Fredholm criteria for a class of regular hypoelliptic operators in multianisotropic spaces in $\mathbb{R}^n$

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Abstract. We study the Fredholm property of regular hypoelliptic operators, which are the special subclass of Hyormander's hypoelliptic operators. In this paper necessary and sufficient conditions are obtained for the Fredholm property of regular hypoelliptic operators with special variable coefficients in multianisotropic weighted spaces in $\mathbb{R}^n$.

Keywords: Fredholm property, regular hypoelliptic operator, a priori estimate, multianisotropic weighted space.

1. Introduction

In this paper, we study the Fredholm property of regular hypoelliptic operators in multianisotropic weighted Sobolev spaces. The class of regular hypoelliptic operators is an important subclass of Hyormander's hypoelliptic operators (see [1]). They were introduced in late 60s-70s and studied by many authors: V. P. Mikhailov [2], J. Friberg [3], L. R. Volevich, S. G. Gindikin. [4]. Corresponding characteristic polynomials of regular hypoelliptic operators are "multi-quasi-elliptic", so they are natural generalization of elliptic and quasielliptic polynomials.

The analysis of the Fredholm property of regular hypoelliptic operators in Sobolev spaces in $\mathbb{R}^n$ has certain difficulties - characteristic polynomials of such operators are not homogeneous as in elliptic case and Fredholm theorems for compact manifolds cannot always be used in this case.

The Fredholm property of elliptic operators in special weighted spaces is studied in the works of L. A. Bagirov [5], R. B. Lockhart, R. C. McOwen [6, 7], E. Schrohe [8] and others.

L. A. Bagirov [9], G. A. Karapetyan, A. A. Darbinyan [10], A. G. Tumanyan [11, 12] studied the Fredholm property of quiasielliptic operators in weighted anisotropic spaces. For quasielliptic operators with constant coefficients isomorphism properties in some special scales of weighted Sobolev spaces are studied in G. V. Demidenko's works (see [13, 14]). In the works of L. Rodino, P. Boggiatto, E. Buzano (see [15]) the Fredholm property is studied for the special classes of pseudodifferential operators acting in multianisotropic spaces with special polynomial weights.

In this work, necessary conditions are obtained for fulfillment of special a priori estimates for differential operators acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$ (Theorem 3.1 and Theorem 3.4). Necessary and sufficient conditions are obtained for the Fredholm property of regular hypoelliptic operators with variable coefficients acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$ with certain weight functions (Theorem 3.6).

2. Basic notions and definitions

Definition 2.1. *A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called an n–normal operator, if the following conditions hold:*

1. *the image of operator A is closed* $(\mathrm{Im}(A) = \overline{\mathrm{Im}(A)})$*;*
2. *the kernel of operator A is finite dimensional* $(\dim \mathrm{Ker}(A) < \infty)$.

An operator A is called a Fredholm operator if conditions 1-2 hold and

3. *the cokernel of operator A is finite dimensional* $(\dim \mathrm{coker}(A) = \dim Y/\mathrm{Im}(A) < \infty)$.

The difference between the dimension of the kernel and the cokernel of operator A is called index of the operator:

$$\mathrm{ind}(A) = \dim \mathrm{Ker}(A) - \dim \mathrm{coker}(A).$$

Definition 2.2. *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operator $R_1 : Y \to X$ and $R_2 : Y \to X$ are called respectively left and right regularizers if the following holds: $R_1 A = I_X + T_1, AR_2 = I_Y + T_2$, where I_X, I_Y– identity operators, $T_1 : X \to X$ and $T_2 : Y \to Y$ are compact operators.*

Definition 2.3. *For a bounded linear operator A, acting from a whole Banach space X to a Banach space Y, bounded linear operator $R : Y \to X$ is called a regularizer for operator A, if it is left and right regularizer.*

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be Euclidean n-dimensional space, $\mathbb{Z}^n_+$, $\mathbb{N}^n$ be the sets of n-dimensional multi-indices and multi-indices with natural components respectively.

Let $\mathcal{N} \subset \mathbb{Z}^n_+$ be a finite set of multi-indices, $\mathcal{R} = \mathcal{R}(\mathcal{N})$ be a minimum convex polyhedron containing all the points $\mathcal{N}$.

Definition 2.4. *A polyhedron $\mathcal{R}$ is called a completely regular if the following holds: a) $\mathcal{R}$ is a complete polyhedron: $\mathcal{R}$ has a vertex at the origin and further vertices on each coordinate axes in $\mathbb{R}^n$; b) all components of the outer normals of $(n-1)$–dimensional non-coordinate faces of $\mathcal{R}$ are positive.*

Let $\mathcal{R}$ be a completely regular polyhedron. Denote by $\mathcal{R}_j^{n-1}(j=1,\dots,I_{n-1})$ $(n-1)$–dimensional non-coordinate faces of $\mathcal{R}$ with corresponding outer normal μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha:\mu^j)=\frac{\alpha_1}{\mu_1^j}+\ldots+\frac{\alpha_n}{\mu_n^j}=1$, $\partial'\mathcal{R}=\bigcup_{j=1}^{I_{n-1}}\mathcal{R}_j^{n-1}$. For $k>0$ denote by $k\mathcal{R}:=\{k\alpha=(k\alpha_1,k\alpha_2\dots,k\alpha_n):\alpha\in\mathcal{R}\}$.

Consider the differential form

$$P(x,\mathbb{D})=\sum_{\alpha\in\mathcal{R}}a_\alpha(x)D^\alpha, \tag{1}$$

where $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n},\ D_j=i^{-1}\frac{\partial}{\partial x_j}, x=(x_1,\dots,x_n)\in\mathbb{R}^n, a_\alpha(x)\in C(\mathbb{R}^n)$.

Denote by

$$P(x,\xi)=\sum_{\alpha\in\mathcal{R}}a_\alpha(x)\xi^\alpha. \tag{2}$$

For each $(n-1)$–dimensional non-coordinate face $\mathcal{R}_j^{n-1}(j=1,\dots,I_{n-1})$ denote by

$$P_j(x,\mathbb{D})=\sum_{\alpha\in\mathcal{R}_j^{n-1}}a_\alpha(x)D^\alpha, P_j(x,\xi)=\sum_{\alpha\in\mathcal{R}_j^{n-1}}a_\alpha(x)\xi^\alpha.$$

For $\xi\in\mathbb{R}^n$ denote by

$$|\xi|_{\mathcal{R}}=\sum_{\alpha\in\mathcal{R}}|\xi^\alpha|, |\xi|_{\partial'\mathcal{R}}=\sum_{\alpha\in\partial'\mathcal{R}}|\xi^\alpha|.$$

Definition 2.5. *A differential form $P(x,\mathbb{D})$ is called regular at a point $x_0\in\mathbb{R}^n$, if there exists a constant $\delta>0$ such that:*

$$1+|P(x_0,\xi)|\geq\delta|\xi|_{\mathcal{R}},\forall\xi\in\mathbb{R}^n.$$

Definition 2.6. *A differential form $P(x,\mathbb{D})$ is called regular in $\mathbb{R}^n$, if $P(x,\mathbb{D})$ is regular at each point $x\in\mathbb{R}^n$.*

Definition 2.7. *A differential form $P(x,\mathbb{D})$ is called uniformly regular in $\mathbb{R}^n$, if there exists a constant $\delta>0$ such that:*

$$1+|P(x,\xi)|\geq\delta|\xi|_{\mathcal{R}},\forall\xi\in\mathbb{R}^n,\forall x\in\mathbb{R}^n.$$

Let's consider some examples of regular differential forms.

1. Let $m\in\mathbb{N}$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0,0,\dots,0)$, $(m,0,\dots,0),\dots,(0,0,\dots,m)$. In this case conditions from definitions 2.5–2.7 coincide with ellipticity conditions and differential form $P(x,\mathbb{D})$ is elliptical.

2. Let $\nu \in \mathbb{N}^n$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0,0,\dots,0)$, $(\nu_1,0,\dots,0),\dots,(0,0,\dots,\nu_n)$. In this case conditions from definitions 2.5–2.7 coincide with quasiellipticity of differential form $P(x,\mathbb{D})$.

3. Let $n=2$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0)$, $(8,0)$, $(0,8)$ and $(6,4)$. Then,
$$P(x,\mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$
is a regular differential form in $\mathbb{R}^2$ with some $a_1,a_2,a_3>0$ and $q\in C(\mathbb{R}^2)$.

4. Let $n=3$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0,0)$, $(8,0,0)$, $(0,8,0)$, $(6,4,0)$, $(6,0,6)$, $(0,6,6)$ and $(0,0,12)$. Then,
$$P(x,\mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$
is a regular differential form in $\mathbb{R}^3$ with $q\in C(\mathbb{R}^3)$.

For $k\in\mathbb{R}$ and completely regular polyhedron $\mathcal{R}$ denote
$$H^{k,\mathcal{R}}(\mathbb{R}^n) := \{u\in S' : \|u\|_{k,\mathcal{R}} := (\int |\widehat{u}(\xi)|^2 (1+|\xi|_{\partial'\mathcal{R}})^{2k} d\xi)^{\frac{1}{2}} < \infty\},$$
where S' is the set of tempered distributions, $\widehat{u}$ is a Fourier transform of function u.

Denote
$$Q := \{g\in C(\mathbb{R}^n) : \exists c>0 \text{ such that } g(x)\ge c>0, \forall x\in\mathbb{R}^n\}.$$
For $m\in\mathbb{Z}_+$ and completely regular polyhedron $\mathcal{R}$ denote
$$Q^{m,\mathcal{R}} := \{g(x)\in Q : D^\beta g(x)\in C(\mathbb{R}^n), \frac{1}{g(x)} \rightrightarrows 0, \max_{|x-y|\le 1} \frac{|g(x)-g(y)|}{g(y)} \rightrightarrows 0,$$
$$\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\mu^j)}} \rightrightarrows 0 \text{ as } |x|\to\infty, \forall \beta\in m\mathcal{R}, \beta\ne 0, j=1,\dots,I_{n-1}\}.$$

The examples of weight functions from $Q^{m,\mathcal{R}}$ include polynomial functions as well as special exponential functions, for example:
$$(1+|x|_{\mathcal{R}})^l, l>0, \exp{(1+|x|_{\mathcal{R}})^r}, 0<r<\frac{1}{\mu_{max}},$$
where $\mu_{max} = \max_{1\le i\le I_{n-1}} \max_{1\le s\le n}\{\mu_s^i\}$. For $k\in\mathbb{Z}_+$, $q\in Q$, completely regular polyhedron $\mathcal{R}$ and $\Omega\subset\mathbb{R}^n$ denote
$$H_q^{k,\mathcal{R}}(\mathbb{R}^n) :=$$
$$\{u : \|u\|_{H_q^{k,\mathcal{R}}(\mathbb{R}^n)} := \|u\|_{k,\mathcal{R},q} := \sum_{\alpha\in k\mathcal{R}} \|D^\alpha u\cdot q^{k-\max_i(\alpha:\mu^i)}\|_{L_2(\mathbb{R}^n)} < \infty\},$$

$$H_q^{k,\mathcal{R}}(\Omega) := \{u : \|u\|_{H_q^{k,\mathcal{R}}(\Omega)} := \sum_{\alpha\in k\mathcal{R}} \|D^\alpha u \cdot q^{k-\max_i(\alpha:\mu^i)}\|_{L_2(\Omega)} < \infty\}.$$

Let $k \in \mathbb{Z}_+, q \in Q$ and the coefficients of differential expression $P(x,\mathbb{D})$ of the form (1) satisfy the following conditions:

$$|D^\beta a_\alpha(x)| \le C_{\alpha,\beta}\, q(x)^{1-\max_i(\alpha-\beta:\mu^i)} \;\; \forall \alpha \in \mathcal{R}, \beta \in k\mathcal{R}. \tag{3}$$

Then, it is easy to check that $P(x,\mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$.

3. Main results

For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \le N\}, K_N := K_N(0).$$

Theorem 3.1. *Let $k \in \mathbb{Z}_+, q \in Q$ and the differential form $P(x,\mathbb{D})$ with some constant $\kappa > 0$ satisfies the following estimate:*

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(\mathbb{R}^n)}), \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n). \tag{4}$$

Then, $P(x,\mathbb{D})$ is uniformly regular in $\mathbb{R}^n$.

Proof. Let $x_0, \xi \in \mathbb{R}^n$, $|\xi| \ne 0$, $N > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset K_N(x_0)$ and $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$. For $\lambda > 0$ and $j \in \{1, \ldots, I_{n-1}\}$ denote by $\lambda^{\frac{1}{\mu^j}}\xi = (\lambda^{\frac{1}{\mu_1^j}}\xi_1, \ldots, \lambda^{\frac{1}{\mu_n^j}}\xi_n)$, where μ^j is outer normal of $\mathcal{R}_j^{n-1}$. Denote by $u_{\lambda,j}(x) = \exp(i(\lambda^{\frac{1}{\mu^j}}\xi, x))\varphi(x)$. For $\beta \in (k+1)\mathcal{R}$, we have

$$\begin{aligned} D^\beta u_{\lambda,j}(x) &= \lambda^{(\beta:\mu^j)}\xi^\beta \exp(i(\lambda^{\frac{1}{\mu^j}}\xi, x))\varphi(x) \\ &\quad + \sum_{0\le\gamma<\beta} C_\beta^\gamma \lambda^{(\gamma:\mu^j)}\xi^\gamma \exp(i(\lambda^{\frac{1}{\mu^j}}\xi, x))D^{\beta-\gamma}\varphi(x). \end{aligned}$$

From last equality, for $\beta \in (k+1)\mathcal{R}$, we get

$$\begin{aligned} \|D^\beta u_{\lambda,j} \cdot q^{k+1-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} &= \lambda^{(\beta:\mu^j)}|\xi^\beta|\|\varphi \cdot q^{k+1-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} \\ &\quad + o(\lambda^{(\beta:\mu^j)}), \end{aligned}$$

when $\lambda \to \infty$. Then, we get

$$\|u_{\lambda,j}\|_{k+1,\mathcal{R},q} = \sum_{\beta\in(k+1)\mathcal{R}} \lambda^{(\beta:\mu^j)}|\xi^\beta|\|\varphi \cdot q^{k+1-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}), \tag{5}$$

when $\lambda \to \infty$.

Notice that, for $\alpha \in \mathcal{R}$ the following holds: $(\alpha : \mu^j) \leq 1$ and $(\alpha : \mu^j) = 1$ if only if $\alpha \in \mathcal{R}_j^{n-1}$. Taking it into account along with the fact that $\varphi \in C_0^\infty(\mathbb{R}^n), \operatorname{supp}\varphi \subset K_N(x_0)$ for $\beta \in k\mathcal{R}$ we get

$$\begin{aligned}&\|D^\beta(P(x,\mathbb{D})u_{\lambda,j})q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)}\\ &\leq \lambda^{(\beta:\mu^j)}\lambda|\xi^\beta| \max_{x\in K_N(x_0)}|P_j(x,\xi)|\|\varphi\cdot q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}).\end{aligned}$$

From the last estimate, we get

$$\begin{aligned}\|Pu_{\lambda,j}\|_{k,\mathcal{R},q} \leq \sum_{\beta\in k\mathcal{R}} \lambda^{(\beta:\mu^j)+1}|\xi^\beta| \max_{x\in K_N(x_0)}|P_j(x,\xi)|&\\ \times\|\varphi\cdot q^{k-\max_i(\beta:\mu^i)}\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}),&\end{aligned} \tag{6}$$

when $\lambda \to \infty$. Then, from (4) and (5)–(6) we get

$$\begin{aligned}&\sum_{\beta\in(k+1)\mathcal{R}_j^{n-1}} \lambda^{k+1}|\xi^\beta|\|\varphi\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1})\\ &\leq \kappa \sum_{\beta\in k\mathcal{R}_j^{n-1}} \lambda^{k+1}|\xi^\beta| \max_{x\in K_N(x_0)}|P_j(x,\xi)|\|\varphi\|_{L_2(\mathbb{R}^n)} + o(\lambda^{k+1}),\end{aligned}$$

when $\lambda \to \infty$.

In last inequality we take into account that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$, divide by λ^{k+1} and tend $\lambda \to \infty$. Then the following is obtained:

$$\sum_{\beta\in(k+1)\mathcal{R}_j^{n-1}} |\xi^\beta| \leq \kappa \sum_{\beta\in k\mathcal{R}_j^{n-1}} |\xi^\beta| \max_{x\in K_N(x_0)}|P_j(x,\xi)|.$$

Since $\mathcal{R}$ is completely regular polyhedron for $k \in \mathbb{Z}_+$ and $j \in \{1,\dots,I_{n-1}\}$, there exist such constants $\delta_1 > 0, \delta_2 > 0$ such tat

$$\begin{aligned}\sum_{\beta\in(k+1)\mathcal{R}_j^{n-1}} |\xi^\beta| \geq \delta_1(\sum_{\beta\in\mathcal{R}_j^{n-1}}|\xi^\beta|)^{k+1},\\ \sum_{\beta\in k\mathcal{R}_j^{n-1}} |\xi^\beta| \leq \delta_2(\sum_{\beta\in\mathcal{R}_j^{n-1}}|\xi^\beta|)^{k}.\end{aligned} \tag{7}$$

From (7), we get

$$\delta_1 \sum_{\beta\in\mathcal{R}_j^{n-1}} |\xi^\beta| \leq \kappa\delta_2 \max_{x\in K_N(x_0)}|P_j(x,\xi)|.$$

Since the coefficients of $P(x,\mathbb{D})$ are continuous, tending $N \to 0$, we obtain

$$|P_j(x_0,\xi)| \geq \delta_3 \sum_{\beta\in\mathcal{R}_j^{n-1}} |\xi^\beta|, \forall\xi \in \mathbb{R}^n, \tag{8}$$

where $\delta_3 = \frac{\delta_1}{\kappa\delta_2} > 0$.

Since constant δ_3 does not depend on the choice of $x_0 \in \mathbb{R}^n$, we get

$$|P_j(x,\xi)| \geq \delta_3 \sum_{\beta \in \mathcal{R}_j^{n-1}} |\xi^\beta|, \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n. \tag{9}$$

Similarly analogous inequalities can be obtained for all $j \in \{1, \ldots, I_{n-1}\}$. Then applying Theorem 6.1 from [2] we obtain that $P(x,\mathbb{D})$ is uniformly regular in $\mathbb{R}^n$. □

Theorem 3.2. (Theorem 7.1 [19]) *Let E, F and E_0 be Banach spaces such that E is compactly embedded in E_0. Let A be a bounded linear operator acting from E to F. An operator $A : E \to F$ is an $n-$normal operator if and only if there exists a constant $C > 0$ such that*

$$\|x\|_E \leq C(\|Ax\|_F + \|x\|_{E_0}),\ \forall x \in E.$$

Applying the last theorem for operator $P(x,\mathbb{D})$, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$, we get

Theorem 3.3. *Let $k \in \mathbb{Z}_+, q \in Q$ and $P(x,\mathbb{D})$ be differential form (1). Then operator $P(x,\mathbb{D})$, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$, is an $n-$normal operator if and only if there exist constants $\kappa > 0$ and $N > 0$ such that the following holds*

$$\|u\|_{k+1,\mathcal{R},q} \leq \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(K_N)})\ , \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n).$$

Corollary 3.1. *Let $k \in \mathbb{Z}_+$, $q \in Q$ and operator $P(x,\mathbb{D})$, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$, be a Fredholm operator. Then $P(x,\mathbb{D})$ is uniformly regular in $\mathbb{R}^n$.*

Proof. Since operator $P(x,\mathbb{D})$, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator, then it is an $n-$normal operator. From Theorem 3.3, we get that there exist such constants $\kappa > 0$ and $N > 0$ that the following estimate holds

$$\|u\|_{k+1,\mathcal{R},q} \leq \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(K_N)}) \leq \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(\mathbb{R}^n)}),$$

for all $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$. From last estimate using Theorem 3.1 we obtain that $P(x,\mathbb{D})$ is uniformly regular in $\mathbb{R}^n$. □

Remark 3.1. It is easy to check that in the case $q \equiv 1$ inverse statement is true with some smoothness conditions on the coefficients of $P(x,\mathbb{D})$. In next theorem 3.4 it is proved that under the special conditions on the weight function q and coefficients of the differential form $P(x,\mathbb{D})$ uniform regularity in $\mathbb{R}^n$ (in the sense of definition (2.7)) does not imply the fulfillment of a priori estimate of the form (4) and stronger conditions are necessary for it.

Let $k \in \mathbb{Z}_+$ and $q \in Q^{k,\mathcal{R}}$. Consider the differential form $P(x,\mathbb{D})$ (see (1)), which is expressed in the following way:

$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha = \sum_{\alpha\in\mathcal{R}} (a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} + a^1_\alpha(x)) D^\alpha, \tag{10}$$

where $a_\alpha(x) = a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} + a^1_\alpha(x)$, $D^\beta(a^0_\alpha(x)) = O(q(x)^{\min_i(\beta:\mu^i)})$ and $D^\beta(a^1_\alpha(x)) = o(q(x)^{1-\max_i(\alpha-\beta:\mu^i)})$, when $|x| \to \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$.

Theorem 3.4. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (10) *with the coefficients that satisfy $\lim_{|x|\to\infty} \max_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Let there exists a constant $\kappa > 0$ such that:

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(\mathbb{R}^n)}), \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n). \tag{11}$$

Then, $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) \lambda^{1-\max_i(\alpha:\mu^i)} \xi^\alpha| \ge \delta(\lambda + |\xi|_{\partial'\mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{12}$$

Proof. Regularity of $P(x,\mathbb{D})$ in $\mathbb{R}^n$ follows from Theorem 3.1. It remains to prove that inequality (12) holds.

Let $M \in \mathbb{R}_+, x_M \in \mathbb{R}^n \backslash K_M, \varphi \in C^\infty_0(\mathbb{R}^n), \operatorname{supp}\varphi \subset K_1(x_M), \|\varphi\|_{L_2(\mathbb{R}^n)} = 1$ and $\xi \in \mathbb{R}^n$. Let $j \in \{1,\dots,I_{n-1}\}$.

Consider the function $\tilde{u}_j(x) = \exp(i(q(x_M)^{\frac{1}{\mu^j}}\xi, x))\varphi(x)$ where μ^j is an outer normal of non-coordinate face $\mathcal{R}^{n-1}_j$ such that all multi-indices $\alpha \in \mathcal{R}^{n-1}_j$ satisfy $(\alpha : \mu^j) = 1$.

Denote by $\mathcal{R}_j = \{\alpha \in \mathcal{R} : (\alpha : \mu^j) = \max_{1\le i\le I_{n-1}}(\alpha : \mu^i)\}$.

Since $\lim_{|x|\to\infty} \max_{|x-y|\le 1} \frac{|q(x)-q(y)|}{q(y)} = 0$, then for any $r \in \mathbb{R}_+$ the following inequality is fulfilled

$$|q(x)^r - q(x_M)^r| \le \varepsilon_r(M) q(x_M)^r, \forall x \in K_1(x_M), \tag{13}$$

where $\varepsilon_r(M) \to 0$ when $M \to \infty$.

Using inequality (13) and the fact that $\operatorname{supp}\widetilde{u}_j \subset K_1(x_M)$ it is easy to see that there exists a function $\varepsilon(M)$ such that $\varepsilon(M) \to 0$ when $M \to \infty$ and the following inequalities hold:

$$\|\tilde{u}_j\|_{k+1,\mathcal{R},q} \ge (1 - \varepsilon(M))\|\tilde{u}_j\|_{k+1,\mathcal{R},q(x_M)}, \tag{14}$$

$$\|P\tilde{u}_j\|_{k,\mathcal{R},q} \le (1 + \varepsilon(M))\|P\tilde{u}_j\|_{k,\mathcal{R},q(x_M)}. \tag{15}$$

Taking into consideration the definition of function $\tilde{u}_j$ one can check that for any $\beta \in (k+1)\mathcal{R}_j$ with some constants $C_1 > 0$ and $\sigma = \sigma(\mathcal{R}) > 0$ the following holds

$$\|D^\beta \tilde{u}_j\|_{L_2(\mathbb{R}^n)} q(x_M)^{k+1-(\beta:\mu^j)} \geq |\xi^\beta| q(x_M)^{k+1} \|\varphi\|_{L_2(\mathbb{R}^n)} - C_1 \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_M)^{k+1-\sigma}.$$

For $\beta \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j)$ with some constants $C_2 > 0$ and $\sigma = \sigma(\mathcal{R}) > 0$ the following holds

$$\|D^\beta \tilde{u}_j\|_{L_2(\mathbb{R}^n)} q(x_M)^{k+1-\max_i(\beta:\mu^i)} \geq |\xi^\beta| q(x_M)^{k+1-\sigma} \|\varphi\|_{L_2(\mathbb{R}^n)} - C_2 \sum_{0 \leq \gamma < \beta} |\xi^\gamma| q(x_M)^{k+1-\sigma}.$$

From last two inequalities, taking into account that $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$, with some constants $C_3 > 0$ and $\sigma > 0$ we get

$$\|\tilde{u}_j\|_{k+1,\mathcal{R},q(x_M)} \geq \sum_{\beta \in (k+1)\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} - C_3 \sum_{\gamma \in (k+1)(\mathcal{R} \setminus \mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}. \tag{16}$$

Taking into account, conditions (3) on the coefficients of $P(x, \mathbb{D})$ and inequality (13) we get that for $\beta \in k\mathcal{R}$ and $\alpha \in \mathcal{R}$ with some $C_{\alpha,\beta}, C'_{\alpha,\beta} > 0$ holds

$$\begin{aligned} &|D^\beta(a_\alpha(x) D^\alpha \tilde{u}_j(x))| q(x_M)^{k-\max_i(\beta:\mu^i)} \\ &\leq C_{\alpha,\beta} \sum_{\beta_1+\beta_2=\beta} |D^{\beta_1}(a_\alpha(x)) D^{\alpha+\beta_2} \tilde{u}_j(x)| q(x_M)^{k-\max_i(\beta:\mu^i)} \\ &\leq C'_{\alpha,\beta} \sum_{\beta_1+\beta_2=\beta} |D^{\alpha+\beta_2} \tilde{u}_j(x)| q(x_M)^{1-\max_i(\alpha-\beta_1:\mu^i)} q(x_M)^{k-\max_i(\beta:\mu^i)}. \end{aligned}$$

Let $\beta \in k(\mathcal{R} \setminus \mathcal{R}_j)$ and $\alpha \in \mathcal{R}$. Then, for $\beta_1, \beta_2 \in k\mathcal{R}, \beta_1 + \beta_2 = \beta$ with some $\sigma > 0$ the following holds

$$\begin{aligned} &\max_i(\alpha - \beta_1 : \mu^i) + \max_i(\beta : \mu^i) - (\alpha + \beta_2 : \mu^j) \\ &\geq (\max_i(\alpha - \beta_1 : \mu^i) - (\alpha - \beta_1 : \mu^j)) + (\max_i(\beta : \mu^i) - (\beta : \mu^j)) \geq \sigma > 0. \end{aligned}$$

So, for $\beta \in k(\mathcal{R} \setminus \mathcal{R}_j)$ and $\alpha \in \mathcal{R}$ with some constants $C''_{\alpha,\beta}, C'''_{\alpha,\beta} > 0$ and $\sigma > 0$ we get

$$|D^\beta(a_\alpha(x) D^\alpha \tilde{u}_j(x))| q(x_M)^{k-\max_i(\beta:\mu^i)}$$

$$\leq C''_{\alpha,\beta} \sum_{\beta_1+\beta_2=\beta} |D^{\alpha+\beta_2}\tilde{u}_j(x)| q(x_M)^{k+1-\sigma-\max_i(\alpha+\beta_2:\mu^i)},$$

$$\|D^\beta(a_\alpha(x)D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-\max_i(\beta:\mu^i)}$$
$$\leq C'''_{\alpha,\beta} \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}.$$

From the last estimate we obtain that for $\beta \in k(\mathcal{R}\setminus\mathcal{R}_j)$ with some constants $C_4 > 0$ and $\sigma > 0$

$$\begin{aligned} &\|D^\beta(P(x,\mathbb{D})\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-\max_i(\beta:\mu^i)} \\ &\leq C_4 q(x_M)^{k+1-\sigma} \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma| \end{aligned} \tag{17}$$

For $\beta \in k\mathcal{R}_j$, we have

$$\begin{aligned} &\|D^\beta(P(x,\mathbb{D})\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)} \\ &\leq \|D^\beta(\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x) q(x)^{1-(\alpha:\mu^j)} D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)} \\ &+ \|D^\beta(\sum_{\alpha\in\mathcal{R}\setminus\mathcal{R}_j} a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)} \\ &+ \|D^\beta(\sum_{\alpha\in\mathcal{R}_j} a^1_\alpha(x) D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)}. \end{aligned} \tag{18}$$

Similarly, to (17) with some constants $C_5 > 0$ and $\sigma > 0$ holds

$$\begin{aligned} &\|D^\beta(\sum_{\alpha\in\mathcal{R}\setminus\mathcal{R}_j} a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)} \\ &\leq C_5 \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}_j^{n-1})} |\xi^\gamma| q(x_M)^{k+1-\sigma}. \end{aligned} \tag{19}$$

Using conditions $D^\beta(a^1_\alpha(x)) = o(q(x)^{1-\max_i(\alpha-\beta:\mu^i)})$ when $|x| \to \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$ it is easy to check that for $\beta \in k\mathcal{R}_j$ there exists such a function $\tau_1(M)$ that $\tau_1(M) \to 0$ when $M \to \infty$ and the following holds

$$\begin{aligned} &\|D^\beta(\sum_{\alpha\in\mathcal{R}} a^1_\alpha(x) D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)} q(x_M)^{k-(\beta:\mu^j)} \\ &\leq \tau_1(M) \sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma| q(x_M)^{k+1}. \end{aligned} \tag{20}$$

For the first term in (18) the following estimate holds

$$\begin{aligned}&\|D^\beta(\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x)q(x)^{1-(\alpha:\mu^j)}D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta:\mu^j)}\\&\le \|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_M)q(x_M)^{1-(\alpha:\mu^j)}D^{\alpha+\beta}\tilde{u}_j\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta:\mu^j)}\\&+\|\sum_{\alpha\in\mathcal{R}_j} D^\beta((a^0_\alpha(x)q(x)^{1-(\alpha:\mu^j)}-a^0_\alpha(x_M)q(x_M)^{1-(\alpha:\mu^j)})D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}\\&\times q(x_M)^{k-(\beta:\mu^j)}.\end{aligned}$$

It is easy to check that with some constants $C_6>0$ and $\sigma>0$ the following holds

$$\begin{aligned}&\|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_M)q(x_M)^{1-(\alpha:\mu^j)}D^{\alpha+\beta}\tilde{u}_j\|_{L_2(\mathbb{R}^n)}q(x_M)^{k-(\beta:\mu^j)}\\&\le |\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_M)\xi^\alpha||\xi^\beta|q(x_M)^{k+1}\\&+C_6\sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}^{n-1}_j)}|\xi^\gamma|q(x_M)^{k+1-\sigma}.\end{aligned}\tag{21}$$

Taking into account, conditions in (10), $\lim_{|x|\to\infty}\max_{|x-y|\le 1}|a^0_\alpha(x)-a^0_\alpha(y)|=0$ for $\alpha\in\mathcal{R}$, $q\in Q^{k,\mathcal{R}}$ and inequality (13) we get that for $\alpha\in\mathcal{R}$ and $\beta\in k\mathcal{R}$ there exists $\tau_{\alpha,\beta}(M)$ such that $\tau_{\alpha,\beta}(M)\to 0$ when $M\to\infty$ and the following holds

$$|D^\beta(a^0_\alpha(x)q(x)^{1-(\alpha:\mu^j)}-a^0_\alpha(x_M)q(x_M)^{1-(\alpha:\mu^j)})|\le\tau_{\alpha,\beta}(M)q(x_M)^{1-(\alpha:\mu^j)+(\beta:\mu^j)}.$$

From last inequality we get that there exist a function $\tau_2(M)$ such that $\tau_2(M)\to 0$ when $M\to\infty$, constants $C_7>0$ and $\sigma>0$ that the following estimate is satisfied

$$\begin{aligned}&\|\sum_{\alpha\in\mathcal{R}_j} D^\beta(a^0_\alpha(x)q(x)^{1-(\alpha:\mu^j)}-a^0_\alpha(x_M)q(x_M)^{1-(\alpha:\mu^j)}D^\alpha\tilde{u}_j)\|_{L_2(\mathbb{R}^n)}\\&\times q(x_M)^{k-(\beta:\mu^j)}\le\tau_2(M)\sum_{\alpha\in\mathcal{R}_j}|\xi^{\alpha+\beta}|q(x_M)^{k+1}\\&+C_7\sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}^{n-1}_j)}|\xi^\gamma|q(x_M)^{k+1-\sigma}.\end{aligned}\tag{22}$$

So, using last estimate and estimates (17)–(22) we obtain that there exist such a function $\tau_3(M)$, $\tau_3(M)\to 0$ when $M\to\infty$ and constants $C_8>0$ and

$\sigma > 0$ that the following holds

$$\|P\tilde{u}_j\|_{k,\mathcal{R},q(x_M)} \le |\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_M)\xi^\alpha| \sum_{\beta\in k\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1}$$
$$+ C_8 \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}^{n-1}_j)} |\xi^\gamma| q(x_M)^{k+1-\sigma} + \tau_3(M) \sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma| q(x_M)^{k+1}. \tag{23}$$

Then, from (11) using (16) and (23) we get

$$\sum_{\beta\in(k+1)\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1} - C_3 \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}^{n-1}_j)} |\xi^\gamma| q(x_M)^{k+1-\sigma}$$
$$\le \kappa(|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_M)\xi^\alpha| \sum_{\beta\in k\mathcal{R}_j} |\xi^\beta| q(x_M)^{k+1}$$
$$+ C_8 \sum_{\gamma\in(k+1)(\mathcal{R}\setminus\mathcal{R}^{n-1}_j)} |\xi^\gamma| q(x_M)^{k+1-\sigma} + \tau_3(M) \sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma| q(x_M)^{k+1}).$$

Since $\{a^0_\alpha(x) : \alpha \in \mathcal{R}_j\}$ are bounded functions and $x_M \to \infty$ when $M \to \infty$, there exist convergent subsequences of sequences $\{a^0_\alpha(x_M) : \alpha \in \mathcal{R}_j\}$. Without loss of generality assume that sequences $\{a^0_\alpha(x_M) : \alpha \in \mathcal{R}_j\}$ are convergent, so for each $\alpha \in \mathcal{R}_j$ there exists a constant $\tilde{a}^0_\alpha$ such that $a^0_\alpha(x_M) \rightrightarrows \tilde{a}^0_\alpha$ when $M \to \infty$.

Dividing by $q(x_M)^{k+1}$ and tending M to infinity we get

$$\sum_{\alpha\in(k+1)\mathcal{R}_j} |\xi^\alpha| \le \kappa |\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha| \sum_{\beta\in k\mathcal{R}_j} |\xi^\beta|.$$

From last inequality similarly to (7) from Theorem 3.1 we obtain that for each $j \in 1, \dots, I_{n-1}$ there exists a constant $\delta_j > 0$ such that

$$|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha| \ge \delta_j (1 + |\xi|_{\mathcal{R}^{n-1}_j}),$$

where $|\xi|_{\mathcal{R}^{n-1}_j} = \sum_{\beta\in\mathcal{R}^{n-1}_j} |\xi^\beta|$. For $\lambda > 0$ substituting $\xi = (\xi_1, \dots, \xi_n)$ with $\lambda^{-\frac{1}{\mu^j}}\xi = (\lambda^{-\frac{1}{\mu^j_1}}\xi_1, \dots, \lambda^{-\frac{1}{\mu^j_n}}\xi_n)$ we get

$$|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha \lambda^{1-(\alpha:\mu^j)}| \ge \delta_j (\lambda + |\xi|_{\mathcal{R}^{n-1}_j}).$$

Similarly, analogous inequalities can be obtained for all $j \in \{1, \dots, I_{n-1}\}$.

Using Theorem 6.1, from [2], we get

$$|\sum_{\alpha\in\mathcal{R}} \tilde{a}^0_\alpha \xi^\alpha \lambda^{1-\max_i(\alpha:\mu^i)}| \ge \delta(\lambda + |\xi|_{\partial'\mathcal{R}}), \forall\lambda > 0, \forall\xi \in \mathbb{R}^n.$$

Since the last inequality holds for all subsequential limit values of sequences $\{a^0_\alpha(x_M) : \alpha \in \mathcal{R}\}$ where $x_M \to \infty$, we obtain that there exist such constants $\delta > 0$ and $M > 0$ that the following holds

$$|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\xi^\alpha \lambda^{1-\max_i(\alpha:\mu^i)}| \geq \delta(\lambda + |\xi|_{\partial'\mathcal{R}}), \forall\lambda > 0, \forall\xi \in \mathbb{R}^n, |x| > M.$$

□

Further, we use the following criteria for the Fredholm property (equivalent formulation and proof can be found for example in work [16], Theorem 3.14):

Theorem 3.5. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds:*

1. *if operator A has left regularizer, then kernel of operator A in X is finite dimensional;*
2. *if operator A has right regularizer, then the image of operator A is closed in Y and cokernel is finite dimensional;*
3. *operator A has left and right regularizers if and only if A is a Fredholm operator.*

It is easy to check that the following proposition holds:

Proposition 3.1. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $P(x,\mathbb{D})$ be the differential form* (1) *with the coefficients that satisfy conditions* (3). *Then operator*

$$Tu := P(u\varphi) - \varphi Pu, u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$$

is a compact operator acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$.

Lemma 3.1. *Let $k \in \mathbb{Z}_+, q \equiv 1, x_0 \in \mathbb{R}^n$ and $P(x,\mathbb{D})$ be regular in $\mathbb{R}^n$ differential form* (1) *with the coefficients that satisfy conditions* (3). *Then, there exists $\eta_0 > 0$, such that when $\max_{\alpha\in\partial'\mathcal{R}} \sup_{x\in\mathbb{R}^n} |a_\alpha(x) - a_\alpha(x_0)| < \eta_0$, for operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k,\mathcal{R}}(\mathbb{R}^n)$ exists an operator $R : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ such that*

$$RP(x,\mathbb{D}) = I + T_1, P(x,\mathbb{D})R = I + T_2,$$

where $T_1 : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1+\sigma,\mathcal{R}}(\mathbb{R}^n)$, $T_2 : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$ are bounded linear operators with some $\sigma = \sigma(\mathcal{R}) > 0$.

Proof. For differential form $P(x,\mathbb{D})$ (see (1)) denote

$$P^0(x,\mathbb{D}) = \sum_{\alpha\in\partial'\mathcal{R}} a_\alpha(x)D^\alpha, Q(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}\setminus\partial'\mathcal{R}} a_\alpha(x)D^\alpha.$$

Consider

$$R^0 := F^{-1} \frac{|\xi|_{\partial'\mathcal{R}}}{(1+|\xi|_{\partial'\mathcal{R}})P^0(x_0,\xi)} F.$$

Since $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, it is easy to check that $|\xi|_{\partial'\mathcal{R}}/P^0(x_0,\xi)$ is a Fourier multiplier (see [17]) and for operator $R^0 : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ holds

$$R^0 P^0(x_0,\mathbb{D}) = I + T', \tag{24}$$

where $T' : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+2,\mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator.

Using (24) and estimates similar to the ones used in the proof of Lemma 4.2 from [18], it can be checked that there exists $\eta_0 > 0$ such that when $\max_{\alpha\in\partial'\mathcal{R}} \sup_{x\in\mathbb{R}^n} |a_\alpha(x) - a_\alpha(x_0)| < \eta_0$ the following holds

$$\begin{aligned} R^0P(x,\mathbb{D}) = R^0P^0(x_0,\mathbb{D}) + R^0(P^0(x,\mathbb{D}) - P^0(x_0,\mathbb{D})) + R^0Q(x,\mathbb{D}) \\ = I + T_1' + T_2', \end{aligned} \tag{25}$$

where $T_1' : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1+\sigma,\mathcal{R}}(\mathbb{R}^n)$ with some $\sigma = \sigma(\mathcal{R}) > 0$ and operator $T_2' : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ satisfies $\|T_2'\| < 1$.

Consider

$$R := (I + T_2')^{-1} R^0.$$

From (25), we get

$$RP(x,\mathbb{D}) = I + T_1,$$

where $T_1 : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1+\sigma,\mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator with some $\sigma = \sigma(\mathcal{R}) > 0$.

Similarly, we get

$$P(x,\mathbb{D})R = I + T_2,$$

where $T_2 : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$ is a bounded linear operator with some $\sigma = \sigma(\mathcal{R}) > 0$. □

Theorem 3.6. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (10) *with the coefficients that satisfy $\lim_{|x|\to\infty} \max_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$. Then, operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max_i(\alpha:\mu^i)}\xi^\alpha| \ge \delta(\lambda + |\xi|_{\partial'\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{26}$$

Proof. Let's first prove sufficient part.

Let $\delta_0 > 0$,$\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in K_{\frac{\delta_0}{2}}$, $\varphi(x) = 0$ for $|x| \ge \delta_0$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset K_{2\delta_0}$ and $\psi(x) = 1$ for $x \in K_{\delta_0}$. Let $\omega > 0$ be such that $\omega\sqrt{n} < \delta_0$. Let's denote $\{z_m\}_{m=0}^\infty$ points on the lattice in $\mathbb{R}^n$ with a side equals to ω.

Denote

$$\varphi_m(x) := \varphi(x-z_m)(\sum_{l=0}^{\infty}\varphi(x-z_l))^{-1}, \psi_m(x) := \psi(x-z_m), \ \ m \in \mathbb{Z}_+.$$

Then, $\{\varphi_m\}_{m=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$ satisfy the following conditions:

(i) $\max_{x,y\in \operatorname{supp}\varphi_m}|x-y| < \delta_0$,

(ii) there exists $r \in \mathbb{N}$ such that for any number i there are no more than r functions $\varphi_j(x)$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \neq \emptyset$;

(iii) $\varphi_m(x)\psi_m(x) \equiv \varphi_m(x)$ for all $m \in \mathbb{Z}_+$;

(iv) for any $\alpha \in \mathbb{Z}_+^n$ there exists some constant $C_\alpha > 0$ such that $|D^\alpha\varphi_m(x)| \le C_\alpha$, $|D^\alpha\psi_m(x)| \le C_\alpha$, $\forall x \in \mathbb{R}^n, \forall m \in \mathbb{Z}_+$;

(v) $\sum_{m=0}^{\infty}\varphi_m(x) \equiv 1$.

Denote $W_m = \operatorname{supp}\varphi_m, m \in \mathbb{Z}_+$. Let $x_m \in W_m$ and $m_0 \in \mathbb{N}$. For $m \le m_0$ denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}}(\psi_m(x)(a_\alpha(x)-a_\alpha(x_m)) + a_\alpha(x_m))D^\alpha.$$

For $m > m_0$, denote

$$\begin{aligned}P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}}[&\psi_m(x)(a^0_\alpha(x)q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)})\\ &+ a^0_\alpha(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}]D^\alpha.\end{aligned}$$

Since $q \in Q^{k,\mathcal{R}}$ and $\lim_{m\to\infty}\max_{|x-x_m|\le 1}|a^0_\alpha(x) - a^0_\alpha(x_m)| = 0$, according to Theorem 2.2, from [10], we can choose m_0 big enough such that for $m > m_0$ operator $P^m : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ has the inverse operator $R^m : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Since $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, according to Lemma 3.1, we get that for a small enough δ_0 from the condition (i) and any $m \le m_0$ there exists operator $R^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ such that

$$R^mP^m = I + T^m, \tag{27}$$

where $T^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$ with some number $\sigma = \sigma(\mathcal{R}) > 0$.

Denote

$$Rf := \sum_{l=0}^{\infty}\psi_l R^l(\varphi_l f), f \in H_q^{k,\mathcal{R}}(\mathbb{R}^n).$$

Since (26) holds one can check that the norms of operators R^l, acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, are uniformly bounded. From this fact, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$ and properties (i)–(v) of the functions

$\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that R is a bounded linear operator, acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

In (10) denote

$$L(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha^1(x) D^\alpha.$$

For $P(x,\mathbb{D})$ and $RP(x,\mathbb{D})$, using properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty$ and $\{\psi_m\}_{m=0}^\infty$, we get

$$\begin{aligned} P(x,\mathbb{D})u &= \sum_{m=0}^\infty \varphi_m P(x,\mathbb{D})(\psi_m u) = \sum_{m=0}^{m_0} \varphi_m P^m(x,\mathbb{D})(\psi_m u) \\ &\quad + \sum_{m=m_0+1}^\infty \varphi_m P^m(x,\mathbb{D})(\psi_m u) + \sum_{m=m_0+1}^\infty \varphi_m L(x,\mathbb{D})(\psi_m u), \end{aligned}$$

$$\begin{aligned} RP(x,\mathbb{D})u &= \sum_{l=0}^{m_0}\sum_{m=0}^{m_0} \psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) \\ &+ \sum_{l=0}^{m_0}\sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) + \sum_{l=m_0+1}^\infty\sum_{m=0}^{m_0} \psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) \\ &+ \sum_{l=m_0+1}^\infty\sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) \\ &+ \sum_{l=0}^\infty\sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l\varphi_m L(\psi_m u)), \end{aligned} \tag{28}$$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Based on definitions of $P^m(x,\mathbb{D})$ and properties of functions $\{\varphi_m\}_{m=0}^\infty$, $\{\psi_m\}_{m=0}^\infty$, the following equalities hold

a) for $0 \le l \le m_0$ and $0 \le m \le m_0$,
$\varphi_l\varphi_m P^m(x,\mathbb{D})(\psi_m u) = \varphi_l\varphi_m P(x,\mathbb{D})(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})(\psi_m u)$;

b) for $0 \le l \le m_0$ and $m > m_0$,
$$\begin{aligned}\varphi_l\varphi_m P^m(x,\mathbb{D})(\psi_m u) &= \varphi_l\varphi_m(P(x,\mathbb{D}) - L(x,\mathbb{D}))(\psi_m u) \\ &= \varphi_l\varphi_m P^l(x,\mathbb{D})(\psi_m u) - \varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u);\end{aligned}$$

c) for $l > m_0$ and $0 \le m \le m_0$,
$\varphi_l\varphi_m P^m(x,\mathbb{D})(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})(\psi_m u) + \varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)$;

d) for $l > m_0$ and $m > m_0$,
$\varphi_l\varphi_m P^m(x,\mathbb{D})(\psi_m u) = \varphi_l\varphi_m P^l(x,\mathbb{D})(\psi_m u)$,

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Consider the first three sums in (28). They contain finite number of terms such that $\varphi_l\varphi_m \neq 0$. Using equalities a)-c), (27) and Proposition 3.1 similarly to the proof of Theorem 2.6 from work [12] it can be checked that the following holds:

$$\sum_{l=0}^{m_0}\sum_{m=0}^{m_0}\psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) = \sum_{l=0}^{m_0}\sum_{m=0}^{m_0}\varphi_l\varphi_m u + T_1 u, \tag{29}$$

$$\begin{aligned}\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) &= \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l R^l(\varphi_l\varphi_m P^l(\psi_m u))\\ -\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{m_1}\psi_l R^l(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)) &= \sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u + T_2 u\\ -\sum_{l=0}^{m_0}\sum_{m=m_0+1}^{\infty}\psi_l R^l(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)),&\end{aligned} \tag{30}$$

$$\begin{aligned}\sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u)) &= \sum_{l=m_0+1}^{m_1}\sum_{m=0}^{m_0}\psi_l R^l(\varphi_l\varphi_m P^l(\psi_m u))\\ +\sum_{l=m_0+1}^{m_1}\sum_{m=0}^{m_0}\psi_l R^l(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)) &= \sum_{l=m_0+1}^{\infty}\sum_{m=0}^{m_0}\varphi_l\varphi_m u + T_3 u\\ +\sum_{l=m_0+1}^{m_1}\sum_{m=0}^{m_0}\psi_l R^l(\varphi_l\varphi_m L(x,\mathbb{D})(\psi_m u)),&\end{aligned} \tag{31}$$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, $m_1 := \max_{m>m_0}\{m : \operatorname{supp}\varphi_m \cap (\bigcup_{s=0}^{m_0}\operatorname{supp}\varphi_s) \neq \emptyset\}$ and operators T_1, T_2 and T_3 are compact operators acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

From equality d), and the fact that, for $m > m_0$ operators $R^m : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ are the inverse operators of $P^m : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ we get

$$\begin{aligned}&\sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l(\varphi_l\varphi_m P^m(\psi_m u))\\ &= \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l(\varphi_l\varphi_m P^l(\psi_m u)) = \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\varphi_l\varphi_m u\\ &+ \sum_{l=m_0+1}^{\infty}\sum_{m=m_0+1}^{\infty}\psi_l R^l(\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)),\end{aligned}$$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$.

Taking into account (26), definitions of $P^l(x,\mathbb{D})$ and properties of functions $\{\varphi_m\}_{m=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$, for $l > m_0$ and $m > m_0$ with some constant $C_1 > 0$

we get

$$
\begin{aligned}
&\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k,\mathcal{R},q} \\
&\leq C_1 \| \sum_{\alpha\in\mathcal{R}} \sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|>0}} a^0_\alpha(x) D^\beta(\psi_m u) D^\gamma(\varphi_l\varphi_m) q(x)^{1-\max_i(\alpha:\mu^i)} \|_{k,\mathcal{R},q} \\
&\leq C_1 \| \sum_{\alpha\in\mathcal{R}} \sum_{\substack{\beta+\gamma=\alpha,\\ |\gamma|>0}} a^0_\alpha(x) D^\gamma(\varphi_l\varphi_m) \frac{1}{q(x)^{\min_i(\gamma:\mu^i)}} D^\beta(\psi_m u) q(x)^{1-\max_i(\beta:\mu^i)} \|_{k,\mathcal{R},q}.
\end{aligned}
$$

From the last inequliaty, taking into account that $\frac{1}{q(x)} \rightrightarrows 0$ when $|x| \to \infty$, properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$ and the conditions on the coefficients $\{a^0_\alpha(x)\}$ (see (10)) we get

$$
\|\varphi_l\varphi_m P^l(\psi_m u) - P^l(\varphi_l\varphi_m\psi_m u)\|_{k,\mathcal{R},q} \leq \omega(m_0)\|u\|_{H_q^{k+1,\mathcal{R}}(W_l\cap W_m)}, \tag{32}
$$

where $\omega(m_0)$ is such a function that $\omega(m_0) \to 0$ when $m_0 \to \infty$. Since (26) holds, the norms of operators R^l, acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, are uniformly bounded. Using this fact, inequality (32) and properties (i)–(v) of the functions $\{\varphi_m\}_{m=0}^\infty, \{\psi_m\}_{m=0}^\infty$, it is easy to check that for a big enough m_0 the operator

$$
T_4 := \sum_{l=m_0+1}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l[\varphi_l\varphi_m P^l(\psi_m \cdot) - P^l(\varphi_l\varphi_m\psi_m \cdot)],
$$

acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, satisfies $\|T_4\| < \frac{1}{2}$.

Similarly, for remained terms from (28), (30) and (31), taking into account that $D^\beta(a^1_\alpha(x)) = o(q(x)^{1-\max_i(\alpha-\beta:\mu^i)})$ when $|x| \to \infty$, $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$, for a big enough m_0 we get that the operator

$$
\begin{aligned}
T_5 := &\sum_{l=0}^\infty \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l\varphi_m L(\psi_m \cdot)) - \sum_{l=0}^{m_0} \sum_{m=m_0+1}^\infty \psi_l R^l(\varphi_l\varphi_m L(\psi_m \cdot)) \\
&+ \sum_{l=m_0+1}^\infty \sum_{m=0}^{m_0} \psi_l R^l(\varphi_l\varphi_m L(\psi_m \cdot)),
\end{aligned}
$$

acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, has a norm that satisfies $\|T_5\| < \frac{1}{2}$.

Denote

$$
T' := T_1 + T_2 + T_3, T'' := T_4 + T_5.
$$

From the representation (28), we get

$$
RPu = \sum_{l=0}^\infty \sum_{m=0}^\infty \varphi_l\varphi_m u + T_1 u + T_2 u + T_3 u + T_4 u + T_5 u = u + T'u + T''u,
$$

where $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$, $T' : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ is a compact operator and for operator $T'' : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ we have $\|T''\| < 1$. Therefore,

$$(I + T'')^{-1} RP = I + (I + T'')^{-1} T',$$

where $T := (I+T'')^{-1}T' : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ is a compact operator. So we get that operator $(I+T'')^{-1}R : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ is a left regularizer.

Analogously we can construct a right regularizer.

Since right and left regularizers exist, applying Theorem 3.5, we obtain the Fredholm property for operator $P(x, \mathbb{D})$ acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$. For necessary part regularity of $P(x, \mathbb{D})$ follows from Corollary 3.1. Necessity of condition (26) for the Fredholm property of operator $P(x, \mathbb{D})$, acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$, follows from Theorem 3.3 and Theorem 3.4. □

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