Strong block-block domination of a graph

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Abstract. Let $B(G)$ denotes the set of all blocks of a graph G. Two blocks in G are adjacent if there is a common cutvertex incident on them. Two blocks $b_1, b_2 \in B(G)$ are said to *bb-dominate* each other if there is a common vertex incident with b_1 and b_2. A set $L \subseteq B(G)$ is said to be a *bb-dominating set (BBD set)* if every block in G is bb-dominated by some block in L. *The bb-domination number* $\gamma_{bb} = \gamma_{bb}(G)$ is the cardinality of a minimum bb-dominating set of G.

In this paper we define strong (weak) bb-dominating set and strong (weak) bb-full set and obtained relationship between them. We also obtain the relation with existing graph parameters.

Keywords: strong bb-domination, weak bb-domination, strong bb-full domination, weak bb-full domination.

1. Introduction

The terminologies and notations used here are as in [10]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. The independence number $\beta_0 = \beta_0(G)$ is the maximum order of a set $S \subseteq V$ in which no two vertices are adjacent. On the other hand, minimum number of vertices that cover all the edges of G is called vertex covering number $\alpha_0 = \alpha_0(G)$. These

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two parameters are related by $\alpha_0(G) + \beta_0(G) = p$ which is now referred as the classical Gallai's Theorem [10]. A set $S \subseteq V$ is a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number $\gamma = \gamma(G)$ is the order of a minimum dominating set of G. The domination number is a well studied parameter in literature and for a survey refer [2], [7], [8]. Sampathkumar and Pushpa Latha [6] introduced strong (weak) dominating sets. For any two adjacent vertices u and v, we say that u strongly dominates v if $deg(u) \geq deg(v)$. A set $D \subseteq V$ is a strong dominating set (sd-set) if every vertex $v \in V - D$ is strongly dominated by some $u \in D$. The strong domination number $\gamma_s = \gamma_s(G)$ is the order of a minimum sd-set of G. Similarly, weak domination number $\gamma_w = \gamma_w(G)$ is defined. The strong domination is later studied in [3], [4]. Similar to strong (weak) domination S. S. Kamath and R. S. bhat [5] studied strong (weak) independent sets. A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non separable. A maximal non-separable subgraph is a block of G. Let $B(G)$ and $C(G)$ respectively denote the set of all blocks and cutvertices of G with $|B(G)| = m$ and $|C(G)| = n$. Two blocks in G are adjacent if there is a common cutvertex incident on them. A block-graph $B_G(G)$ is a graph with vertex set $B(G)$ and any two vertices in $B_G(G)$ are adjacent if and only if corresponding blocks are adjacent in G. P. G. Bhat and R. S. Bhat [1] defined bb-dominating sets. Two blocks $b_1, b_2 \in B(G)$ are said to *bb-dominate* each other if there is a common vertex incident with b_1 and b_2. A set $L \subseteq B(G)$ is said to be a *bb-dominating set (BBD set)* if every block in G is bb-dominated by some block in L. *The bb-domination number* $\gamma_{bb} = \gamma_{bb}(G)$ is the cardinality of a minimum bb-dominating set of G. For any tree T, $\gamma_{bb}(T) = \gamma'(T)$ where $\gamma'(T)$ is the line domination number of T. A set $L \subseteq B(G)$ is *bb-full* if every block in L is adjacent to some block in $B(G) - L$. *Then bb-full number* $f_{bb} = f_{bb}(G)$ is the cardinality of a maximum bb-full set of G.

2. Strong bb-degree, weak bb-degree and regular bb-degree of a vertex

A block $g \in B(G)$ strongly (weakly) b-dominates a block $h \in B(G)$ if g is adjacent to h and $d_{bb}(g) \geq d_{bb}(h)$ $(d_{bb}(g) \leq d_{bb}(h))$. Then strong (weak) bb-degree of a block g, denoted as $d_{sbb}(g)$ $(d_{wbb}(g))$is the number of blocks strongly (weakly) b-dominated by a block g. A block $g \in B(G)$ regularly b-dominates a block $h \in B(G)$ if g is adjacent to h and $d_{bb}(g) = d_{bb}(h)$. Then regular bb-degree of a block g, denoted as $d_{rbb}(g)$ is the number of blocks regularly b-dominated by a block g. Also $\triangle_{sbb}(G) = \max_{h \in B(G)}\{d_{sbb}(h)\}$ and $\delta_{sbb}(G) = \min_{h \in B(G)}\{d_{sbb}(h)\}$. Analogously other parameters are defined.

Example 2.1. In the Figure 1, The first, second and third elements of a vertex label represent strong bb-degree, weak bb-degree and regular bb-degree of the corresponding vertices respectively. For the graph G in the above Figure 1, $\triangle_{sbb}(G) = 4$, $\triangle_{wbb}(G) = 1$, $\triangle_{rbb}(G) = 0$ and $\delta_{svb}(G) = \delta_{wvb}(G) = \delta_{rvb}(G) = 0$.

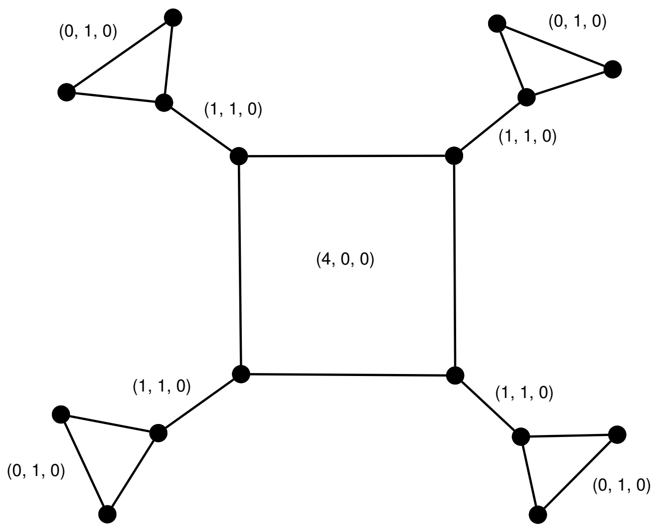

Figure 1: Strong bb-degree, Weak bb-degree and Regular bb-degree of a vertex

These various degrees are related by the following proposition.

Proposition 2.1. *Let $G = (V, E)$ be a graph. Then for any block $h \in B(G)$*

$$d_{bb}(h) = d_{sbb}(h) + d_{wbb}(h) - d_{rbb}(h). \tag{1}$$

Proof. For any block $h \in B(G)$, Let A=set of blocks bb-dominated by h, S=set of blocks strongly bb-dominated by h, W=set of blocks weakly bb-dominated by h and R=set of blocks regularly bb-dominated by h. We observe that $A = S \cup W$, $S \cap W = R$. Therefore $d_{bb}(h) = |A| = |S \cup W| = |S| + |W| - |S \cap W| = d_{sbb}(h) + d_{wbb}(h) - d_{rbb}(h)$. □

2.1 bb-strong number, bb-weak number, bb-balanced number and bb-regular number

A block $g \in B(G)$ is said to be bb-strong block (bb-weak block) if $d_{bb}(g) \geq d_{bb}(h)$ ($d_{bb}(g) \leq d_{bb}(h)$) for every h adjacent g. A block $g \in B(G)$ is said to be bb-balanced block if it is neither bb-strong block nor b-weak block. A block $g \in B(G)$ is said to be bb-regular block if it is both bb-strong block and bb-weak block.

A set $L \subseteq B(G)$ is said to be bb-strong, bb-weak, bb-balanced and bb-regular set if every block in L is respectively bb-strong, bb-weak, bb-balanced and bb-regular block in G. The bb-strong number $s_{bb} = s_{bb}(G)$, The bb-weak number $w_{bb} = w_{bb}(G)$, The bb-balanced number $b_{bb} = b_{bb}(G)$ and The bb-regular number $r_{bb} = r_{bb}(G)$ are respectively the cardinalities of the bb-strong, bb-weak, bb-balanced and bb-regular set of G. A block $g \in B(G)$ is strictly bb-strong (strictly bb-weak) if $d_{bb}(g) > d_{bb}(h)$ ($d_{bb}(g) < d_{bb}(h)$) for every h adjacent to g.

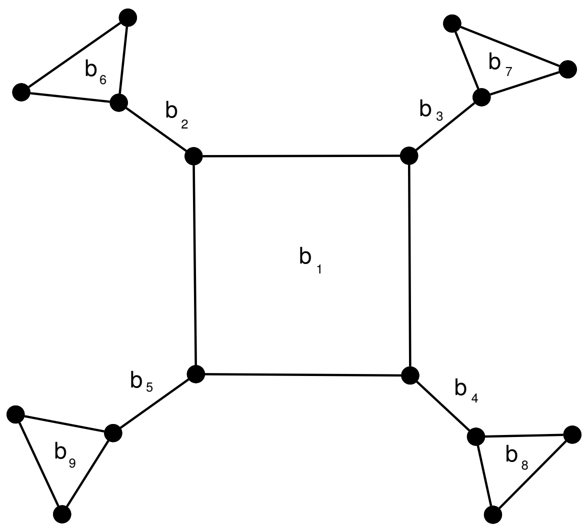

Figure 2: Graph G

Example 2.2. In the Figure 2, the bb-strong set is $S = \{b_1\}$ and $s_{bb}(G) = 1$. The bb-weak set is $W = \{b_6, b_7, b_8, b_9\}$ and $w_{bb}(G) = 4$. The bb-balanced set is $B = \{b_2, b_3, b_4, b_5\}$ and $b_{bb}(G) = 4$. The bb-regular set is $R = \{\}$ and $r_{bb}(G) = 0$.

Proposition 2.2. *For any graph $G = (V, E)$ with m blocks,*

$$s_{bb}(G) + w_{bb}(G) + b_{bb}(G) - r_{bb}(G) = m. \tag{2}$$

Proof. Let G be a graph with m blocks. Let S_{bb} be the bb-strong set, W_{bb} be the bb-weak set, B_{bb} be the bb-balanced set and R_{bb} be the bb-regular set. We have $S_{bb} \cap W_{bb} = R_{bb}$, by definition. Since any bb-balanced block is neither bb-strong nor bb-weak, therefore it cannot be either in S_{bb} set or in W_{bb} set. Therefore $(S_{bb} \cup W_{bb}) \cap B_{bb} = \phi$. Also we have $B(G) = (S_{bb} \cup W_{bb}) \cup B_{bb}$. Then $|B(G)| = m = |S_{bb} \cup W_{bb}| + |B_{bb}|=|S_{bb}| + |W_{bb}| - |S_{bb} \cap W_{bb}| + |B_{bb}|=s_{bb}(G) + w_{bb}(G) - r_{bb}(G) + b_{bb}(G)$. □

3. Strong (weak) bb-dominating sets of a graph

Two blocks $b_1, b_2 \in B(G)$ strongly (weakly) bb-dominates each other if there is a common cutvertex incident on b_1 and b_2, and $d_{bb}(b_1) \geq d_{bb}(b_2)$ ($d_{bb}(b_1) \leq d_{bb}(b_2)$). A set $L \subseteq B(G)$ is said to be a strong block-block dominating set (SBBD-set) (weak block-block dominating set(WBBD-set)) if every block in $B(G) - L$ is strongly (weakly) bb-dominated by some block in L. The strong block-block domination number $\gamma_{sbb} = \gamma_{sbb}(G)$ (weak block-block domination number $\gamma_{wbb} = \gamma_{wbb}(G)$) is the cardinality of a minimum SBBD-set (WBBD-set) of G.

A set $L \subseteq B(G)$ is said to be strong bb-full set (SBBF-set) (weak bb-full set (WBBF-set)) if every block in L strongly (weakly) bb-dominates at least one block in $B(G) - L$. The strong bb-full number $f_{sbb} = f_{sbb}(G)$ (weak bb-full number $f_{wbb} = f_{wbb}(G)$) is the cardinality of a maximum SBBF-set (WBBF-set) of G.

Example 3.1. In the Figure 2, BBD-set is $S_1 = \{b_2, b_3, b_4, b_5\}$, and BBF-set is $B(G) - S_1$. Therefore $\gamma_{bb}(G) = 4$ and $f_{bb}(G) = 5$. Also SBBD-set is $S_2 = \{b_1, b_2, b_3, b_4, b_5\}$ and WBBF-set is $B(G) - S_2$. Therefore $\gamma_{sbb}(G) = 5$ and $f_{wbb}(G) = 4$. Also WBBD-set is $S_3 = \{b_6, b_7, b_8, b_9, b_2\}$ and SBBF-set is $B(G) - S_3$. Therefore $\gamma_{wbb}(G) = 5$ and $f_{sbb}(G) = 4$.

3.1 Gallai type results

It is known that for any graph G with m blocks, $\gamma_{bb}(G) + f_{bb}(G) = m$. We now obtain similar results for the new parameters defined.

Observation 3.1. For any set $L \subseteq B(G)$,

(i) L is SBBF-set if, and only if, $B(G) - L$ is a WBBD-set,

(ii) L is WBBF-set if, and only if, $B(G) - L$ is a SBBD-set.

Analogous to Gallai's theorem, we now prove the following.

Proposition 3.1. *For any graph $G = (V, E)$ with m blocks,*

$$\gamma_{sbb}(G) + f_{wbb}(G) = m, \tag{3}$$

$$\gamma_{wbb}(G) + f_{sbb}(G) = m. \tag{4}$$

Proof. Let S be a minimum SBBD set of G. Then $B(G)-S$ is an WBBF set by Observation 3.1. Hence $f_{wbb}(G) \geq |B(G)-S|$. Therefore $\gamma_{sbb}(G)+f_{wbb}(G) \geq m$. Again, if D is a maximum WBBF set of G, then $B(G) - D$ is an SBBD set by Observation 3.1. Hence $\gamma_{sbb}(G)+f_{wbb}(G) \leq m$. Then (3) follows from the above inequalities. The proof of (4) is similar. □

Proposition 3.2. *For any graph $G = (V, E)$,*

$$\gamma_{sbb}(G) \leq \gamma_{wbb}(G), \tag{5}$$

$$f_{sbb}(G) \leq f_{wbb}(G). \tag{6}$$

Proof. Proof is by induction on the number of cutvertices n of G. If G contains only one cutvertex, then clearly $\gamma_{sbb}(G) = \gamma_{wbb}(G) = 1$. If G contains two cutvertices say u_1 and u_2. Then observe that there is only one support block say b which is incident on u_1 and u_2, and is adjacent to all the remaining blocks of G. Hence b strongly bb-dominates all the blocks of G. Therefore $\gamma_{sbb}(G) = 1$. On the other hand, in order to weakly bb-dominates all the blocks of G, we have to consider two pendant blocks say b_1 and b_2 which are incident on u_1 and u_2 respectively. Hence $\gamma_{wbb}(G) = 2$. Therefore $\gamma_{sbb}(G) < \gamma_{wbb}(G)$. Assume that the result is true for $n = k$. Now we prove the result for $n = k + 1$ with $n \geq 3$. Now, consider a graph G with $n = k + 1$ cutvertices. Observe that there is at least one cutvertex u incident on only one support block and any number of pendant blocks. Let b_1 is any one pendant block incident on the cutvertex u. Now, removal of all pendant blocks incident on a cutvertex u from G results in a graph G' with $n = k$ cutvertices. Let W' be γ_{wbb}-set of G', then $W = W' \cup \{b_1\}$ is the γ_{wbb}-set of G. Observe that, number of blocks in γ_{wbb}-set of G is one greater than the number of blocks in γ_{wbb}-set of G'. Now , Let S' be the γ_{sbb}-set of G'. Then $S = S' \cup \{b\}$ is the γ_{sbb}-set of G with cardinality one greater than that of S' of G'. Therefore $|S| \leq |W|$, i.e. $\gamma_{svb}(G) \leq \gamma_{wvb}(G)$. Hence the result is true for $n = k + 1$. Therefore by principle of mathematical induction, the result is true for all n. (6) follows from the Proposition (3.1). □

Corollary 3.1. *For any graph $G = (V, E)$ with m blocks,*

$$\gamma_{wbb}(G) + f_{wbb}(G) \geq m, \tag{7}$$

$$\gamma_{sbb}(G) + f_{sbb}(G) \leq m. \tag{8}$$

Proof. Since $\gamma_{sbb}(G) + f_{wbb}(G) = m$ from the Proposition 3.1. Also $\gamma_{sbb}(G) \leq \gamma_{wbb}(G)$ from the Proposition 3.2. Then (7) follows from the above inequalities. Proof of (8) is similar. □

3.2 Bounds on strong (weak) bb-dominating sets

We now obtain some elementary bounds for $\gamma_{sbb}(G)$ and $\gamma_{wbb}(G)$.

For any $h \in B(G)$, $N_{bb}(h) = \{g \in B(G) \mid g \text{ is adjacent to } h\}$. Let $\triangle_{bb}(G)$ and $\delta_{bb}(G)$ respectively be the maximum and minimum bb-degrees of a graph G.

Proposition 3.3. *For any graph G with m vertices,*

$$\gamma_{bb}(G) \leq \gamma_{sbb}(G) \leq m - \triangle_{bb}(G), \tag{9}$$

$$\gamma_{bb}(G) \leq \gamma_{wbb}(G) \leq m - \delta_{bb}(G), \tag{10}$$

$$\triangle_{bb}(G) \leq f_{wbb}(G) \leq f_{bb}(G), \tag{11}$$

$$\delta_{bb}(G) \leq f_{sbb}(G) \leq f_{bb}(G). \tag{12}$$

Proof. Since every SBBD-set or WBBD-set is a BBD-set, we have $\gamma_{bb}(G) \leq \gamma_{sbb}(G)$ and $\gamma_{bb}(G) \leq \gamma_{wbb}(G)$. For any $g, h \in B(G)$, let $d_{bb}(g) = \triangle_{bb}(G)$ and $d_{bb}(h) = \delta_{bb}(G)$. It is clear that $B(G) - N_{bb}(g)$ is a SBBD-set and $B(G) - N_{bb}(h)$ is a WBBD-set. Hence bound in (9) and (10) follows. (11) and (12) follows from the Proposition 3.1. □

For any block $h \in B(G)$, be-degree $d_{be}(h)$ is the number of edges in the block h and $\delta_{be}(G) = min_{h \in B(G)} d_{be}(h)$.

Proposition 3.4. *For any graph $G = (V, E)$ with q edges,*

$$\gamma_{bb}(G) \le \gamma_{sbb}(G) \le \frac{q}{\delta_{be}(G)}. \tag{13}$$

Proof. Since every SBBD-set is a BBD-set, we have $\gamma_{bb}(G) \le \gamma_{sbb}(G)$. Since edges of G can be partitioned into blocks of G, we have $\sum_{h\in B(G)} d_{be}(h) = q \ge m\delta_{be}(G)$. As set of all blocks $B(G)$ is a SBBD-set of G, we have $\gamma_{sbb}(G) \le |B(G)| = m \le \frac{q}{\delta_{be}(G)}$ □

4. Conclusion

Block domination is a well-studied parameter in literature. We modified and studied strong block-block domination in graphs. Few Gallai's theorem type results are obtained. Several bounds for strong block-block domination parameters are obtained. We also obtained the relationship between these newly defined parameters. Characterization of the graphs attaining these bounds are not studied in full and one may take this as an open problem for further research.

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