 1, 2$. For any integers a_1, a_2, q_1, q_2 satisfying

$$1 \leqslant a_1 \leqslant q_1 \leqslant P_1, (a_1, q_1) = 1, 1 \leqslant a_2 \leqslant q_2 \leqslant P_2, (a_2, q_2) = 1,$$

we define the major arcs $\mathcal{M}_1$, $\mathcal{M}_2$ and minor arcs $C(\mathcal{M}_1)$, $C(\mathcal{M}_2)$ as usual, namely

$$\mathcal{M}_i = \bigcup_{q_i \leqslant P_i} \bigcup_{\substack{1 \leqslant a_i \leqslant q_i \\ (a_i, q_i) = 1}} \mathcal{M}_i(a_i, q_i), \quad C(\mathcal{M}_i) = \left[\frac{1}{Q_i}, 1 + \frac{1}{Q_i}\right] \backslash \mathcal{M}_i, \tag{2.4}$$

where $i = 1, 2$ and

$$\mathcal{M}_i(a_i, q_i) = \left\{ \alpha_i \in [0,1] : \left| \alpha_i - \frac{a_i}{q_i} \right| \leqslant \frac{1}{q_i Q_i} \right\}.$$

It follows from $2P_i \leqslant Q_i$ that the arcs $\mathcal{M}_1(a_1, q_1)$ and $\mathcal{M}_2(a_2, q_2)$ are mutually disjoint respectively.

Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_{12}$$

be the weighted number of solutions of (1.2) in $(p_1, \cdots, p_{12}, v_1, \cdots, v_k)$ with

$$\begin{aligned}
&p_1, p_2 \sim U_1, \quad p_3, p_4 \sim V_1, \quad p_5, p_6 \sim W_1, \\
&p_7, p_8 \sim U_2, \quad p_9, p_{10} \sim V_2, \quad p_{11}, p_{12} \sim W_2, \\
&v_j \leqslant L,
\end{aligned}$$

for $j = 1, 2, \cdots, k$. Then $R(N_1, N_2)$ can be written as

$$\begin{aligned}
R(N_1, N_2) = &\int_0^1 \int_0^1 f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\
&\times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\
&= \left\{ \int_{\mathcal{M}_1} + \int_{C(\mathcal{M}_1) \bigcap \mathcal{E}_\lambda} + \int_{C(\mathcal{M}_1) \setminus \mathcal{E}_\lambda} \right\} \left\{ \int_{\mathcal{M}_2} + \int_{C(\mathcal{M}_2) \bigcap \mathcal{E}_\lambda} + \int_{C(\mathcal{M}_2) \setminus \mathcal{E}_\lambda} \right\} \\
&\times f^2(\alpha_1, U_1) g^2(\alpha_1, V_1) h^2(\alpha_1, W_1) f^2(\alpha_2, U_2) g^2(\alpha_2, V_2) \\
&\times h^2(\alpha_2, W_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\
&:= \sum_{s=1}^{3} \sum_{t=1}^{3} R_{st}(N_1, N_2),
\end{aligned}$$

where $R_{st}(N_1, N_2)$ denotes the combination of s-th term in the first bracket and the t-th term in the second bracket.

We will establish Theorem 1.1 by estimating the term $R_{st}(N_1, N_2)$ for all $1 \leqslant s, t \leqslant 3$. We need to show that $R(N_1, N_2) > 0$ for every pair of sufficiently large even positive integers $N_2 \gg N_1 > N_2$.

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1. *Let* $\mathcal{E}_\lambda = \{\alpha \in (0,1] : |G(\alpha) \geqslant \lambda L|\}$. *We have* $\operatorname{meas}(\mathcal{E}_\lambda) \ll N_i^{-E(\lambda)}$, *with* $E(0.90365) > 131/168 + 10^{-10}$.

Proof. This is Lemma 2.3 in Liu [13]. □

Let

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q), \tag{2.5}$$

where

$$A(n,q)=\frac{1}{\varphi^6(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}C_2^2(q,a)C_3^2(q,a)C_4^2(q,a)e\left(-\frac{an}{q}\right),$$

$$C_k(q,a)=\sum_{\substack{m=1\\(m,q)=1}}^{q}e\left(\frac{am^k}{q}\right).$$

Lemma 2.2. *Let $\mathcal{M}_i$ be as in (2.4), with P_i and Q_i determined by (2.3). Then for $2\leqslant n\leqslant N_i$, we have*

$$\int_{\mathcal{M}_i}f^2(\alpha_i,U_i)g^2(\alpha_i,V_i)h^2(\alpha_i,W_i)e(-\alpha n)\mathrm{d}\alpha=\frac{1}{2^2\cdot 3^2\cdot 4^2}\mathfrak{S}(n)\mathfrak{J}(n)+O(N_i^{7/6}L^{-1}).$$

Here, the singular series $\mathfrak{S}(n)$ is defined as in (2.5) and satisfies $\mathfrak{S}(n)\gg 1$ for $n\equiv 0(\text{mod }2)$. $\mathfrak{J}(n)$ is defined as

$$\mathfrak{J}(n):=\sum_{\substack{m_1+\ldots+m_6=n\\(U_i/2)^2<m_1,m_2\leqslant U_i^2\\(V_i/2)^2<m_3,m_4\leqslant V_i^2\\(W_i/2)^2<m_5,m_6\leqslant W_i^2}}(m_1m_2)^{-1/2}(m_3m_4)^{-2/3}(m_5m_6)^{-3/4},$$

and satisfies $N_i^{7/6}\ll\mathfrak{J}(n)\ll N_i^{7/6}$.

Proof. This is Lemma 2.1 in Liu [13]. □

Lemma 2.3. *For all integers $n\equiv 0(\text{mod }2)$, we have $\mathfrak{S}(n)>1.072808$.*

Proof. This result can be found in Section 3 in Liu [13]. □

Lemma 2.4. *Let $\mathcal{B}(N_i,k)=\{(1-\eta)N_i\leqslant n_i\leqslant N_i: n_i=N_i-2^{v_1}-\cdots-2^{v_k}\}$ with $k\geqslant 2$. Then, for $N_1\equiv N_2\equiv 0(\text{mod }2)$, we have*

$$\sum_{\substack{n_1\,\in\,\mathcal{B}(N_1,\,k)\\n_2\,\in\,\mathcal{B}(N_2,\,k)\\n_1\,\equiv\,n_2\,\equiv\,0(\text{mod }2)}}\mathfrak{J}(n_1)J(n_2)\geqslant 57.877N_1^{7/6}N_2^{7/6}L^k.$$

Proof. Using [13, Lemma 4.2], we have

$$\sum_{\substack{n_1\,\in\,\mathcal{B}(N_1,\,k)\\n_2\,\in\,\mathcal{B}(N_2,\,k)\\n_1\,\equiv\,n_2\,\equiv\,0(\text{mod }2)}}\mathfrak{J}(n_1)\mathfrak{J}(n_2)\geqslant(7.607695)^2N_1^{7/6}N_2^{7/6}\sum_{((v))}1,$$

where $((v))$ means that $v_1,\cdots,v_k$ satisfies

$$1\leqslant v_1,\cdots,v_k\leqslant L,\quad 2^{v_1}+\cdots+2^{v_k}\equiv N_1(\text{mod }2).$$

Then following the argument of [13, Lemma 4.1], we have

$$\sum_{((v))} 1 \geqslant (1-\epsilon)L^k.$$

Then we get the proof of this lemma. □

Lemma 2.5. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$, be defined by (2.1), $C(\mathcal{M}_i)$ by (2.4). Then*

$$\sup_{\alpha \in C(\mathcal{M}_i)} |f(\alpha_i, U_i)| \ll N_i^{7/16+\epsilon}, \qquad \sup_{\alpha \in C(\mathcal{M}_i)} |g(\alpha_i, V_i)| \ll N_i^{13/42+\epsilon}.$$

Proof. The proof of this lemma can be found in [13, Lemma 2.4],which is based on the estimate of exponential sums over primes. □

Lemma 2.6. *Let $f(\alpha_i, U_i)$, $g(\alpha_i, V_i)$ and $h(\alpha_i, W_i)$ be defined by (2.1). Then we have*

$$\int_0^1 |f(\alpha, U_i)g(\alpha, V_i)h(\alpha, W_i)|^2 \mathrm{d}\alpha_i \leqslant 0.8485 N_i^{7/6}.$$

Proof. This is Lemma 2.2 in [13]. □

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

We begin with the estimate for $R_{11}(N_1, N_2)$. Applying Lemmas 2.2, 2.3 and 2.4 and introducing the notation $\mathcal{B}(N_i, k)$, we can get

$$\begin{aligned}
&R_{11}(N_1, N_2) \\
&= \int_{\mathcal{M}_1} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^k(\alpha_1)e(-\alpha_1 N_1)\mathrm{d}\alpha_1 \\
&\times \int_{\mathcal{M}_2} f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^k(\alpha_2)e(-\alpha_2 N_2)\mathrm{d}\alpha_2 \\
&= \sum_{\substack{n_1 \in \mathcal{B}(N_1,k) \\ n_2 \in \mathcal{B}(N_2,k)}} \int_{\mathcal{M}_1} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)e(-\alpha_1 n_1)\mathrm{d}\alpha_1 \qquad (3.1) \\
&\times \int_{\mathcal{M}_2} f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)e(-\alpha_2 n_2)\mathrm{d}\alpha_2 \\
&\geqslant \left(\frac{1}{2^2 \cdot 3^2 \cdot 4^2}\right)^2 \sum_{\substack{n_1 \in \mathcal{B}(N_1,k) \\ n_2 \in \mathcal{B}(N_2,k)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)J(n_1)J(n_2) + O(N_1^{7/6}N_2^{7/6}L^{k-1}) \\
&\geqslant \frac{1}{331776} \cdot (1.072808)^2 \cdot 57.8771 N_1^{7/6}N_2^{7/6}L^k,
\end{aligned}$$

where we used $\frac{n_i}{N_i} = 1 + O(L^{-1})$ for $n_i \in \mathcal{B}(N_i, k)$.

Now, we turn to give an upper bound for $R_{12}(N_1, N_2)$. The estimate for $R_{21}(N_1, N_2)$ is similar. By Cauchy's inequality, we can get $|G(\alpha_1 + \alpha_2)| \leqslant \sqrt{|G(2\alpha_1)G(2\alpha_2)|}$. For $\alpha_2 \in C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda$ and sufficiently large N_1, we have

$$|G(2\alpha_i)| \leqslant |G(\alpha_i)| + 2 \leqslant \lambda L + 2 \leqslant (1 + o(1))\lambda L.$$

Then, using the definition of $\mathcal{E}_\lambda$, the trivial bound of $G(\alpha_i)$, Lemmas 2.1, 2.5 and 2.6, we have

$$\begin{aligned}
&R_{12}(N_1, N_2)\\
&= \int_{\mathcal{M}_1}\int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)\\
&\times h^2(\alpha_2, W_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\
&\ll \int_{\mathcal{M}_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^{k/2}(2\alpha_1)|\mathrm{d}\alpha_1\\
&\times \int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2\\
&\ll L^{k/2-2}\int_{\mathcal{M}_1} |f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)G^2(2\alpha_1)|\mathrm{d}\alpha_1\\
&\times \int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)h^2(\alpha_2, W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2\\
&\ll N_1^{7/6}L^{k/2}L^{k/2}\Bigg(\max_{\alpha_2\in C(\mathcal{M}_2)} |f(\alpha_2, U_2)g^2(\alpha_2, V_2)|\\
&\times \left(\int_0^1 |f^2(\alpha_2, U_2)g^2(\alpha_2, V_2) \times h^2(\alpha_2, W_2)|\mathrm{d}\alpha_2\right)^{1/2}\\
&\times \left(\int_0^1 |f^2(\alpha_2, U_2)h^4(\alpha_2, W_2)|\mathrm{d}\alpha_2\right)^{1/2}\Bigg)^{2/3}\\
&\times \left(\int_{\mathcal{E}_\lambda} 1d\alpha_2\right)^{1/3} \ll N_1^{7/6}L^k N_2^{719/504+\epsilon}N_2^{-E(\lambda)/3} \ll N_1^{7/6}N_2^{7/6}L^{k-1}.
\end{aligned} \tag{3.2}$$

Similarly, we can get

$$R_{21}(N_1, N_2) \ll N_1^{7/6}N_2^{7/6}L^{k-1}. \tag{3.3}$$

Next, we give an upper bound for $R_{13}(N_1, N_2)$. By Lemma 2.6, using the trivial bound $|G(2\alpha)| \leqslant L$ when $\alpha \in \mathcal{M}_1$ and the bound $|G(2\alpha)| \leqslant (1 + o(1))\lambda L$ when $\alpha \in C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda$, we have

$$\begin{aligned}
&|R_{13}(N_1, N_2)|\\
&= \Bigg|\int_{\mathcal{M}_1}\int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} f^2(\alpha_1, U_1)g^2(\alpha_1, V_1)h^2(\alpha_1, W_1)f^2(\alpha_2, U_2)g^2(\alpha_2, V_2)\\
&\times h^2(\alpha_2, W_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{\mathcal{M}_1} |f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)G^{k/2}(2\alpha_1)|\mathrm{d}\alpha_1 \\
&\times \int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} |f^2(\alpha_2,U_2)g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2 \\
&\leqslant L^{k/2}\int_{\mathcal{M}_1} |f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)|\mathrm{d}\alpha_1 \\
&\times (\lambda L)^{k/2}\int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} |f^2(\alpha_2,U_2)g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)|\mathrm{d}\alpha_2 \\
&\leqslant (0.8485)^2\lambda^{k/2}N_1^{7/6}N_2^{7/6}L^k.
\end{aligned} \tag{3.4}$$

We can obtain the estimate for $R_{31}(N_1,N_2)$ analogously,

$$|R_{31}(N_1,N_2)| \leqslant (0.8485)^2\lambda^{k/2}N_1^{7/6}N_2^{7/6}L^k. \tag{3.5}$$

Similar with the estimate of $R_{12}(N_1.N_2)$, We give the estimate for $R_{22}(N_1,N_2)$ by the trivial bound for $G(\alpha)$, Lemma 2.1, 2.5, 2.6 and the definition of $\mathcal{E}_\lambda$,

$$\begin{aligned}
&R_{22}(N_1,N_2) \\
&= \int_{C(\mathcal{M}_1)\bigcap\mathcal{E}_\lambda}\int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)f^2(\alpha_2,U_2) \\
&\times g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
&\ll \int_{C(\mathcal{M}_1)\bigcap\mathcal{E}_\lambda} |f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)G^{k/2}(2\alpha_1)|\mathrm{d}\alpha_1 \\
&\times \int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} |f^2(\alpha_2,U_2)g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2 \\
&\ll N_1^{7/6}L^{k/2-1}N_2^{7/6}L^{k/2-1} \ll N_1^{7/6}N_2^{7/6}L^{k-1}.
\end{aligned} \tag{3.6}$$

For $R_{23}(N_1,N_2)$, we can easily get

$$\begin{aligned}
&R_{23}(N_1,N_2) \\
&= \int_{C(\mathcal{M}_1)\bigcap\mathcal{E}_\lambda}\int_{C(\mathcal{M}_2)\bigcap\mathcal{E}_\lambda} f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)f^2(\alpha_2,U_2) \\
&\times g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
&\ll \int_{C(\mathcal{M}_1)\bigcap\mathcal{E}_\lambda} |f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)G^{k/2}(2\alpha_1)|\mathrm{d}\alpha_1 \\
&\times \int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} |f^2(\alpha_2,U_2)g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2 \\
&\ll N_1^{7/6}L^{k/2-1}N_2^{7/6}L^{k/2} \ll N_1^{7/6}N_2^{7/6}L^{k-1}.
\end{aligned} \tag{3.7}$$

Similarly, we have

$$R_{32}(N_1,N_2) \ll N_1^{7/6}N_2^{7/6}L^{k-1}. \tag{3.8}$$

In the end, we provide the upper bound for $R_{33}(N_1, N_2)$.

$$
\begin{aligned}
&|R_{33}(N_1,N_2)| \\
&= \bigg| \int_{C(\mathcal{M}_1)\backslash\mathcal{E}_\lambda} \int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)f^2(\alpha_2,U_2) \\
&\times g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\bigg| \\
&\leqslant \int_{C(\mathcal{M}_1)\backslash\mathcal{E}_\lambda} |f^2(\alpha_1,U_1)g^2(\alpha_1,V_1)h^2(\alpha_1,W_1)G^{k/2}(2\alpha_1)|\mathrm{d}\alpha_1 \\
&\times \int_{C(\mathcal{M}_2)\backslash\mathcal{E}_\lambda} |f^2(\alpha_2,U_2)g^2(\alpha_2,V_2)h^2(\alpha_2,W_2)G^{k/2}(2\alpha_2)|\mathrm{d}\alpha_2 \\
&\leqslant \left((\lambda L)^{k/2}\times 0.8485N_1^{7/6}\right)\left((\lambda L)^{k/2}\times 0.8485N_2^{7/6}\right) \\
&\leqslant \lambda^k(0.8485)^2N_1^{7/6}N_2^{7/6}L^k.
\end{aligned} \tag{3.9}
$$

Combining (3.1)-(3.9), we can obtain

$$
\begin{aligned}
R(N_1,N_2) &> R_{11}(N_1,N_2) - R_{13}(N_1,N_2) - R_{31}(N_1,N_2) - R_{33}(N_1,N_2) \\
&\quad + O(N_1^{7/6}N_2^{7/6}L^{k-1}) \\
&> \frac{1}{331776}\cdot(1.072808)^2\cdot 57.8771N_1^{7/6}N_2^{7/6}L^k \\
&\quad - 2\times(0.8485)^2\lambda^{k/2}N_1^{7/6}N_2^{7/6}L^k \\
&\quad - \lambda^k(0.8485)^2N_1^{7/6}N_2^{7/6}L^k + O(N_1^{7/6}N_2^{7/6}L^{k-1}).
\end{aligned}
$$

We therefore solve the inequality $R(N_1, N_2) > 0$ and get $k \geqslant 176$. Consequently we deduce that every pair of large even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$ and $N_1 \equiv N_2 \equiv 0 (\mathrm{mod}\ 2)$ can be written in the form of (1.3) for $k \geqslant 176$. Thus, Theorem 1.1 follows.

Acknowledgements

This work is supported by Social Science Planning Project of Jiangxi Province of 2020 Annual (Grant Nos. 20JY06), Tender subject for key research base of Humanities and Social Sciences in Colleges and universities in Jiangxi Province (Grant Nos.JD20109), Culture and Art Science Planning Project of Jiangxi Province of 2020 Annual (Grant Nos. YG2020129) , Basic education research project of Jiangxi Province (Grant Nos. SZUNDSX2019-984) and Natural Science Foundation of Jiangxi Province for Distinguished Young Scholars (Grant Nos. 20212ACB211007).

The authors would like to express their thanks to the referee for many useful suggestions and comments on the manuscript.

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Accepted: November 04, 2020