Power gamma extending modules

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Abstract. A submodule N of Gamma module M is $PE\Gamma-$submodule, if for each m in M, ideal I in R if $I\Gamma m \neq 0$, then $I\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$. $PE\Gamma-$submodule is a proper generalization of essential $R_\Gamma-$submodule. An $R_\Gamma-$module M is called power Gamma extending module ($PE\Gamma-$ module), if every $R_\Gamma-$submodule is $PE\Gamma-$submodule in a direct summand.

Keywords: extending gamma module, power essential gamma submodule, frail closed gamma submodule.

1. Introduction

Let R be a Γ-ring, an additive abelian group M with $\cdot : R \times \Gamma \times M \longrightarrow M$ by $(r, \gamma, m) \longmapsto r\gamma m$ such that $(k_1 + k_2)\rho x = k_1\rho x + k_2\rho x$, $k(\rho + \eta)x = k\rho x + k\eta x$, $k\rho(x_1 + x_2) = k\rho x_1 + k\rho x_2$ and $(k_1\rho k_2)\eta x = k_1\rho(k_2\eta x)$ where $k, k_1, k_2 \in R$, $\rho, \eta \in \Gamma$ and $x, x_1, x_2 \in M$ is called a left $R_\Gamma-$module [10]. A Γ-ring R is commutative if $s\beta r = r\beta s$ for any $s, r \in R$ and $\beta \in \Gamma$. M is unitary if there exist $1 \in R$ and $\gamma_\circ \in \Gamma$ such that $1\gamma_\circ a = a$ for every $a \in M$ [2]. A nonempty set K is $R_\Gamma-$submodule, $K \leq M$ if $R\Gamma K = \{s\gamma k : s \in R, \gamma \in \Gamma, k \in K\} \leq K$ [8]. For $a \in M$, cyclic $R_\Gamma-$submodule $Ra =< a >= \{\sum_{s=1}^{n} t_s\gamma_s a : n \in N, \gamma_s \in \Gamma\}$ [9]. The set of all $r \in R$ such that $r\Gamma x \subseteq N$ denoted by $(N :_{R_\Gamma} x)$ [4]. An $R_\Gamma-$submodule N is $R_\Gamma-$idempotent, $N \leq_{id} M$ if $N = (N :_{R_\Gamma} M)\Gamma N$ and M is fully idempotent if $N \leq_{id} M$ for each $R_\Gamma-$submodule N [3]. An $R_\Gamma-$submodule N of $R_\Gamma-$module M is essential, $N \leq_e M$ if for each $a \neq 0 \in M$,

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there is $t_1, t_2, ..., t_n \in R$ and $\nu_1, \nu_2, ..., \nu_n \in \Gamma$ such that $\sum_{s=1}^{n} t_s \gamma_s a (\neq 0) \in N$ [2]. For more basics refer to [1,2,5,6,7,8] and [9]. All modules in this work are unitary left gamma module over commutative $\Gamma-$ring .

2. Power essential gamma module

In this section, we introduce definition of power essential Gamma submodule and some properties that play an important role in our next definitions.

Definition 2.1. *An $R_\Gamma-$submodule N of M is called power essential Gamma submodule ($PE\Gamma-$submodule) in M denoted by $N \leq_{pe} M$, for each $m \in M$ and for each ideal I in R with $I\Gamma m \neq 0$, then $I\Gamma\,(N :_{R_\Gamma} m)\,\Gamma m \neq 0$. Power essential Gamma ideal is $PE\Gamma-$submodule in R as $R_\Gamma-$module.*

Clearly that $N \leq_{pe} N$ for each $N(\neq 0) \leq M$. Every $PE\Gamma-$submodule is essential. The converse not true see examples 3.9(1).

Proposition 2.2. *The following are equivalent for R_Γ-module:*

1. *N is $PE\Gamma-$submodule in M.*
2. *$\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, we have $s\Gamma\,(N :_{R_\Gamma} m)\,\Gamma m \neq 0$.*
3. *$\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, there exists $r \in R$ such that $r\Gamma m \leq N$ and $s\Gamma r\Gamma m \neq 0$.*
4. *$\forall m \in M$, $(N :_{R_\Gamma} m)\,\Gamma m \leq_{pe} Rm$.*

Proof. (1) → (2) Assume $N \leq_{pe} M$ and $\forall m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, then $\langle s\rangle\Gamma m \neq 0$, hence $\langle s\rangle\Gamma\,(N :_{R_\Gamma} m)\,\Gamma m \neq 0$, so $0 \neq \sum_{i=1}^{n}(r_i\gamma_i s)\gamma r\beta m \in \langle s\rangle\Gamma\,(N :_{R_\Gamma} m)\,\Gamma m$ for some $\beta, \gamma, \gamma_i \in \Gamma$, $r \in (N :_{R_\Gamma} m)$, $r_i \in R$. Let $0 \neq \sum_{i=1}^{n}(r_i\gamma_i s)\gamma r\ \beta m = \sum_{i=1}^{n}(s\gamma_i r_i)\gamma r\beta m = s\gamma_\circ \sum_{i=1}^{n}(1\gamma_i r_i)\gamma r\beta m$, so $s\Gamma(N :_{R_\Gamma} m)\ \Gamma m \neq 0$ since $\sum_{i=1}^{n}\ (1\gamma_i r_i)\ \gamma r \in (N :_{R_\Gamma} m)$.

(2) → (3) Clear.

(3) → (4) Let I be ideal of R with $I\Gamma(\sum_{i=1}^{n} r_i\gamma_i m) \neq 0$, then there is $s\gamma(\sum_{i=1}^{n} r_i\gamma_i m)\ (\neq 0) \in s\Gamma(\sum_{i=1}^{n} r_i\gamma_i m)$ for some $s \in I$, hence $s\Gamma(\sum_{i=1}^{n} r_i\gamma_i m) \neq 0$. By (3) there is $r \in R$ such that $r\Gamma \sum_{i=1}^{n} r_i\gamma_i m \leq N$ and $s\Gamma r\Gamma \sum_{i=1}^{n} r_i\gamma_i m \neq 0$. Since $r \in (N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)$, so $r\Gamma \sum_{i=1}^{n} r_i\gamma_i m \leq (N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\,\Gamma \sum_{i=1}^{n} r_i\gamma_i m$, hence $s\Gamma r\Gamma \sum_{i=1}^{n} r_i\gamma_i m(\neq 0) \leq I\Gamma\,(N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\,\Gamma \sum_{i=1}^{n} r_i\gamma_i m$.

(4) → (1) Let $m \in M$ and I is an ideal of R with $I\Gamma \sum_{i=1}^{n} r_i\gamma_i m \neq 0$. Since $(N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\Gamma \sum_{i=1}^{n} r_i\gamma_i m \leq N$, then $((N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\Gamma \sum_{i=1}^{n} r_i\gamma_i m :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m) \leq (N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)$. By (4) $(N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\Gamma \sum_{i=1}^{n} r_i\gamma_i m \leq_{pe} Rm$, then $I\Gamma((N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\Gamma \sum_{i=1}^{n} r_i\gamma_i m :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)(\neq 0) \leq I\Gamma\,(N :_{R_\Gamma} \sum_{i=1}^{n} r_i\gamma_i m)\ \Gamma \sum_{i=1}^{n} r_i\gamma_i m$. □

Proposition 2.3. *Let N and K two R_Γ-submodules of M, then N is $PE\Gamma-$submodule in K if and only if $(N :_{R_\Gamma} m)\Gamma m$ is $PE\Gamma-$submodule in $(K :_{R_\Gamma} m)\Gamma m$ for each m in M.*

Proof. Assume that $N \leq_{pe} M$ and $m \in M$, for each $x = r\beta m \in (K :_{R_\Gamma} m)\Gamma m$ for some $r \in (K :_{R_\Gamma} m)$, $\beta \in \Gamma$ and each $s \in R$ with $s\Gamma(r\beta m) \neq 0$, by proposition 2.2(2), $0 \neq s\Gamma(N :_{R_\Gamma} x)\Gamma x = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta m = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta 1\gamma_\circ m$ where $x = r\beta m$. To show that $(N :_{R_\Gamma} x)\Gamma r\beta 1 \subseteq (N :_{R_\Gamma} m)$, for each $t\gamma r\beta 1 \in (N :_{R_\Gamma} x)\Gamma r\beta 1$ for some $t \in (N :_{R_\Gamma} x)$ and $\gamma \in \Gamma$, if $\theta \in \Gamma$, then $(t\gamma r\beta 1)\theta m = t\gamma(r\beta 1)\theta 1\gamma_\circ m = t\gamma 1\theta(r\beta 1)\gamma_\circ m = 1\gamma t\theta(r\beta m) = 1\gamma t\theta x \in R\Gamma N \leq N$, so $(N :_{R_\Gamma} x)\Gamma r\beta 1 \subseteq (N :_{R_\Gamma} m)$. Thus $0 \neq s\Gamma(N :_{R_\Gamma} x)\Gamma x = s\Gamma(N :_{R_\Gamma} x)\Gamma r\beta 1\gamma_\circ m \leq s\Gamma(N :_{R_\Gamma} m)\Gamma m$. For other direction, let $k \in K$, then $(N :_{R_\Gamma} k)\Gamma k \leq_{pe} (K :_{R_\Gamma} k)\Gamma k = Rk$, by proposition 2.2(4) $N \leq_{pe} K$. □

Theorem 2.4. *Let M be an $R_\Gamma-$module, then:*

1. *Let N be $R_\Gamma-$submodule of M, then $N \leq_{pe} M$ if and only if $N \leq_{pe} K$ and $K \leq_{pe} M$.*
2. *If $N_1 \leq_{pe} K_1$ and $N_2 \leq_{pe} K_2$, then $N_1 \cap N_2 \leq_{pe} K_1 \cap K_2$.*
3. *If $f : A \to M$ is an $R_\Gamma-$homomorphism and $N \leq_{pe} M$, then $f^{-1}(N) \leq_{pe} A$.*
4. *If $\{N_\alpha\}$ is an independent family of M and $N_\alpha \leq_{pe} K_\alpha \leq M$ for each α, then $\{K_\alpha\}$ is an independent family of M and $\oplus_{\alpha\in\Lambda} N_\alpha \leq_{pe} \oplus_{\alpha\in\Lambda} K_\alpha$.*

Proof. 1. Assume that $N \leq_{pe} M$, for each $k \in K, m \in M$ and $s \in R$ with $s\Gamma k \neq 0$, then $s\Gamma(N :_{R_\Gamma} k)\Gamma k \neq 0$, so $N \leq_{pe} K$. Also if $s\Gamma m \neq 0$, then $s\Gamma(N :_{R_\Gamma} m)\Gamma m \neq 0$, hence $s\Gamma(K :_{R_\Gamma} m)\Gamma m \neq 0$, since $(N :_{R_\Gamma} m) \leq (K :_{R_\Gamma} m)$, then $K \leq_{pe} M$. Assume $N \leq_{pe} K \leq_{pe} M$, for each $m \in M$ and $s \in R$ with $s\Gamma m \neq 0$, then $s\Gamma(K :_{R_\Gamma} m)\Gamma m \neq 0$, so there is $s\gamma r\beta m(\neq 0) \in s\Gamma(K :_{R_\Gamma} m)\Gamma m$ for some $r \in (K :_{R_\Gamma} m)$ and $\gamma, \beta \in \Gamma$, so $r\beta m(\neq 0) \in (K :_{R_\Gamma} m)\Gamma m$. By proposition (2.3)$(N :_{R_\Gamma} m)\Gamma m \leq_{pe} (K :_{R_\Gamma} m)\Gamma m$, hence $s\Gamma r\Gamma m \neq 0$.

2. For each $k \in K_1 \cap K_2$ and $s \in R$ with $s\Gamma k \neq 0$, there is $r_1 \in R$ such that $r_1\Gamma k \leq N_1$ and $s\Gamma r_1\Gamma k \neq 0$. Take $0 \neq s\gamma r_1\beta k \in s\Gamma r_1\Gamma k$ for some $\gamma, \beta \in \Gamma$, since $s\gamma r_1 \in R$ and $0 \neq s\gamma r_1\beta k \in (s\gamma r_1)\Gamma k$, then there exists $r_2 \in R$ such that $r_2\Gamma k \leq N_2$ and $(s\gamma r_1)\Gamma r_2\Gamma k \neq 0$, so there is $0 \neq s\gamma r_1\rho r_2\lambda k \in (s\gamma r_1)\Gamma r_2\Gamma k$, since $(r_1\rho r_2)\Gamma k \leq N_1 \cap N_2$ and $0 \neq s\gamma r_1\rho r_2\lambda k \in s\Gamma(r_1\rho r_2)\Gamma k$, then by proposition(2.2)(3), $N_1 \cap N_2 \leq_{pe} K_1 \cap K_2$.

3. For each $a \in A$ and $s \in R$ with $s\Gamma a \neq 0$.

Case (1). If $s\Gamma f(a) \neq 0$, since $N \leq_{pe} M$, there is $r \in R$ such that $r\Gamma f(a) \leq N$ and $s\Gamma r\Gamma f(a) \neq 0$, so $r\Gamma a \leq f^{-1}(N)$ and $s\Gamma r\Gamma a \neq 0$.

Case (2). If $s\Gamma f(a) = 0$, then $s\Gamma a \leq kerf \leq f^{-1}(N)$, since $s\Gamma a \neq 0$, so there is $0 \neq s\beta a \in s\Gamma a \leq f^{-1}(N)$. Take $r = 1$, then $0 \neq s\Gamma a = 1\gamma_\circ s\Gamma a = s\gamma_\circ 1\Gamma a \leq s\Gamma 1\Gamma a$. Thus, $f^{-1}(N) \leq_{pe} A$.

4. First if $\Lambda = \{1,2\}$, by (2) $N_1 \cap N_2 = (0) \leq_{pe} K_1 \cap K_2$, then $0 \leq_e K_1 \cap K_2$, so $K_1 \cap K_2 = 0$, that is $\{K_1, K_2\}$ is an independent family of M. Let $\pi_1 : K_1 \oplus K_2 \to K_1$ and $\pi_2 : K_1 \oplus K_2 \to K_2$ the projections of $K_1 \oplus K_2$ on K_1 and K_2, then by (3) $\pi_1^{-1}(N_1) = N_1 \oplus K_2 \leq_{pe} K_1 \oplus K_2$ and $\pi_2^{-1}(N_2) = K_1 \oplus N_2 \leq_{pe} K_1 \oplus K_2$, so by (2) $(K_1 \oplus N_2) \cap (N_1 \oplus K_2) = N_1 \oplus [(K_1 \oplus N_2) \cap K_2] = N_1 \oplus [N_2 \oplus (K_1 \oplus K_2)] = N_1 \oplus N_2 \leq_{pe} K_1 \oplus K_2$. Consider $\oplus_{\alpha\in\Lambda} N_\alpha \leq_{pe} \oplus_{\alpha\in\Lambda} K_\alpha$ for $\Lambda = \{1, 2, ..., n\}$, using the case above $(K_1 \oplus ... \oplus K_n) \cap K_{n+1} = 0$, so $\{K_1, K_2, ..., K_{n+1}\}$ is independent family. By the same way above $\oplus_{\alpha\in\Lambda} N_\alpha \leq_{pe} \oplus_{\alpha\in\Lambda} K_\alpha$ for all finite index sets. Take distinct indices $\alpha(0), \alpha(1), ..., \alpha(n)$, then $\{K_{\alpha(0)}, K_{\alpha(1)}, ..., K_{\alpha(n)}\}$ is independent, for each $k \in \oplus K_\alpha$ and each $s \in R$ such that $s\Gamma k \neq 0$, then $k \in K_{\alpha(0)} \oplus ... \oplus K_{\alpha(n)}$ for some $\alpha(i)$. Hence there is $r \in R$ such that $r\Gamma k \leq N_{\alpha(0)} \oplus ... \oplus N_{\alpha(n)}$ and $s\Gamma r\Gamma k \neq 0$, thus $r\Gamma k \leq \oplus N_\alpha$. This prove for general case. □

Proposition 2.5. *Let $f : M \to N$ be an $R_\Gamma-$monomorphism and A, B be $R_\Gamma-$subm- odules of M with $A \leq B$. If A is $PE\Gamma-$submodule of B, then $f(A)$ is $PE\Gamma-$submodule of $f(B)$.*

Proof. Let $b \in B$ and $s \in R$ such that $s\Gamma f(b) \neq 0$, then $f(s\Gamma b) \neq 0$, so $s\Gamma b \neq 0$. By hypothesis there is $r \in R$ such that $r\Gamma b \leq A$ and $s\Gamma r\Gamma b \neq 0$. Hence $f(r\Gamma b) = r\Gamma f(b) \leq f(A)$ and $0 \neq f(s\Gamma r\Gamma b) = s\Gamma r\Gamma f(b)$. □

Proposition 2.6. *Let $N \leq_{pe} L \leq M$, then:*

1. *If $N \cap K = 0$, then $L \cap K = 0$ for each $K \leq M$.*
2. *If $I\Gamma N = 0$, then $I\Gamma L = 0$ for each ideal I in R.*

Proof. 1. If $L \cap K \neq 0$, there is $l = k(\neq 0) \in L \cap K$, so $l = 1\gamma_\circ l \in 1\Gamma l \neq 0$, there exists $r \in R$ such that $r\Gamma l \leq N$ and $1\Gamma r\Gamma l \neq 0$, but $r\Gamma l = r\Gamma k \in K \cap N = 0$, then $1\Gamma r\Gamma l = 0$ a contradiction.

2. If $I\Gamma L \neq 0$, then there exists $i\gamma l \neq 0$ for some $i \in I, \gamma \in \Gamma$ and $l \in L$, so $i\Gamma l \neq 0$, then there exists $r \in R$ such that $r\Gamma l \leq N$ and $0 \neq i\Gamma r\Gamma l \leq I\Gamma N = 0$, a contradiction. □

3. Power gamma extending module

In this section, the definitions of frail closed gamma submodule and power Gamma extending module was introduce as a proper generalization of closed and Gamma extending respectively.

Definition 3.1. *An $R_\Gamma-$submodule N of M is called frail closed ($f-$closed), $N \leq_{fc} M$, if N has no proper power essential Gamma extension in M, that is $N \leq_{pe} L \leq M$, then $L = M$.*

Clearly that every closed is $f-$closed but the converse is not true, see Example (3.9)(1).

Proposition 3.2. *Direct summand of an R_Γ-module is $f-$closed.*

Proof. Let $M = N \oplus K$ be an R_Γ-module such that N is $PE\Gamma-$submodule in a submodule $L \leq M$. Hence $L \cap K = 0$ by proposition 2.6, so $L = L \cap (N + K) = N + (L \cap K) = N$. Therefore N has no proper power essential Gamma extensions in M. □

Lemma 3.3. *Let N be an R_Γ-submodule of R_Γ-module M. Then the union of any chain of frail closed extensions of N is also frail closed extensions of N in M.*

Proof. Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a family of $R_\Gamma-$submodules of M such that N is a $PE\Gamma-$ submodule in L_α for each $\alpha \in \Lambda$. For each $x \in \cup_{\alpha\in\Lambda} L_\alpha$ and each $s \in R$, if $s\Gamma x \neq 0$, then there is $\beta \in \Lambda$ such that $x \in L_\beta$, so there is $r \in R$ such that $r\Gamma x \leq L_\beta \leq \cup_{\alpha\in\Lambda} L_\alpha$ and $s\Gamma r\Gamma x \neq 0$. □

Theorem 3.4. *Every $R_\Gamma-$submodule N of an $R_\Gamma-$module M is contained in an $f-$closed $R_\Gamma-$submodule L of M in which N is a power essential Gamma submodule in L.*

Proof. Let N be an $R_\Gamma-$submodule of M and let $\Omega = \{K \leq M : N \leq_{pe} K\}$. Order Ω by inclusion, then by lemma 3.3, Ω is inductive, and hence by Zorn's Lemma Ω has a maximal element L. By Lemma 3.3 L is $f-$closed in M, that is $N \leq_{pe} L \leq_{fc} M$. □

Proposition 3.5. *Let N be an $f-$closed submodule of an $R_\Gamma-$module and K is $PE\Gamma-$submodule in M, then $N \cap K$ is $f-$closed in K.*

Proof. Assume $N \cap K \leq_{pe} L \leq K$. First we show that $N \leq_{pe} N + L$, for each $n + l \in N + L$ where $n \in N, l \in L$ and for each $s \in R$ with $s\Gamma(n + l) \neq 0$. Since $K \leq_{pe} M$, there exists $r \in R$ such that $r\Gamma(n + l) \leq K$ and $s\Gamma r\Gamma(n + l) \neq 0$, so $r\Gamma n \leq K$, hence $r\Gamma n \leq N \cap K \leq L$ with $s\Gamma r\Gamma(n + l) \neq 0$, so $r\Gamma(n + l) \neq 0$. Let $x = r\beta(n + l) \neq 0 \in r\Gamma(n + l)$, since $N \cap K \leq_{pe} L$, there exists $r_1 \in R$ such that $r_1\Gamma x \leq N \cap K \leq N$ and $0 \neq s\Gamma r_1\Gamma x = s\Gamma r_1\Gamma r\beta(n + l)$. Take $t = r_1\beta r$, then $t\Gamma(n + l) = r_1\beta r\Gamma(n + l) = r_1\gamma_\circ 1\beta r\Gamma 1\gamma_\circ(n + l) = r_1\gamma_\circ 1\Gamma 1\beta r\gamma_\circ(n + l) = r_1\Gamma r\beta 1\gamma_\circ(n + l) = r_1\Gamma r\beta(n + l) = r_1\Gamma x \leq N$ and $s\Gamma t\Gamma(n + l) = s\Gamma r_1\beta r\Gamma(n + l) = s\Gamma r_1\gamma_\circ 1\beta r\Gamma 1\gamma_\circ(n+l) = s\Gamma r_1\gamma_\circ 1\Gamma 1\beta r\gamma_\circ(n+l) = s\Gamma r_1\gamma_\circ 1\Gamma r\beta 1\gamma_\circ(n+l) = s\Gamma r_1\Gamma r\beta(n + l) = s\Gamma r_1\Gamma x \neq 0$, hence $N \leq_{pe} N + L$. Since $N \leq_{fc} M$ it follows that $N = N + L$. Hence $L \leq (N \cap K) + L = (N + L) \cap K = N \cap K \leq L$, so $N \cap K = L$ and hence $N \cap K \leq_{pe} K$. □

Lemma 3.6. *If N is an R_Γ-submodule of M and K is a complement of N in M. Then:*

1. *There is a maximal $PE\Gamma-$submodule extension L such that $L \cap K = 0$.*
2. *If $N \leq_{fc} M$, then $N = L$.*

Proof. 1. By theorem 3.4 there is a maximal R_Γ-submodule L of M such that $N \leq_{pe} L$. For each $x(\neq 0) \in L \cap K$ and each $s \in R$, if $s\Gamma x \neq 0$, there exists $r \in R$ such that $r\Gamma x \leq N$ and $0 \neq s\Gamma r\Gamma x \leq N \cap K$, a contradiction.

2. Clear. □

Lemma 3.7. *Let N be an R_Γ-submodule of M. If N is $f-$closed in a direct summand of M, then N is $f-$closed in M.*

Proof. Let $M = L_1 \oplus L_2$ with $N \leq_{fc} L_1$ and $N \leq_{pe} K \leq M$. Let $\pi : M \to L_1$ be the projection, then $N = \pi(N) \leq \pi(K) \leq L_1$. To show that $N \leq_{pe} \pi(K)$, for each $\pi(k) \in \pi(K)$ and $s \in R$ such that $s\Gamma\pi(k) \neq 0$, so $s\Gamma k \neq 0$, hence there is $0 \neq s\gamma k \in s\Gamma k \leq K$. Hence there is $r \in R$ such that $r\Gamma k \leq N$ and $s\Gamma r\Gamma k \neq 0$. Now $r\Gamma k = r\Gamma l_1 + r\Gamma l_2 \leq N$ with $k = l_1 + l_2$ for some $l_1 \in L_1$ and $l_2 \in L_2$, so $r\Gamma l_1 = r\Gamma k \leq N$, hence $s\Gamma\pi(k) = s\Gamma l_1 = s\Gamma k \leq N$ and $s\Gamma r\Gamma\pi(k) = s\Gamma r\Gamma l_1 = s\Gamma r\Gamma k \neq 0$. Since $N \leq_{pe} \pi(K) \leq L_1$ and $N \leq_{fc} K$, then $N = \pi(K) \leq K$, so $(1-\pi)K \leq K$. For each $x = (1-\pi)k \in N \cap (1-\pi)K$, $x = (1-\pi)(l_1 + l_2)$, then $x = l_2 \in L_1 \cap L_2 = 0$. Hence $N \cap (1-\pi)K = 0$ but $N \leq_e K$, then $(1-\pi)K = 0$, so $K \leq \pi(K)$ which implice that $K \leq \pi(K) = N$, thus $N \leq_{fc} M$. □

$\Gamma-$Extending R_Γ-module was introduce in [4]. An R_Γ-module M is called $\Gamma-$Extending if each closed R_Γ-submodule of M is a direct summand of M.

Definition 3.8. *An $R_\Gamma-$module M is called power Gamma extending module ($PE\Gamma-$ module), if for each R_Γ-submodule is $PE\Gamma-$submodule in a direct summand. A $\Gamma-$ring R is power Gamma extending if it is power Gamma extending module as $R_\Gamma-$module.*

Clearly that every $PE\Gamma-$module is $\Gamma-$Extending but the converse is not true, see Example 3.9(1).

Example 3.9. 1. Let $R = \Gamma = Z$ and $M = Z_4 \oplus Z_2$, then M is $\Gamma-$-Extending $R_\Gamma-$module while M not $PE\Gamma-$module. Take $N = \langle(0,1)\rangle$, $K = \langle(2,1)\rangle$, $L = N \oplus K$ and $I = 2Z$, then $L \nleq_{pe} M$, since $I\Gamma M \neq 0$ and $I\Gamma L = 0$. Note that $L \leq_e M$ and $L \nleq_{pe} M$, aslo L is $f-$closed but not closed in M, in fact every $R_\Gamma-$submodule of M is $f-$closed.

2. Direct sum of $PE\Gamma-$module may not $PE\Gamma-$module (see, example (1)).

3. Let $R = \{(n,n), n \in Z\}$ and $\Gamma = \left\{\begin{pmatrix}\gamma\\0\end{pmatrix}, \gamma \in Z\right\}$. Then R is $\Gamma-$ring with $\cdot : R \times \Gamma \times R \to R$ defined by $(n \quad n)\begin{pmatrix}\gamma\\0\end{pmatrix}(m \quad m) = (n\gamma m \quad n\gamma m)$. For each $(m \quad m)$, and $(s \quad s) \in R$ with $(s \quad s)\Gamma(m \quad m) \neq 0$. Then there is $\begin{pmatrix}\gamma\\0\end{pmatrix} \in \Gamma$ such that $0 \neq (s \quad s)\begin{pmatrix}\gamma\\0\end{pmatrix}(m \quad m) = (n\gamma m \quad n\gamma m)$, so $s \neq 0, \gamma \neq 0, m \neq 0$. To show that every ideal $J \neq 0$ in R is $PE\Gamma-$submodule, let $(j \quad j)(\neq 0) \in J$. Choose $(s\gamma j \quad s\gamma j) \in R$, for each $(s\gamma j \quad s\gamma j)\begin{pmatrix}\beta\\0\end{pmatrix}(m \quad m) \in (s\gamma j \quad s\gamma j)\Gamma(m \quad m)$, then $(s\gamma j \quad s\gamma j)\begin{pmatrix}\beta\\0\end{pmatrix}(m \quad m) = (s\gamma j\beta m \quad s\gamma j\beta m) =$

$(s\gamma m\beta j \;\; s\gamma m\beta j) = (s\gamma m \;\; s\gamma m)\binom{\beta}{0}(j \;\; j) \in J$, hence $(s\gamma j \;\; s\gamma j)\Gamma(m \;\; m) \leq J$. Since $s \neq 0, \gamma \neq 0, m \neq 0$ and $j \neq 0$, then $s\gamma s\gamma j\gamma m \neq 0$, hence $0 \neq (s \;\; s)\binom{\gamma}{0}(s\gamma j \;\; s\gamma j)\binom{\gamma}{0}(m \;\; m) \in (s \;\; s)\Gamma(s\gamma j \;\; s\gamma j)\Gamma(m \;\; m)$, therefore $J \leq_{pe} R$. Thus every ideal in R is $PE\Gamma-$submodule, so R is $PE\Gamma-$module.

Lemma 3.10. *An R_Γ-module M is $PE\Gamma-$module if and only if for each $R_\Gamma-$submodule N of M, there exists decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L \leq_{pe} M$.*

Proof. For each $R_\Gamma-$submodule N of a $PE\Gamma-$module M, then $N \leq_{pe} K \leq_\oplus M$, by theorem 2.3, $N \oplus L \leq_{pe} K \oplus L = M$. For each $N \leq M$, there exists decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L \leq_{pe} M$. To show that $N \leq_{pe} K$, for each $k \in K$ and $s \in R$ such that $s\Gamma k \neq 0$, then there exists $r \in R$ such that $r\Gamma k \leq N \oplus L$ and $s\Gamma r\Gamma k \neq 0$. For each each $r\beta k \in r\Gamma k$, then $r\beta k = n + l$ for some $n \in N$ and $l \in L$, so $r\beta k - n = l \in K \cap L = 0$, thus $r\Gamma k \leq N$. □

Easy to prove the following two results.

Lemma 3.11. *Let M be $PE\Gamma-$module, then an $R_\Gamma-$submodule N of M is $PE\Gamma-$submodule if and only if N is essential in M.*

Proposition 3.12. *An R_Γ-module M is $PE\Gamma-$module if and only if every $f-$closed is a direct summand.*

The following lemma is used in proof of next result.

Lemma 3.13. *Let $M = L_1 \oplus L_2$ be R_Γ-module, if N is $R_\Gamma-$idempotent in M, then $N = (N \cap L_1) \oplus (N \cap L_2)$, Furthermore, $\pi_1(N)$ and $\pi_2(N)$ are $R_\Gamma-$idempotent in L_1 and L_2 recpectivly where π_1 and π_2 are the projections of M on to L_1 and L_2.*

Proof. Let $N = (N :_{R_\Gamma} M)\Gamma N$ and let $\pi_1 : L_1 \oplus L_2 \to L_1$ and $\pi_2 : L_1 \oplus L_2 \to L_2$ be the projections of M on to L_1 and L_2, then $\pi_1(N) = \pi_1((N :_{R_\Gamma} M)\Gamma N) = (N :_{R_\Gamma} M)\Gamma\pi_1(N) \leq N \cap L_1$, similar $\pi_2(N) \leq N \cap L_2$. Now $N \subseteq \pi_1(N) \oplus \pi_2(N) \subseteq (N \cap L_1) \oplus (N \cap L_2) \subseteq N$. Moreover if $r \in (N :_{R_\Gamma} M)$, then $r\Gamma M \leq N$, so $r\Gamma L_1 = \pi_1(r\Gamma M) \subseteq \pi_1(N)$, that is $r \in (\pi_1(N) :_{R_\Gamma} L_1)$. Thus $\pi_1(N) = (N :_{R_\Gamma} M)\Gamma N\pi_1(N) \subseteq (\pi_1(N) :_{R_\Gamma} L_1)\Gamma\pi_1(N) \subseteq \pi_1(N)$. □

Proposition 3.14. *Let M be power Gamma extending module and N be $R_\Gamma-$idempotent submodule of M, then N is a power Gamma extending module.*

Proof. For each submodule K of N, by lemma 3.10 there exists a decomposition $M = L_1 \oplus L_2$ such that $K \leq L_1$ and $K \oplus L_2 \leq_{pe} M$. By lemma 3.13, $N = (N \cap L_1) \oplus (N \cap L_2)$ and by theorem 2.4, $N \cap (K \oplus L_2) = K \oplus (N \cap L_2) \leq_{pe} N$. Since $K \leq N \cap L_1$, then $K \leq_{pe} N$ by lemma 3.10. □

Corollary 3.15. *Let M be fully $R_\Gamma-$idempotent. If M is a power Gamma extending module, then every $R_\Gamma-$submodule is a power Gamma extending module.*

Theorem 3.16. *Let M be a power Gamma extending module. Then every direct summand of M is a power Gamma extending.*

Proof. Let N be a direct summand of M and K an $f-$closed $R_\Gamma-$submodule of N. Then K ia also $f-$closed in M by lemma 3.7. Since M is a power Gamma extending, then $M = K \oplus L$ for some $L \leq M$, so $N = K \oplus (N \cap L)$. Thus N is $PE\Gamma-$module. □

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