A generalized BL-ring

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Abstract. The purpose of this work is to extend the commutative rings whose lattice of ideals can be equipped with a structure of BL-algebra as carried out by Heubo-Kwegna et al in 2018, to non-commutative ones which are called pseudo BL-rings in

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this paper. We study and characterize rings whose ideals form a pseudo BL-algebra, we describe them in terms of their subdirectly irreductible factors. We obtain that every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of special primary ring and is also isomorphic to a subring of a direct product of discrete valuation rings.

Keywords: BL-rings, pseudo-BL algebras, pseudo BL-ring, BL-algebras, multiplication ring.

1. Introduction

It is well known that the algebraic study of classical logic proceeds via Boolean algebras [1]. For the infinite valued logic of Łukasiewicz, the algebraic analogues of Boolean algebras are the MV-algebras. It is also well known that Boolean algebras can be subsumed within the theory of rings. From their inception there was the question of whether or not the same is true of MV-algebras. In 2009 Belluce and Di Nola wrote an article on commutative rings whose ideals form an MV-algebra [1], in that work they introduced and gave some properties of a class of commutative rings, whose lattice of ideals forms an MV-algebra, which were called Łukasiewicz rings. An important non-commutative generalization of MV-algebra, known as pseudo MV-algebra was introduced by Georgescu and Iorgulescu [5]. The natural question that arises is what happens if one drops the commutativity assumption on Łukasiewicz rings. The answer of that question has been the goal of Kadji, Lele and Nganou [7], where they studied and characterized all rings whose ideals form a pseudo MV-algebra, which were called generalized Łukasiewicz rings, this was in 2016. Since the class of BL-algebras contains the MV-algebras, from then in 2018, the authors in [8] initiated the study of commutative rings, whose lattice of ideals can be equipped with a structure of BL-algebra. Recall that BL-algebras introduced by Peter Hájek in 1998 as algebraic structures for the basic logic, he used it to formalize many-valued logics (in short MV logics) induced by continuous t-norms on the real unit interval [0,1] and they were generalized to pseudo BL-algebras by Di Nola, Georgescu and Iorgulescu [3].

In the idea to continue the investigation of the classes of rings for which the residuated lattice $A(R)$ of their ideals is an algebra of a well-known subvariety of residuated lattices, in this work we generalize the notion of commutative BL-ring to non commutative case, and we call it *pseudo BL-ring.*

The paper is organized as follows. In Section 2, we introduce and recall some definitions and preliminary results that we will use in the paper. In Section 3, we introduce and study the main properties of pseudo BL-ring, also its connections with some class of non commutative rings. In Section 4, we proved that this class is closed under finite direct products, arbitrary direct sums and homomorphic images. Furthermore, a description of subdirectly irreducible pseudo BL-rings is obtained.

2. General preliminaries

We recall here some definitions and properties that will be used throughout the article.

Definition 2.1 ([6])**.** A *BL-algebra* is a structure $(A, \vee, \wedge, \odot, 0, 1)$ such that

(1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(A, \odot, 1)$ is an abelian monoid, that is, $\odot$ is commutative, associative and $x \odot 1 = 1 \odot x = x$,

(3) the following conditions hold for all $x, y, z \in A$,

(BL-1) $x \odot y \leq z$ iff $x \leq y \to z$ (residuation),

(BL-2) $x \wedge y = x \odot (x \to y)$ (divisibility),

(BL-3) $(x \to y) \vee (y \to x) = 1$ (prelinearity).

An *MV-algebra* is a BL-algebra A satisfying the double negation law: $x^{**} = x$, for all $x \in A$, where $x^* = x \to 0$.

Given any BL-algebra A, $MV(A) := \{x^* \mid x \in A\}$ is an MV-algebra, and is indeed the largest BL-subalgebra of A satisfying the double negation. This MV-algebra is called the *MV-center of A.*

We recall that a BL-algebra A is an MV-algebra iff $(x^*)^* = x$, for any $x \in A$.

Definition 2.2 ([5])**.** A pseudo MV-algebra is a structure $(A, \odot, \oplus, (^*), (^-), 0, 1)$ of type (2, 2, 1, 1, 0, 0), such that the following axioms are satisfied for all $x, y, z \in A$,

(1) $x \odot (y \odot z) = (x \odot y) \odot z$,

(2) $x \odot 1 = 1 \odot x = x$,

(3) $x \odot 0 = 0 \odot x = 0$,

(4) $0^* = 1$, $0^- = 1$,

(5) $(x^- \odot y^-)^* = (x^* \odot y^*)^-$,

(6) $x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^*) \odot y = (y \oplus x^*) \odot x$,

(7) $x \oplus (x^* \odot y) = (x \odot y^*) \oplus y$,

(8) $(x^*)^- = x$, where $y \oplus x := (x^* \odot y^*)^- = (x^* \odot y^*)^-$.

Definition 2.3 ([3])**.** A pseudo BL-algebra is a structure $\mathcal{A} = (A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ of type (2, 2, 2, 2, 2, 0, 0) which satisfies the following axioms, for all $x, y, z \in A$,

(1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(A, \odot, 1)$ is a monoid, that is, $\odot$ is associative and $x \odot 1 = 1 \odot x = x$,

(3) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,

(4) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$,

(5) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

A ring R is said to be *generated by idempotents*, if for every $x \in R$, there exists an idempotent element $e \in R$ (that is $e^2 = e$) such that $e \cdot x = x \cdot e = x$. There is a residuated lattice formed by the two-sided ideals of the ring R, this lattice is defined by

$$A(R) := \langle Id(R), \wedge, \vee, \otimes, \to, \rightsquigarrow, \{0\}, R \rangle,$$

where

$$I \wedge J = I \cap J,\ I \vee J = I + J,\ I \otimes J = I \cdot J,$$
$$I \to J := \{x \in R \mid xI \subseteq J\},\ I \rightsquigarrow J := \{x \in R \mid Ix \subseteq J\}.$$

Note that $I^* := \{x \in R \mid xI = \{0\}\}$ denotes the right annihilator of I in R and $I^- := \{x \in R \mid Ix = \{0\}\}$ denotes the left annihilator of I in R, where I is an ideal of R. Given a ring R, recall that an ideal I of R is called an annihilator ideal (resp. a dense ideal) if $I = J^*$ and $I = K^-$ for some ideals J, K of R (resp. $I^* = I^- = \{0\}$).

$AN^*(R) := \{I^* \mid I \in A(R)\}$ and $AN^-(R) := \{I^- \mid I \in A(R)\}$ denote the sets of left and right annihilator ideals of R, respectively.

$D^*(R) := \{I \in A(R) \mid I^* = \{0\}\}$ and $D^-(R) := \{I \in A(R) \mid I^- = \{0\}\}$ denote the sets of left and right dense ideals of R, respectively.

Definition 2.4 ([11]). A ring R is called a multiplication ring, if for every ideals I, J of R such that $I \subseteq J$, there exist ideals K, K' such that $I = J \cdot K = K' \cdot J$.

3. Pseudo BL-rings

Definition 3.1. A ring R generated by idempotents is called a *pseudo BL-ring* if it satisfies,

PBLR-1 $I \cap J = I \cdot (I \rightsquigarrow J) = (I \to J) \cdot I$,

PBLR-2 $(I \to J) + (J \to I) = (I \rightsquigarrow J) + (J \rightsquigarrow I) = R$,

for all ideals I, J of R.

Lemma 3.1. *Let I and J be two ideals of a ring R. Then*
(1) $I \to J = I \to (I \cap J)$ and $I \rightsquigarrow J = I \rightsquigarrow (I \cap J)$.
(2) $(I + J) \to J = I \to J$ and $(I + J) \to I = J \to I$.

Proof. These are easily derived from the definitions and the operations involved. □

Lemma 3.2. *Let I and J be two ideals of a ring R. Then $(I \to J) \cdot I \subseteq I \cap J$ and $I \cdot (I \rightsquigarrow J) \subseteq I \cap J$.*

Proof. Let $x \in (I \to J) \cdot I$, then $x = \sum_{i=1}^{n} a_i b_i$ with $a_i \in I \to J$, $b_i \in I$ and $n \in \mathbb{N}$. By the definition of ideal and the definition of $I \to J$, we have that $x \in I$ and $x \in J$. Therefore, $x \in I \cap J$ and $(I \to J) \cdot I \subseteq I \cap J$.

Let $y \in I \cdot (I \rightsquigarrow J)$, then $y = \sum_{i=1}^{n} a_i b_i$ with $a_i \in I$, $b_i \in I \rightsquigarrow J$ and $n \in \mathbb{N}$. By the definition of ideal and the definition of $I \rightsquigarrow J$, we have that $y \in I$ and $y \in J$. Therefore, $y \in I \cap J$ and $I \cdot (I \rightsquigarrow J) \subseteq I \cap J$. □

The following proposition gives some axioms which are equivalent to PBLR-1 or PBLR-2.

Proposition 3.1. *Let R be a pseudo BL-ring. For all ideals I, J, K of R, we have that,*

(1) *PBLR-1 iff $I \cap J \subseteq (I \to J) \cdot I$ and $I \cap J \subseteq I \cdot (I \rightsquigarrow J)$.*

(2) *PBLR-2 iff $(I \cap J) \to K = (I \to K) + (J \to K)$ iff $(I \cap J) \rightsquigarrow K = (I \rightsquigarrow K) + (J \rightsquigarrow K)$.*

(3) *PBLR-2 iff $I \to (J + K) = (I \to J) + (I \to K)$ iff $I \rightsquigarrow (J + K) = (I \rightsquigarrow J) + (I \rightsquigarrow K)$.*

Proof. (1) By using Lemma 3.2.

(2) Assume that $(I \cap J) \to K = (I \to K) + (J \to K)$. Let's take $K = I \cap J$, then $(I \cap J) \to (I \cap J) = (I \to (I \cap J)) + (J \to (I \cap J))$. Since $(I \cap J) \to (I \cap J) = R$, $I \to (I \cap J) = I \to J$ and $J \to (I \cap J) = J \to I$, by Lemma 3.1, we have $R = (I \to J) + (J \to I)$.

Assume now that $R = (I \to J) + (J \to I)$.

Let $x \in (I \cap J) \to K$, then x can be written as $x = x \cdot i + x \cdot j$, with $i \in I$, $j \in J$, and $i + j = 1$ (1 is an idempotent). Thus, $x \cdot j \in I \to K$ and $x \cdot i \in J \to K$. Therefore, $(I \cap J) \to K \subseteq (I \to K) + (J \to K)$.

Let $z \in (I \to K) + (J \to K)$, then $z = x + y$ with $x \in I \to K$ and $y \in J \to K$, that is $xI \subseteq K$ and $yJ \subseteq K$. Thus, $x(I \cap J) \subseteq K$ and $y(I \cap J) \subseteq K$, and we have that $z(I \cap J) = (x + y)(I \cap J) \subseteq K$. Therefore, $(I \to K) + (J \to K) \subseteq (I \cap J) \to K$.

Using the same method with $\rightsquigarrow$ it is straightforward that PBLR-2 holds iff $(I \cap J) \rightsquigarrow K = (I \rightsquigarrow K) + (J \rightsquigarrow K)$.

(3) Assume that PBLR-2 is true, then for ideals J and K of R, we have $J \to K + K \to J = R$. Let $y \in I \to (J + K)$, then $y \in R$. This means that y can be written as $y = y \cdot j + y \cdot k$, with $j \in J$ and $k \in K$, such that $j + k = 1$ (1 is an idempotent). Since $y \cdot j \in I \to J$ and $y \cdot k \in I \to K$, then $I \to (J + K) \subseteq (I \to J) + (I \to K)$.

Let $x \in (I \to J) + (I \to K)$, then $x = a + b$ with $a \in I \to J$ and $b \in I \to K$. Thus, $aI \subseteq J$ and $bI \subseteq K$, so $aI + bI \subseteq J + K$. Therefore, $xI \subseteq J + K$, and $I \to (J + K) \supseteq (I \to J) + (I \to K)$.

Conversely, assume now that $I \to (J + K) = (I \to J) + (I \to K)$. Set $I = J + K$, then by Lemma 3.1, we have $J \to K + K \to J = R$ for all ideals J, K of R. Thus, PBLR-2 holds.

Using the same method with " $\rightsquigarrow$ " it is straightforward that PBLR-2 holds iff $I \rightsquigarrow (J + K) = (I \rightsquigarrow J) + (I \rightsquigarrow K)$. □

Remark 3.1. Let R be a commutative ring. If $I \to J = I \rightsquigarrow J$, then R is a BL-ring.

By Definitions 3.1 and 2.4, we have the following results that give a characterization of the pseudo BL-rings.

Proposition 3.2. *Let R be a ring. R satisfies PBLR-1 if and only if it is a multiplication ring.*

Proof. Let R be a ring. Assume that R satisfies PBLR-1, then $I \cap J = I \cdot (I \rightsquigarrow J) = (I \to J) \cdot I$. Let I and J be two ideals of R such that $I \subseteq J$. We can find two ideals K and K' of R, such that $I = J \cdot K = K' \cdot J$.

By $PBLR-1$, $I = I \cap J = J \cdot (J \rightsquigarrow I) = (J \to I) \cdot J$. Then $K = J \rightsquigarrow I$ and $K' = J \to I$.

Conversely, assume that R is a multiplication ring and let I, J be ideals of R. Since $I \cap J \subseteq J$, there exist two ideals K and K' of R such that $I \cap J = J \cdot K = K' \cdot J$. Hence $J \cdot K \subseteq I$, $K' \cdot J \subseteq I$ and it follows that $K \subseteq J \rightsquigarrow I$ and $K' \subseteq J \to I$.

Thus, $I \cap J \subseteq J \cdot (J \rightsquigarrow I)$ and $I \cap J \subseteq (J \to I) \cdot I$. Therefore, PBLR-1 holds. □

Proposition 3.3. *Let R be a ring. Then the followings are equivalent,*

(1) *R is a pseudo BL-ring.*

(2) *$A(R)$ is a pseudo BL-algebra.*

Proof. Assume that R is a pseudo BL-ring. Then PBLR-1 and PBLR-2 hold.

By the definition of $A(R)$, $(A(R); \wedge, \vee, \{0\}, R)$ is a bounded lattice and $(A(R); \otimes, R)$ is an associative monoid.

Let I, J and K be ideals of R, then $I \otimes J \subseteq K \Leftrightarrow I \subseteq J \to K \Leftrightarrow J \subseteq I \rightsquigarrow K$. Since PBLR-1 and PBLR-2 hold, then prelinearity and divisibility hold. Thus, $A(R)$ is a pseudo BL-algebra.

Conversely assume that $A(R)$ is a pseudo BL-algebra.

Since for all x, y in R, $x \wedge y = (x \to y) \cdot x = x \cdot (x \rightsquigarrow y)$ and $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, then PBLR-1 and PBLR-2 hold. Therefore, by Proposition 3.2, R is a multiplication ring, so generated by idempotents. Thus, R is a pseudo BL-ring. □

We know that if R is a ring and $M_n(R)$ the ring of $n \times n$ matrices over R, then any ideal $\mathcal{I}$ of $M_n(R)$ has the form $M_n(I)$ for a uniquely determined ideal I of R [9].

Lemma 3.3. *If R is a multiplication ring, then $M_n(R)$ is also a multiplication ring.*

Proof. Let $M_n(I)$ and $M_n(J)$ be two ideals of $M_n(R)$ such that $M_n(I) \subseteq M_n(J)$, with I, J two ideals of R.

It is easy to verify that $M_n(I) \subseteq M_n(J)$ if and only if $I \subseteq J$. Since R is a multiplicative ring, then there exist two ideals K and K' of R, such that $I = J \cdot K = K' \cdot J$ which allows that $M_n(I) = M_n(J) \cdot M_n(K) = M_n(K') \cdot M_n(J)$. Thus, $M_n(R)$ is a multiplicative ring. □

Lemma 3.4. *If R is a multiplication ring, then $M_n(R)$ satisfies PBLR-1.*

Proof. Let R be a multiplication ring, then by using Lemma 3.3 and Proposition 3.2, we have that PBLR-1 holds for $M_n(R)$. □

Lemma 3.5. *Let I and J be two ideals of a ring R, then*

(1) $M_n(I+J) = M_n(I) + M_n(J)$.

(2) $M_n(I \to J) = M_n(I) \to M_n(J)$ *and* $M_n(I \rightsquigarrow J) = M_n(I) \rightsquigarrow M_n(J)$.

Proof. Straightforward, just use the definition of the operations involved. □

Example 3.1. (1) Let R be a Notherian multiplication ring, then $M_n(R)$ is a pseudo BL-ring.

(2) Let R be a discrete valuation ring or a Łukasiewicz ring, then $M_n(R)$ is a pseudo BL-ring.

Now given a ring R and a proper ideal I of R, let's consider the ring R/I. An ideal of R/I is a set $J/I = \{x/I \mid x \in J\}$, where J is an ideal of R and $I \subseteq J$. We also define the left annihilator (respectively the right annihilator) of an ideal I by $I^* = \{x \in R \mid xI = \{0\}\} = \{x \to 0 \mid x \in I\}$ (respectively $I^- = \{x \in R \mid Ix = \{0\}\} = \{x \rightsquigarrow 0 \mid x \in I\}$).

Lemma 3.6. *Let I, J and K be ideals of a ring R, such that $I \subseteq J$ and $I \subseteq K$. Then,*

(1) $(J/I)^* = (J \to I)/I$ *and* $(J/I)^- = (J \rightsquigarrow I)/I$.

(2) $(J/I) \to (K/I) = (J \to K)/I$ *and* $(J/I) \rightsquigarrow (K/I) = (J \rightsquigarrow K)/I$.

Proof. (1) Let $x/I \in (J/I)^*$, then for all a/I with $a \in J$, $xa \in I$. This means that $xJ \subseteq I$. Thus, $x/I \in (J \to I)/I$. Therefore, $(J/I)^* \subseteq (J \to I)/I$.

Conversely, let $y/I \in (J \to I)/I$, then for any $a/I \in J/I$, $ya \in I$ because, $yJ \subseteq I$ by the definition of $y/I \in (J \to I)/I$. This means that $y/I(J/I) = I$. Therefore, $(J/I)^* \supseteq (J \to I)/I$.

To show that $(J/I)^- = (J \rightsquigarrow I)/I$ holds, we just have to use the same idea as in $(J/I)^* = (J \to I)/I$.

(2) Let $y/I \in (J \to K)/I$, then $yJ \subseteq K$. This implies that $(xJ)/I \subseteq K/I$ which is $(y/I) \cdot (J/I) \subseteq K/I$. Thus, $(J \to K)/I \subseteq (J/I) \to (K/I)$.

Conversely, let $x/I \in (J/I) \to (K/I)$, then for all $a \in J$ $xa \in K$, which implies that $x/I \in (J \to K)/I$. Therefore, $(J/I) \to (K/I) \subseteq (J \to K)/I$.

With the same idea as for $(J/I) \to (K/I) = (J \to K)/I$, we can show that $(J/I) \rightsquigarrow (K/I) = (J \rightsquigarrow K)/I$. □

It is possible to observe if the property PBLR-2 holds in a ring R by observing its quotient by a proper ideal. This is the main purpose of the following propositions.

Proposition 3.4. *Let I and J be two ideals of a ring R. If the ring R satisfies PBLR-2, then $I \cap J = \{0\}$ implies that $I^* + J^* = R$ and $I^- + J^- = R$.*

Proof. Assume that $I \cap J = \{0\}$.

From PBLR-2 we have that $I \to J + J \to I = R$ and $I \rightsquigarrow J + J \rightsquigarrow I = R$. Then $I^* + J^* = R$ and $I^- + J^- = R$ by using Lemma 3.1, definitions and operations involved. □

We denote PBLR-3: $I \cap J = \{0\}$ implies that $I^* + J^* = R$ and $I^- + J^- = R$.

Proposition 3.5. *A ring R satisfies PBLR-2 if and only if any quotient of it by an ideal satisfies PBLR-3.*

Proof. Let I be an ideal of the ring R which satisfies PBLR-2. Let J, K be two ideals of R such that $I \subseteq J$, $J \subseteq K$ and $(J/I) \cap (K/I) = I$. It is clear that $J \cap K = I$. Since $(J/I)^- + (K/I)^- = (J \to I)/I + (K \to I)/I = (J \to (J \cap K))/I + (K \to (J \cap K))/I = ((J \to K) + (K \to J))/I = R/I$ (by using Lemmas 3.1, 3.6). In the same way, $(J/I)^* + (K/I)^* = R/I$. Thus, PBLR-3 holds for R/I.

Conversely, suppose that in every factor of R, PBLR-3 holds. Let I, J be ideals of R, then $R/(I \cap J)$ satisfies PBLR-3. Clearly $(I/(I \cap J)) \cap (J/(I \cap J)) = (I \cap J)$, then $(I/(I \cap J))^- + (J/(I \cap J))^- = R/(I \cap J)$ and $(I/(I \cap J))^* + (J/(I \cap J))^* = R/(I \cap J)$ (by using Lemma 3.6). That is $(I \rightsquigarrow J)/(I \cap J) + (J \rightsquigarrow I)/(I \cap J) = (I \to J)/(I \cap J) + (J \to I)/(I \cap J) = R/(I \cap J)$. Thus, $((I \to J) + (J \to I))/(I \cap J) = ((I \rightsquigarrow J) + (J \rightsquigarrow I))/(I \cap J) = R/(I \cap J)$ that implies $(I \to J) + (J \to I) = R$ and $(I \rightsquigarrow J) + (J \rightsquigarrow I) = R$. Finally R satisfies PBLR-2, which concludes the proof. □

Proposition 3.6. *Every quotient of a multiplication ring is a multiplication ring.*

Proof. Let R be a multiplication ring and K an ideal of R. Let I/K and J/K be two ideals of R/K such that $I/K \subseteq J/K$, then $I \subseteq J$. Hence, there exist ideals T and T' of R (because R is a multiplication ring) such that $I = J \cdot T = T' \cdot J$. Thus, $I/K = (J \cdot T)/K = (T' \cdot J)/K$ and $I/K = J/K \cdot T/K = T'/K \cdot J/K$. Therefore, R/K is a multiplication ring. □

Proposition 3.7. *Let R be a multiplication ring such that its quotient ring satisfies PBLR-3, then R satisfies PBLR-3.*

Proof. Let R ba a multiplication ring, then its quotient ring is a multiplication ring (by Proposition 3.6). Since the quotient of R satisfies PBLR-3, then the ring R satisfies PBLR-2 (by Proposition 3.5). We use Proposition 3.4 to conclude that PBLR-3 holds for the ring R. □

Corollary 3.1. *A multiplication ring satisfies PBLR-2 if and only if it satisfies PBLR-3.*

Corollary 3.2. *Any quotient of a pseudo BL-ring by a proper ideal is also a pseudo BL-ring.*

Definition 3.2 ([9])**.** A ring R is called a prime ring if its null ideal is a prime ring.

In the following, we sometimes use 0 to denote the null ideal of the ring R, which can be easily recognized from the context.

Proposition 3.8. *Prime ideals of pseudo BL-rings are maximal.*

Proof. Let I, J be two ideals of a pseudo BL-ring R, such that I is a prime ideal and $I \subseteq J \subseteq R$ (with $I \neq J$). Since I is prime, it is known that R/I is a prime ring, so $(J/I)^* = (J/I)^- = 0/I$.

Since R/I is a pseudo BL-ring (by Corollary 3.2), $(J/I) = ((J/I)^*)^* = ((J/I)^-)^- = (0/I)^* = (0/I)^- = R/P$. Thus, $J = R$ and I is a maximal ideal of R. □

Proposition 3.9. *Let R be a pseudo BL-ring and I an ideal of R such that $I \cap I^* = \{0\}$ and $I \cap I^- = \{0\}$. Then I is a pseudo BL-ring.*

Proof. It is well known that an ideal is a subring, thus $Id(I) \subseteq Id(R)$ because

$$\begin{cases} I \cap I^* = \{0\}, \\ I \cap I^- = \{0\}, \end{cases}$$

imply that,

$$\begin{cases} I + I^* = I \vee I^* = R, \\ I + I^- = I \vee I^- = R, \end{cases}$$

Since R is generated by idempotents, for any $x \in I$, there are right and left idempotents $e, e' \in R$, $m_1, m_3 \in I$ and $m_2 \in I^*$, $m_4 \in I^-$ such that $ex = x, e'x = x$ and

$$\begin{cases} m_1 + m_2 = e, \\ m_3 + m_4 = e', \end{cases}$$

Then $m_1e + m_2e = e^2 = e = m_1 + m_2$ implies $m_1 - m_1e = m_2e - m_2 \in I \cap I^* = \{0\}$ and $e'm_3 + e'm_4 = (e')^2 = e' = m_3 + m_4$ implies $m_3 - e'm_3 = e'm_4 - m_4 \in I \cap I^- = \{0\}$.

It follows that

$$\begin{cases} m_1 = m_1e, \ m_2e = m_2 \\ e'm_3 = m_3, \ e'm_4 = m_4 \end{cases}$$

$x = m_1x$, $x = xm_3$, $(m_1)^2 = m_1$, $(m_3)^2 = m_3$ since $m_2 \in M^*$, $m_4 \in I^-$. Thus, m_1 and m_3 are idempotents in I, and I is generated by idempotents.

Since $Id(M) \subseteq Id(R)$, then by Proposition 3.7, PBLR-2 holds for I. Hence I is a pseudo BL-ring. □

Proposition 3.10. *Let R be a pseudo BL-rings. R is closed under each of the following operations,*

(1) *finite direct products,*
(2) *arbitrary direct sums,*
(3) *homomorphic images.*

Proof. (1) Let $R = \prod_{k=1}^{n} R_k$ where each R_k is a pseudo BL-ring. Since each R_k is generated by idempotents, one gets that any ideal I of R is of the form

$$I = \prod_{k=1}^{n} I_k,$$

where I_k is an ideal of R_k.

Let $I = \prod_{k=1}^{n} I_k$ and $J = \prod_{k=1}^{n} J_k$ to be two ideals of R, then we get,

$$I \cdot J = \prod_{k=1}^{n} I_k \cdot \prod_{k=1}^{n} J_k = \prod_{k=1}^{n} I_k \cdot J_k.$$

Analogously, we also have $I \to J = \prod_{k=1}^{n} I_k \to J_k$ and $I \rightsquigarrow J = \prod_{k=1}^{n} I_k \rightsquigarrow J_k$. Then

$$\begin{aligned} I \cap J &= \prod_{k=1}^{n} I_k \cap \prod_{k=1}^{n} J_k = \prod_{k=1}^{n} (I_k \cap J_k) = \prod_{k=1}^{n} (I_k \cdot (I_k \rightsquigarrow J_k) \\ &= \prod_{k=1}^{n} I_k \cdot \prod_{k=1}^{n} (I_k \rightsquigarrow J_k) = I \cdot (I \rightsquigarrow J) \end{aligned}$$

and

$$\prod_{k=1}^{n} (I_k \cap J_k) = \prod_{k=1}^{n} (I_k \to J_k) \cdot I_k = \prod_{k=1}^{n} (I_k \to J_k) \cdot \prod_{k=1}^{n} I_k = (I \to J) \cdot I.$$

Thus, R satisfies PBLR-1, and because

$$I + J = \prod_{k=1}^{n} I_k + \prod_{k=1}^{n} J_k = \prod_{k=1}^{n} (I_k + J_k),$$

we have,

$$\begin{aligned} (I \to J) + (J \to I) &= \prod_{k=1}^{n} (I_k \to J_k) + \prod_{k=1}^{n} (J_k \to I_k) \\ &= \prod_{k=1}^{n} ((I_k \to J_k) + (J_k \to I_k)) = \prod_{k=1}^{n} R_k = R, \end{aligned}$$

and

$$\begin{aligned} (I \rightsquigarrow J) + (J \rightsquigarrow I) &= \prod_{k=1}^{n} (I_k \rightsquigarrow J_k) + \prod_{k=1}^{n} (J_k \rightsquigarrow I_k) \\ &= \prod_{k=1}^{n} ((I_k \rightsquigarrow J_k) + (J_k \rightsquigarrow I_k)) = \prod_{k=1}^{n} R_k = R. \end{aligned}$$

Thus, R satisfies PBLR-2. Therefore, R is a pseudo BL-ring.

(2) Let $R = \bigoplus_{k=1}^{n} R_k$, where each R_k is a pseudo BL-ring. Since each R_k is generated by idempotents, one gets that any ideal I of R is of the form,

$$I = \bigoplus_{k=1}^{n} I_k,$$

where I_k is an ideal of R_k.

Let $I = \bigoplus_{k=1}^{n} I_k$ and $J = \bigoplus_{k=1}^{n} J_k$ to be two ideals of R, we get,

$$I \cdot J = \bigoplus_{k=1}^{n} I_k \cdot \bigoplus_{k=1}^{n} J_k = \bigoplus_{k=1}^{n} I_k \cdot J_k;$$

analogously, we also have $I \to J = \bigoplus_{k=1}^{n} I_k \to J_k$ and $I \rightsquigarrow J = \bigoplus_{k=1}^{n} I_k \rightsquigarrow J_k$. Then

$$\begin{aligned} I \cap J &= \bigoplus_{k=1}^{n} I_k \cap \bigoplus_{k=1}^{n} J_k = \bigoplus_{k=1}^{n} (I_k \cap J_k) = \bigoplus_{k=1}^{n} (I_k \cdot (I_k \rightsquigarrow J_k) \\ &= \bigoplus_{k=1}^{n} I_k \cdot \bigoplus_{k=1}^{n} (I_k \rightsquigarrow J_k) = I \cdot (I \rightsquigarrow J) \end{aligned}$$

and

$$\bigoplus_{k=1}^{n} (I_k \cap J_k) = \bigoplus_{k=1}^{n} (I_k \to J_k) \cdot I_k = \bigoplus_{k=1}^{n} (I_k \to J_k) \cdot \bigoplus_{k=1}^{n} I_k = (I \to J) \cdot I.$$

Thus, R satisfies PBLR-1, and because $I + J = \bigoplus_{k=1}^{n} I_k + \bigoplus_{k=1}^{n} J_k = \bigoplus_{k=1}^{n} (I_k + J_k)$, we have,

$$\begin{aligned} (I \to J) + (J \to I) &= \bigoplus_{k=1}^{n} (I_k \to J_k) + \bigoplus_{k=1}^{n} (J_k \to I_k) \\ &= \bigoplus_{k=1}^{n} ((I_k \to J_k) + (J_k \to I_k)) = \bigoplus_{k=1}^{n} R_k = R \end{aligned}$$

and

$$\begin{aligned} (I \rightsquigarrow J) + (J \rightsquigarrow I) &= \bigoplus_{k=1}^{n} (I_k \rightsquigarrow J_k) + \bigoplus_{k=1}^{n} (J_k \rightsquigarrow I_k) \\ &= \bigoplus_{k=1}^{n} ((I_k \rightsquigarrow J_k) + (J_k \rightsquigarrow I_k)) = \bigoplus_{k=1}^{n} R_k = R. \end{aligned}$$

Thus, R satisfies PBLR-2. Therefore, R is a pseudo BL-ring.

(3) Let R be a PBL-ring and I be an ideal of R. We recall that the ideals of R/I are of the form J/I, where J is an ideal of R with $I \subseteq J$.

Let J and K be two ideals of R such that $I \subseteq J, K$. Then $(J/I) \cap (K/I) = (J \cap K)/I = ((J \to K) \cdot J)/I = (J \to K)/I \cdot (J/I) = ((J/I) \to (K/I)) \cdot (J/I)$; furthermore, $(J \cap K)/I = J \cdot (J \rightsquigarrow K)/I = (J/I) \cdot (J \rightsquigarrow K)/I = (J/I) \cdot (J/I \rightsquigarrow K/I)$, thus R/I satisfies PBLR-1.

We also have $(J/I \to K/I) + (K/I \to J/I) = (J + K)/I \to (K + J)/I = ((J \to K) + (K \to J))/I = R/I$ and $(J/I \rightsquigarrow K/I) + (K/I \rightsquigarrow J/I) = (J + K)/I \rightsquigarrow (K + J)/I = ((J \rightsquigarrow K) + (K \rightsquigarrow J))/I = R/I$, thus R/I satisfies PBLR-2. Therefore, R/I is a pseudo BL-ring. □

We are going to establish the connection between pseudo BL-ring and some well-known rings.

Definition 3.3. A unitary ring is called a Von Neumann ring if R/P is a division ring for all prime ideals of R.

Definition 3.4. A non commutative Bear ring is a ring in which for every ideal I of R, there exist idempotents $e, e' \in R$ such that $I^* = eR$ and $I^- = Re'$.

An ideal in a Bear ring (commutative or not) is called a *Bear-ideal* if for every $a, b \in R$ such that $a - b \in I$, then $a^* - b^* \in I$ and $a^- - b^- \in I$.

Definition 3.5 ([8])**.** A reducing ring is a ring in which 0 is the only nilpotent element, that is, the only element $x \in R$ for which there exists an integer $n \geq 1$ such that $x^n = 0$.

Lemma 3.7. *Let R be a ring and I, J be ideals of R. If $I \cap J = \{0\}$, then $I \subseteq J^*$, $I \subseteq J^-$.*

Proof. This is easily derived from the definition of ideal and the operations $(^*)$ and $(^-)$ involved. □

Proposition 3.11. *In a reduced ring with identities, PBLR-3 holds if and only if this ring is a Bear ring.*

Proof. Let R be a reduced ring with identities in which PBLR-3 hlods. Let K be an ideal of R, then $K \cap K^* = \{0\}$ and $K \cap K^- = \{0\}$ implies $K^* + (K^*)^* = R$ and $K^- + (K^-)^- = R$. Thus, there are $k_1 \in K^*$, $k_1^{'} \in (K^*)^*$, $k_2 \in K^-$, and $k_2^{'} \in (K^-)^-$ such that $1_L = k_1 + k_1^{'}$ and $1_R = k_2 + k_2^{'}$.So,

$$\begin{cases} k_1 = k_1 \cdot 1_L = k_1(k_1 + k_1^{'}) = k_1^2, \\ k_2 = 1 \cdot k_2 = (k_2 + k_2^{'})k_2 = k_2^2, \end{cases}$$

from which we get that k_1 and k_2 are idempotents.

It remains to show that $K^* = k_1R$ and $K^- = Rk_2$. Let $x \in K^*$, then $x = 1_L \cdot x = (k_1 + k_1^{'})x = k_1x \in k_1R$.

Let $y \in k_1R$, then there is an $r \in R$ such that $y = k_1r \in K^*$ because $k_1 \in K^*$. Therefore, $K^* = k_1R$.

By the similar way, $K^- = Rk_2$. Then, R is a non commutative Bear ring.

Conversely, assume that I and J are two ideals of a Bear ring such that $I \cap J = \{0\}$. We have to show that $I^* + J^* = R = I^- + J^-$.

Since R is a Bear ring, there exist two idempotents e and e' in R, such that $J^* = eR$ and $J^- = Re'$. Thus, $I \subseteq eR$ and $I \subseteq Re'$ (by Lemma 3.7). Hence for all $i \in I$, $i = er = r'e'$ for some $r, r' \in R$ and

$$\begin{cases} 0 = er - er = er - e^2 r = (1_L - e)er, \\ 0 = r'e' - e'r' = r'e' - (e')^2 r' = r'e'(1_R - e'), \end{cases}$$

which implies $1_L - e \in I^*$ and $1_R - e' \in I^-$. Therefore,

$$\begin{cases} 1_L = (1_L - e) + e \in I^* + (I^*)^*, \\ 1_R = (1_R - e') + e' \in I^- + (I^-)^-, \end{cases}$$

Since $(I^*)^*$ and $I^- + (I^-)^-$, then $I^* + J^* = R$ and $I^- + J^- = R$. □

Proposition 3.12. *For any Bear-ideals I, J of a Bear ring R, we have that, $(I \to J) + (J \to I) = R$ and $(I \rightsquigarrow J) + (J \rightsquigarrow I) = R$.*

Proof. Straightforward by using Propositions 3.11 and 3.5. □

The same result as in Proposition 3.6 holds for Bear rings and Bear-ideals.

Proposition 3.13. *Every quotient of a Bear ring is a multiplication Bear ring.*

Lemma 3.8. *Every Von Neumann is a multiplication ring.*

Proof. Let I and J be two ideals of a non commutative Von Neumann ring R and P a prime ideal of R. Since all multiplication ring have the PBLR-1 property and vice-versa (by Proposition 3.2), then by Definition 3.3 of a non commutative Von Neumann ring, we only have to show that $I/P \cap J/P = (I/P) \cdot ((I \rightsquigarrow J)/P)$ and $I/P \cap J/P = ((I \to J)/P) \cdot (I/P)$. Since in the division ring R/P the ideals I/P and J/P are trivials, we just have to check four cases.

Case 1. If $I/P = J/P = \{0/P\}$, then $I/P \cap J/P = \{O/P\}$ and $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \to J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 2. If $I/P = \{0/P\}$ and $J/P = R/P$, then $I = \{0\}$, which implies $I/P \cap J/P = \{0/P\}$ and $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \to J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 3. If $I/P = R/P$ and $J/P = \{0/P\}$, using the definitions of $(I \rightsquigarrow J)/P$ and $(I \to J)/P$, we easily prove that $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \to J)/P) \cdot (I/P) = \{0/P\}$. Hence the property PBLR-1 holds.

Case 4. If $I/P = R/P$ and $J/P = R/P$, using the definitions of $(I \rightsquigarrow J)/P$ and $(I \to J)/P$, we easily prove that $(I/P) \cdot ((I \rightsquigarrow J)/P) = ((I \to J)/P) \cdot (I/P) = R/P$. Hence the property PBLR-1 holds.

We complete the proof. □

Proposition 3.14. *Let R be a unitary Von Neumann ring. Then R is a pseudo BL-ring if and only if it satisfies $I/P = J/P = \{0\}$ implies $(I \to J)/P = R/P$, for all ideals I, J of R and all prime ideals P of R.*

Proof. This result follows directly from Lemma 3.8. □

4. Representation of pseudo-BL algebras

The purpose of this section is to find a representation of pseudo BL-ring in the sense of subdirectly irreducible algebras. We start this section with the definitions of subdirectly irreducible ring and special primary ring.

Definition 4.1. A ring R is said to be subdirectly irreducible if every subdirect product of R is trivial.

Equivalently, a ring R is subdirectly irreducible if and only if the intersection of all non-zero ideals of R is non-zero.

Definition 4.2. A ring R is a special primary ring if R has a unique maximal ideal M and if each proper ideal of R is a power of M.

If P is a prime ideal of ring R and if $S = R \setminus P$, we shall denote 0_S^* (respectively 0_S^-), the S-component of the right zero ideal, by $N^*(P)$ (respectively $N^-(P)$), that is, $N^*(P) = \{x \in R \mid xs = 0 \text{ for some } s \in R \setminus P\}$ (respectively $N^-(P) = \{x \in R \mid sx = 0 \text{ for some } s \in R \setminus P\}$).

Lemma 4.1. *Let R be a ring that is generated by idempotents and P be a prime ideal of R. Then,*

$$\bigcap_P N^*(P) = \{0\} \text{ and } \bigcap_P N^-(P) = \{0\}.$$

Proof. To show that $\bigcap_P N^*(P) = \{0\}$ and $\bigcap_P N^-(P) = \{0\}$, it is sufficient to prove that if $x \neq 0$, then there exists a prime ideal P such that $x \notin N^*(P)$ and $x \notin N^-(P) = 0$.

Let $x \neq 0$, then $(xR)^* \neq R$ and $(Rx)^- \neq R$ (because if we assume that they are all equal, then for all $z \in R$, $z \cdot x = 0$ and $x \cdot z = 0$ with $x \neq 0$, which means that for all $z \in R$, $z = 0$. So $R = \{0\}$, a contradiction). Thus, by Proposition 2.10,[[8]] there exists a prime ideal Q of R such that $(xR)^* \subseteq Q$ and $(Rx)^- \subseteq Q$.

Assume that $x \in N^*(Q)$ and $x \in N^-(Q)$, then $sx = 0$ and $xs = 0$ for some $s \notin Q$.

If $s \cdot x = 0$, then $(s \cdot x)R = \{0\}$, thus $s \in (xR)^* \subseteq Q$. Therefore, $x \notin N^*(Q)$.

If $x \cdot s = 0$, then $R(x \cdot s) = \{0\}$, thus $s \in (xR)^- \subseteq Q$. Therefore, $x \notin N^-(Q)$.

Hence, $\bigcap_P N^*(P) = \bigcap_P N^-(P) = \{0\}$. □

Proposition 4.1. *Every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of special primary rings.*

Proof. Let R be a pseudo BL-ring with left and right unit, and P a prime ideal of R. Then by Proposition 3.2 we know that R is a multiplicative ring.

Let's consider the mappings $f : R \to \prod_P R/P$ such that $f(x) = (\frac{x}{1_R})/P$ a sequence on the prime ideals of R and $g : R \to \prod_P R/P$ such that $g(x) = (\frac{x}{1_L})/P$ a sequence on the prime ideals of R. We easily check that f and g are ring homomorphisms. Using the previous Lemma 4.1, the kernels of f and g are defined to be $\ker(f) = \bigcap_P N^*(P) = \{0\}$ and $\ker(g) = \bigcap_P N^-(P) = \{0\}$. Hence R is isomorphic to $f(R)$ and $g(R)$ two subrings of $\prod_P R/P$. Since R is a multiplicative ring, then by Theorem 9.23 and Propositions 9.25 and 9.26 in [10], each R/P is a special primary ring. □

Proposition 4.2. *Every unitary pseudo BL-ring with left and right unit is isomorphic to a subring of a direct product of discrete valuation rings.*

Proof. We use the same idea as in Proposition 4.1 and [10]. □

Proposition 4.3. *Let R be a subdirectly irreducible pseudo BL-ring with minimal ideal M, then*

(1) *M is an annihilator ideal.*
(2) *the annihilator ideals of R are linearly ordered and finite in number.*

Proof. (1) We know by definition of M^* that it is an ideal of R. Since M is the minimal ideal of R, then $M \subseteq M^*$, and because $M \neq \{0\}$, then $M^* \neq R$. Therefore, by the maximality of M, we obtain that $M = M^*$; similarly, we show that in the case of right ideal M^- of R, we also have $M = M^-$; therefore M is an annihilator ideal.

(2) Let R be a pseudo BL-ring, then $A(R)$ is a pseudo BL-algebra and also a pseudo MV-algebra. Consequently, $AN^*(R)$ and $AN^-(R)$ are respectively the left pseudo MV-center and right pseudo MV-center of $A(R)$. Moreover, for all ideal I of R, we have $(\bigvee I)^* = \bigcap I^*$ and $(\bigvee I)^- = \bigcap I^-$ which imply that every subset of $AN^*(R)$ and $AN^-(R)$ have an infimum. Thus, every subset of $AN^*(R)$ and $AN^-(R)$ also have a supremum because if S is a subset of $AN^*(R)$ (or $AN^-(R)$), then $\bigwedge S = (\bigvee S^*)^*$ (or $\bigwedge S = (\bigvee S^-)^-$). Hence $AN^*(R)$ and $AN^-(R)$ are complete pseudo MV-algebras. Also, for every non zero ideal I of R, $M \subseteq I$ implies that $I^* \subseteq M^*$ and $I^- \subseteq M^-$.

Therefore, for every proper ideals J in $AN^*(R)$ and K in $AN^-(R)$, we have $J^* \subseteq M^*$ and $K^- \subseteq M^-$ which induce that $AN^*(R) \setminus \{R\}$ and $AN^-(R) \setminus \{R\}$ have both of them a maximum element, namely $M^* \neq R$ and $M^- \neq R$ because $M \neq \{0\}$.

Let $X, Y \in AN^*(R)$ such that X is not included in Y and Y is not included in X, then $X \to Y, Y \to X \neq R$ and $X \to Y, Y \to X \subseteq M^*$; hence, by pre-linearity axiom, $R = X \to Y \vee Y \to X \subseteq M^*$; thus $M^* = R$, which is a contradiction, so $(AN^*(R), \subseteq)$ is a pseudo $MV-$chain.

By the same way we prove that $(AN^-(R), \subseteq)$ is a pseudo $MV-$chain. □

The next result is the Representation Theorem for pseudo BL-ring.

Theorem 4.1. *Every pseudo BL-ring R is a subdirect product of a family $\{R_x | x \in R \setminus \{0\}\}$ of subdirectly irreductible pseudo BL-rings satisfying, then*

(1) *$A(R_x)$ is isomorphic to $AN^*(R_x) \oplus D^*(R_x)$ and to $AN^-(R_x) \oplus D^-(R_x)$ for all $x \neq 0$.*

(2) *every ideal of each R_x is either an annihilator ideal or dense.*

(3) *$A(R)$ is a subdirect product of $\{A(R_x)|x \in R \setminus \{0\}\}$.*

(4) *$A(R_x)$ is a pseudo BL-algebra with a unique atom.*

Proof. Let R be a pseudo BL-ring, it is well known that R is a subdirect product of subdirectly irreductible homomorphic images of R. Using the Zorn's lemma, for all $x \in R$ and $x \neq 0$, there is an maximal ideal K_x that does not contain x, which implies that $\bigcap_x K_x = \{0\}$. Hence each factor R/K_x is subdirectly irreductible and R is the product of the family $\{R/K_x \mid x \in R \setminus \{0\}\}$. Since R/K_x is a pseudo BL-ring by Corollary 3.2 and Proposition 3.10. Let pick up the set $\{R_x \mid x \in R \setminus \{0\}\}$ to be $\{R/K_x \mid x \in R \setminus \{0\}\}$. Then the rest of the proof is similar to Theorem 4.2 in [8]. □

The following corollary is similar to Corollary 4.3 in [8].

Corollary 4.1. *Let R be a subdirectly irreductible BL-ring. Then*

(1) *for every annihilator ideal $I \neq R$ and every dense ideal J with $I \subseteq J$, we have $J \to I = I$ and $J \rightsquigarrow I = I$.*

(2) *for ideals I, J of R, either ($I \to J$ and $I \rightsquigarrow J$) or ($J \to I$ and $J \rightsquigarrow I$) are denses.*

Proof. Let R be a subdirectly irreductible BL-ring. As established in the proof of Theorem 4.1, $A(R)$ is isomorphic to $AN^*(R) \oplus D^*(R)$ and to $AN^-(R) \oplus D^-(R)$.

(1) By the definition of the implication in the ordinal sum of hoops, the condition stated holds.

(2) Since $A(R)$ is a pseudo BL-algebra, using the same way as in [2] and [3], we prove that the sets $D^*(M)$ and $D^-(M)$ of dense elements of any pseudo BL-algebra M is an implicative filter, $M/D^*(R)$ is isomorphic to $AN^*(M)$ and $M/D^-(R)$ is isomorphic to $AN^-(M)$.

Therefore, $A(R)/D^*(R)$ is isomorphic to $AN^*(R)$, and $A(R)/D^-(R)$ is isomorphic to $AN^-(R)$. Futhermore $A(R)/D^*(R)$ and $A(R)/D^-(R)$ are linearly ordered because $AN^*(R)$ and $AN^-(R)$ are linearly ordered. Now using the definitions of order, $\rightsquigarrow$ and $\to$ on $A(R)/D^*(R)$ and $A(R)/D^-(R)$, we finish the proof. □

5. Conclusion

It is a well-known result that a t-norm has a residuum if and only if it is left-continuous. Therefore, this shows that Basic Logic is not the most general t-norm-based logic. In fact, a logic weaker than Basic Logic, called *Monoidal t-norm-based logic* (MTL for short), was defined by Esteva and Godo in [4] and proved to be the logic of left-continuous t-norms and their residua. So MTL-algebra generalized BL-algebra. In this paper, we study a ring R, no matter it is commutative or non-commutative, with which $A(R)$ is a BL-algebra or pseudo BL-algebra.

Our future research in that way will consist to study commutative rings and non commutative rings R for which $A(R)$ is a MTL-algebra and pseudo MTL-algebra.

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