Composition of Abel-Grassmann's strong root of band

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Abstract. As the role of band in the study of semigroups, AG-band plays an important role in the study of AG-groupoids. In this paper, the concept of strong root of band which is an extension of AG-band, is proposed and its structure is studied in depth. We investigate decomposition of strong root of band induced by an equivalence relation and prove that every strong root of band is the disjoint union of its sub-AG-groups. Furthermore, two different ways how to make a strong root of band are obtained which illuminate the structure of strong root of band.
Keywords: strong root of band, root of band, AG-3-band, AG-4-band, AG-band.

1. Introduction

The concept of an Abel-Grassmann's groupoid (AG-groupoid) was first given by Kazim and Naseeruddin [1], in 1972. Moreover, further AG-groupoid theoretical studies can be found in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and some important results have been gotten.

As in Semigroup Theory, bands and band decompositions [15, 16, 17, 18] are one of the most useful method for research on AG-groupoids. In [16], the concept of AG-3-band was introduced and its basic properties are investigated. AG-3-band is the disjoint union of its sub Abelian groups. In [18], the concept of root of band and unipotent radical were introduced, and the decomposition theorem of root of band was proved. As a continuation of [18], we propose strong root of band, study its properties, and analyze its relationship with other AG-groupoid bands.

The rest of this paper is arranged as follows. In Section 2, some definitions and properties on AG-groupoid bands are given. The relationships among strong

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root of band and other AG-groupoid bands are discussed in Section 3. We establish the decomposition theorem and construction methods of strong root of band in Section 4. Lastly, Section 5 presents the summary and the direction of future efforts.

2. Preliminaries

This section contains the related research and results of AG-groupoid bands and some related notions are introduced first.

An AG-groupoid is a generalization of commutative semigroup. A groupoid $(G, *)$ is called an AG-groupoid if it holds the left invertive law, that is, for all $a, b, c \in G$, $(a*b)*c = (c*b)*a$. In an AG-groupoid $(G, *)$ the medial law holds, for all $a, b, c, d \in G$, $(a*b)*(c*d) = (a*c)*(b*d)$. In an AG-groupoid $(G, *)$, $\forall x \in G$, $k \in Z^+$, x^k is defined as follows: $x^1 = x, x^2 = x*x, x^3 = x^2*x, x^4 = x^3*x, ..., x^k = x^{k-1}*x$.

Let G be an AG-groupoid, then a is an idempotent in G if $a \in G$, $a^2 = a$. The set of all idempotents in G is denoted by $E(G)$. An AG-groupoid G is called an AG-band if $G = E(G)$. For an AG-groupoid G we define the set $\sqrt{E(G)} = \{a \in G : a^2 * a^2 = a^2\}$.

Definition 2.1 ([18]). *An AG-groupoid G with a unique idempotent $e \in G$ is called a unipotent radical if for all $a \in G$, $a^2 = e$.*

Definition 2.2 ([18]). *The AG-groupoid G is called a root of band if $\sqrt{E(G)} = G$.*

Definition 2.3 ([16]). *An AG-groupoid $(G, *)$ is called an AG-3-band, if for all $a \in G$, $a^2 * a = a * a^2 = a$.*

Definition 2.4 ([19]). *An AG-groupoid G with a left identity e is called an AG-group if every $a \in G$ has an inverse element a'.*

Definition 2.5 ([18]). *An AG-groupoid $(G, *)$ is called an AG-4-band, if for all $a \in G$, $a^3 * a = a * a^3 = a^2 * a^2 = a * (a * a^2) = (a * a^2) * a = a^2$, that is, all products of a of length 4 are equal to a^2.*

3. The relationships among strong root of band and other bands

Definition 3.1. *An AG-groupoid $(G, *)$ is called a strong root of band if it satisfies: $\forall a \in G, a^2 * a^2 = a^2, a^2 * a = a$.*

Theorem 3.1. *Let $(G, *)$ be a groupoid. Then, G is an AG-3-band if it is both a strong root of band and an AG-4-band.*

Proof. Suppose G is both a strong root of band and an AG-4-band, from Definition 3.1 and Definition 2.5, for any $a \in G$, $a^3 * a = a * a^3 = a^2 * a^2 = a * (a * a^2) = (a * a^2) * a = a^2$, $a^2 * a = a$.

we have

$$\begin{aligned} a * a^2 &= (a^2 * a) * (a^2 * a^2) \\ &= (a^2 * a^2) * (a * a^2) \quad (\textit{by the medial law}) \\ &= a^2 * (a * a^2) \\ &= (a * a) * (a * a^2) \\ &= ((a * a^2) * a) * a \quad (\textit{by the left invertive law}) \\ &= a^2 * a = a. \end{aligned}$$

By Definition 2.3, G is an AG-3-band. □

Theorem 3.2. *Let $(G, *)$ be a root of band. Then, G is an AG-3-band if it satisfies: $\forall a \in G, a * a^2 = a$.*

Proof. Suppose G is a root of band, from Definition 2.2, for any $a \in G$, $a^2 * a^2 = a^2$. Being $a * a^2 = a$, we have $a^2 * a = (a^2 * a^2) * a = (a * a^2) * a^2 = a * a^2 = a$. By Definition 2.3, G is an AG-3-band. □

From Definition 2.2, Definition 2.5 and Definition 3.1, it is easy to see that both an AG-4-band and a strong root of band are a root of band. The following example illustrates that a root of band may be neither an AG-4-band nor a strong root of band.

Example 3.1. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 1. From Definition 2.2 G is a root of band. However, $(2*2^2)*2 = 3 \neq 2^2$, $4^2 * 4 = 3 \neq 4$, from Definition 2.5 and Definition 3.1 it is neither an AG-4-band nor a strong root of band.

Table 1: The operation table of Example 3.1.

$*$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	2	3	3	1	1	1
2	3	1	2	2	3	3	3
3	2	3	1	1	2	2	2
4	2	3	1	1	2	2	2
5	1	2	3	3	5	1	1
6	1	2	3	3	1	6	1
7	1	2	3	3	1	1	7

From Definition 2.5 and Definition 3.1, the notions of AG-4-band and strong root of band are different. The following examples illustrate that an AG-4-band may not be a strong root of band and a strong root of band may not be an AG-4-band.

Example 3.2. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 2. From Definition 2.5 G is an AG-4-band. However, $2^2 * 2 = 1 \neq 2$, thus, From Definition 3.1 it is not a strong root of band.

Table 2: The operation table of Example 3.2.

*	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	4	1	1	1
2	1	1	3	4	1	1	3
3	1	2	1	4	1	1	2
4	4	4	4	1	4	4	4
5	1	1	1	4	5	1	1
6	1	1	1	4	1	6	1
7	1	2	3	4	1	1	7

Example 3.3. Let $G = \{1, 2, 3, 4, 5, 6, 7\}$, the definition of operation $*$ on G is shown in Table 3. From Definition 3.1 G is a strong root of band. However, $(3 * 3^2) * 3 = 4 \neq 3^2$, thus, From Definition 2.5 it is not an AG-4-band.

Table 3: The operation table of Example 3.3.

*	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	3	4	1	1	1
2	1	2	3	4	1	1	2
3	4	4	1	3	4	4	4
4	3	3	4	1	3	3	3
5	1	1	3	4	5	1	1
6	1	1	3	4	1	6	6
7	1	2	3	4	1	6	7

Figure 1 can be used to express the relationships among strong root of band and other bands. Here, A represents an AG-band; B represents an AG-3-band, but it is not an AG-band; as shown in Example 3.2, C represents an AG-4-band, but it is not a strong root of band; as shown in Example 3.3, D represents a strong root of band, but it is not an AG-4-band; and as shown in Example 3.1, E represents a root of band, but it is neither a strong root of band nor an AG-4-band. $A + B$ represents an AG-3-band, $A + B + C$ represents an AG-4-band, $A + B + D$ represents a strong root of band, and $A + B + C + D + E$ represents a root of band.

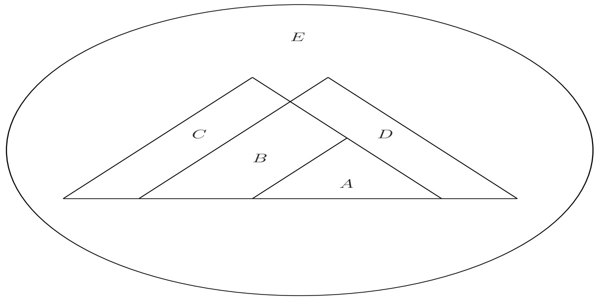

Figure 1: The relationships among strong root of band and other bands.

4. Band decomposition of strong root of band

In [18], a relation $\approx$ on arbitrary AG-groupoids was introduced as follows,

$$a \approx b \Leftrightarrow a^2 = b^2. \tag{1}$$

From the binary $\approx$ definition, it is easy to verify that $\approx$ has the properties of reflexive, symmetric and transitive. Thus, it is an equivalence relation.

From now on we assume that G is an AG-groupoid. Let Y be an AG-band, $Y \subset G$, for any $\alpha \in Y$, the equivalent class contained α will be denoted by S_α, and the elements of S_α will be denoted by a_α, $b_\alpha, \ldots$.

Theorem 4.1. *Let G be an AG-groupoid, Y be an AG-band, $Y \subset G$. G is a strong root of band if and only if $G = \bigcup_{\alpha \in Y} S_\alpha$ where S_α is an AG-group satisfying $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) for all $a_\alpha \in S_\alpha$.*

Proof. Let G be an AG-groupoid, Y be an AG-band, $Y \subset G$. $\forall a_\alpha, b_\alpha, c_\alpha \in S_\alpha$, the left invertive law holds directly.

$\forall \alpha \in Y$, let $e_\alpha = \alpha$, we have $e_\alpha \in S_\alpha$ and e_α is unique idempotent of S_α.

$\forall a_\alpha, b_\alpha \in S_\alpha$, we have $a_\alpha^2 = b_\alpha^2 = e_\alpha$, $(a_\alpha * b_\alpha) * (a_\alpha * b_\alpha) = (a_\alpha * a_\alpha) * (b_\alpha * b_\alpha) = e_\alpha^2 = e_\alpha$. Thus, $a_\alpha * b_\alpha \in S_\alpha$.

$\forall a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha^2 = e_\alpha$, since G is a strong root of band, by Definition 3.1, we have $a_\alpha^2 * a_\alpha = a_\alpha$, $e_\alpha * a_\alpha = a_\alpha^2 * a_\alpha = a_\alpha$. Thus, e_α is a left identity of S_α and a_α is an inverse element of a_α. From Definition 2.4, S_α is a sub-AG-group of G. Since e_α is unique idempotent of S_α, $G = \bigcup_{\alpha \in Y} S_\alpha$.

Conversely, suppose that G is an AG-groupoid, Y is an AG-band, $Y \subset G$. If $G = \bigcup_{\alpha \in Y} S_\alpha$, S_α is an AG-group satisfying $a_\alpha^2 = e_\alpha$ for all $a_\alpha \in S_\alpha$. We are going to prove that G is a strong root of band.

Let $x \in G$ be an arbitrary element, then there exist $\alpha \in Y$ such that $x \in S_\alpha$. Since e_α is the left identity element of S_α, we have $x^2 = e_\alpha$ and $(x * x) * x = e_\alpha * x = x$. Moreover, $x^2 * x^2 = e_\alpha * e_\alpha = e_\alpha = x^2$, thus G is a strong root of band. □

Example 4.1. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the definition of operation $*$ on G is shown in Table 4. From Definition 3.1, G is a strong root of band. An

AG-band $Y = \{1, 6\}$ and $Y \subset G$. $S_1 = \{1, 2, 3, 4, 5\}$ and $S_6 = \{6, 7, 8\}$, both S_1 and S_6 are sub-AG-groups of G. It is easy to verify that $G = S_1 \cup S_6$.

Table 4: The operation table of Example 4.1.

$*$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	4	5	1	1	1
2	3	1	5	2	4	3	3	3
3	2	4	1	5	3	2	2	2
4	5	3	4	1	2	5	5	5
5	4	5	2	3	1	4	4	4
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	8	6	7
8	1	2	3	4	5	7	8	6

Theorem 4.2. *Let $(G, *)$ be a groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta$, $S_\alpha \bigcap S_\beta = \emptyset$. If $\forall a_\alpha \in S_\alpha, \forall b_\beta \in S_\beta$, $a_\alpha * b_\beta = a_\alpha * e_\alpha, b_\beta * a_\alpha = a_\alpha$, then G is a strong root of band.*

Proof. Suppose $G = \bigcup_{\alpha \in Y} S_\alpha$ is the groupoid, Y is an AG-band, for each $\alpha \in Y$, S_α is an AG-group and $S_\alpha \bigcap S_\beta = \emptyset$ if $\alpha \neq \beta$ in Y.

We first prove that $(G, *)$ is an AG-groupoid. Let $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$ and $c_\gamma \in S_\gamma$ be arbitrary elements. Suppose e_α is the left identity of S_α, being S_α is an AG-group, we have $a_\alpha * e_\alpha \in S_\alpha$. Moreover,

$$\begin{aligned}
(a_\alpha * b_\beta) * c_\gamma &= (a_\alpha * e_\alpha) * c_\gamma \\
&= (a_\alpha * e_\alpha) * e_\alpha \\
&= (e_\alpha * e_\alpha) * a_\alpha \quad (\textit{by the left invertive law}) \\
&= e_\alpha * a_\alpha \\
&= a_\alpha,
\end{aligned}$$

$(c_\gamma * b_\beta) * a_\alpha = b_\beta * a_\alpha = a_\alpha = (a_\alpha * b_\beta) * c_\gamma$. Since S_α is an AG-group, the left invertive law holds directly for elements $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$. Thus, G is an AG-groupoid.

Let $a_\alpha \in S_\alpha$ be an arbitrary element. Since e_α is the left identity of S_α and $a_\alpha^2 = e_\alpha$, it follows that $(a_\alpha * a_\alpha) * a_\alpha = e_\alpha * a_\alpha = a_\alpha$, and $a_\alpha^2 * a_\alpha^2 = e_\alpha * e_\alpha = e_\alpha = a_\alpha^2$. Consequently, G is a strong root of band. □

Example 4.2. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, the definition of operation $*$ on G is shown in Table 5. An AG-band $Y = \{1, 4, 7\}$ and $Y \subset G$. $S_1 = \{1, 2, 3\}$, $S_4 = \{4, 5, 6\}$, $S_7 = \{7, 8, 9, 10\}$, S_1, S_4 and S_7 are sub-AG-groups of G. It is easy to verify that $G = S_1 \cup S_4 \cup S_7$. For any $a_1 \in S_1, b_4 \in S_4, c_7 \in S_7$, without losing generality, let $a_1 = 2, b_4 = 5, c_7 = 9$, we can get $2 * 5 = 2 * 1, 5 * 2 = 1 * 2; 2 * 9 = 2 * 1, 9 * 2 = 1 * 2; 5 * 9 = 5 * 4, 9 * 5 = 4 * 5$; and $(2 * 5) * 9 = (9 * 5) * 2$. The other cases can be verified, thus G is an AG-groupoid. $\forall x \in G$, we have $x^2 * x^2 = x^2, x^2 * x = x$. From Definition 3.1, G is a strong root of band.

Table 5: The operation table of Example 4.2.

*	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	1	1	1	1	1	1	1
2	3	1	2	3	3	3	3	3	3	3
3	2	3	1	2	2	2	2	2	2	2
4	1	2	3	4	5	6	4	4	4	4
5	1	2	3	6	4	5	6	6	6	6
6	1	2	3	5	6	4	5	5	5	5
7	1	2	3	4	5	6	7	8	9	10
8	1	2	3	4	5	6	8	7	10	9
9	1	2	3	4	5	6	10	9	7	8
10	1	2	3	4	5	6	9	10	8	7

Theorem 4.3. *Let $(G, *)$ be a groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta$, $S_\alpha \bigcap S_\beta = \emptyset$. If $\forall a_\alpha \in S_\alpha, \forall b_\beta \in S_\beta$, $a_\alpha * b_\beta = b_\beta, b_\beta * a_\alpha = b_\beta * e_\beta$, then G is a strong root of band.*

Proof. It is similar to Theorem 4.2. □

The strong root of band constructed by Theorem 4.2 is not isomorphic to the strong root of band constructed by Theorem 4.3.

Corollary 4.1. *Let $(G, *)$ be an AG-groupoid, Y be an AG-band, $Y \subset G$. $G = \bigcup_{\alpha \in Y} S_\alpha$, $(S_\alpha, *)$ is an AG-group, for all $a_\alpha \in S_\alpha$, $a_\alpha^2 = e_\alpha$ (e_α is the left identity of S_α) and $\forall \alpha, \beta \in Y, \alpha \neq \beta$, $S_\alpha \bigcap S_\beta = \emptyset$. If $|Y| = 2$, then G is a strong root of band with the left identity.*

Proof. Suppose that $G = \bigcup_{\alpha \in Y} S_\alpha$ is an AG-groupoid satisfying $a_\alpha^2 = e_\alpha$ for all $a_\alpha \in S_\alpha$, Y is an AG-band and $|Y| = 2$. By Theorem 4.1, we have G is a strong root of band. If $Y = \{\alpha, \beta\}$, and $\alpha = e_\alpha \in S_\alpha, \beta = e_\beta \in S_\beta$, we have $(e_\alpha * e_\beta) * (e_\alpha * e_\beta) = (e_\alpha * e_\alpha) * (e_\beta * e_\beta) = e_\alpha * e_\beta$, thus, $e_\alpha * e_\beta \in Y$.

Case 1: $e_\alpha * e_\beta = e_\beta$. Let $a_\alpha \in S_\alpha$ be an arbitrary element, $a_\alpha * a_\alpha = e_\alpha, e_\alpha * a_\alpha = a_\alpha$. Let $b_\beta \in S_\beta$ be an arbitrary element, $b_\beta * b_\beta = e_\beta, e_\beta * b_\beta = b_\beta$,

$e_\beta * e_\alpha = (e_\beta * e_\beta) * e_\alpha = (e_\alpha * e_\beta) * e_\beta = e_\beta * e_\beta = e_\beta$.

$$\begin{aligned} e_\alpha * b_\beta &= (e_\alpha * e_\alpha) * (e_\beta * b_\beta) \\ &= (e_\alpha * e_\beta) * (e_\alpha * b_\beta) \quad \textit{(by the medial law)} \\ &= e_\beta * (e_\alpha * b_\beta) \\ &= (e_\beta * e_\beta) * (e_\alpha * b_\beta) \\ &= (e_\beta * e_\alpha) * (e_\beta * b_\beta) \quad \textit{(by the medial law)} \\ &= e_\beta * b_\beta = b_\beta. \end{aligned}$$

Thus, e_α is the left identity of G.

Case 2: $e_\alpha * e_\beta = e_\alpha$. Let $a_\alpha \in S_\alpha$ be an arbitrary element, $a_\alpha * a_\alpha = e_\alpha, e_\alpha * a_\alpha = a_\alpha$. Let $b_\beta \in S_\beta$ be an arbitrary element, $b_\beta * b_\beta = e_\beta, e_\beta * b_\beta = b_\beta$, $e_\beta * e_\alpha = (e_\beta * e_\beta) * e_\alpha = (e_\alpha * e_\beta) * e_\beta = e_\alpha * e_\beta = e_\alpha$.

$$\begin{aligned} e_\beta * a_\alpha &= (e_\beta * e_\beta) * (e_\alpha * a_\alpha) \\ &= (e_\beta * e_\alpha) * (e_\beta * a_\alpha) \quad \textit{(by the medial law)} \\ &= e_\alpha * (e_\beta * a_\alpha) \\ &= (e_\alpha * e_\alpha) * (e_\beta * a_\alpha) \\ &= (e_\alpha * e_\beta) * (e_\alpha * a_\alpha) \quad \textit{(by the medial law)} \\ &= e_\alpha * a_\alpha = a_\alpha. \end{aligned}$$

Thus, e_β is the left identity of G.

From all the above cases, G is a strong root of band with the left identity. □

Example 4.3. Let $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$, the definition of operation $*$ on G is shown in Table 6. An AG-band $Y = \{1, 5\}$ and $Y \subset G$. $S_1 = \{1, 2, 3, 4\}, e_1 = 1$ and $S_5 = \{5, 6, 7, 8\}, e_5 = 5$. Both S_1 and S_5 are sub-AG-groups of G. It is easy to verify that $G = S_1 \cup S_5$. From Definition 3.1, G is a strong root of band. Being $e_1 * e_5 = e_1$, we can get e_5 is the left identity of G.

Table 6: The operation table of Example 4.3.

$*$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	2	3	4	1	1	1	1
2	2	1	4	3	2	2	2	2
3	4	3	1	2	4	4	4	4
4	3	4	2	1	3	3	3	3
5	1	2	3	4	5	6	7	8
6	1	2	3	4	6	5	8	7
7	1	2	3	4	8	7	5	6
8	1	2	3	4	7	8	6	5

5. Conclusion

The construction problem is the basic problem in the study of algebraic systems. In this paper, the construction problem of a kind of AG-groupoid is completely solved. With the introduction to strong root of band and the thorough study of its structure, some important results can be obtained. Figure 1 gives the relationships among strong root of band and other bands. In Figure 1, as strong root of band is a special root of band, strong root of band is a generalization of both AG-band and AG-3-band. AG-3-band is the intersection of AG-4-band and strong root of band. The decomposition theorem of strong root of band has been investigated, where every strong root of band is the disjoint union of its sub-AG-groups (see Theorem 4.1). Furthermore, two different ways how to make a strong root of band are obtained (see Theorem 4.2 and Theorem 4.3) and the structure of strong root of band has been illuminated. As the next research direction, we will discuss the relationship between the strong root of band and other related AG-groupoid, such as Abel-Grassmann's neutrosophic extended triplet loop (AG-NET-Loop), generalized Abel-Grassmann's neutrosophic extended triplet loop (GAG-NET-Loop) (see [20], [21]).

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