Some random iteration processes in modular function space

M. Shahsavari
Department of Mathematics
Qazvin Branch
Islamic Azad University
Qazvin
Iran
shahsavarim1350@gmail.com

A. Razani*
Department of Pure Mathematics
Faculty of Science
Imam Khomeini International University
Postal code: 34149-16818, Qazvin
Iran
razani@sci.ikiu.ac.ir

Gh. Abbasi
Department of Mathematics
Qazvin Branch
Islamic Azad University
Qazvin
Iran
g.abbasi@qiau.ac.ir

Abstract. In this paper, some random iteration processes for some random operators in modular function space are studied. Then the sequences generated by these iterations are strongly convergent to a random fixed point of the ρ-generalized Lipschitzian mapping. Our results extend and improve some recent results in the literature. Finally, some numerical examples are presented to indicate the validity of the results.

Keywords: random iteration, random fixed point, modular function space, strongly convergence.

1. Introduction

The Ishikawa and Mann iteration schemes are two general iterations which have been successfully applied to the operators' fixed point problems and also for obtaining the solutions of operator equations in the linear spaces (see [19, 20]). In recent years, Choudhury [3], has suggested and analyzed random Mann iterative sequence in separable Hilbert spaces for finding random solutions and random fixed points for some kind of random equations and random operators. The study of random fixed point theory is playing an increasing role in mathe-

∗. Corresponding author

matics and applied sciences. At present, it received considerable attention due to enormous applications in many important areas such as Nonlinear Analysis, Probability Theory and for the study of random equations arising in various applied areas. Therefore, random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and their properties and is much needed for the study of various classes of random equations. The random fixed point results is of great value. Random techniques have played a crucial role in pure mathematics as well as applied sciences. No doubt, famously random methods have revolutionized the financial markets.

In 2001, Jung et al. [6] introduced and studied generalized Lipschitzian mappings in Banach space. The importance of generalized Lipschitzian mapping is because of every Lipschitzian mapping is a generalized Lipschitzian mapping, but vice versa is not necessarily true.

Let us recall that, the theory of modular spaces, as a generalization of a metric space, was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and generalized by Musielak and Orlicz [13] in 1959. These spaces were developed following the successful theory of Orlicz spaces, which Orlicz spaces generalize Lebesgue spaces by considering spaces of functions with some growth properties different from the power type growth control provided by the L_p-norm and obtained by replacing the given integral form of the nonlinear functional which controls the growth of the functions, by an abstract functional, the modular. Modular function spaces are natural generalization of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. Even though a metric is not defined, many problems in metric fixed point theory can be reformulated and solved in modular spaces.

Our results (in particular) can be applied to $L_1(\Omega, \lambda)$ or Musielak-Orlicz space, for showing that generalized Lipschitzian mappings have a fixed point when they are defined on a convex subset of $L_1(\Omega, \lambda)$ or Musielak-Orlicz space, which sequences generated by the random operators are strongly convergent to a random fixed point of the ρ-generalized Lipschitzian mapping.

The purpose of this paper is to introduce and construct some random iteration processes for ρ-generalized Lipschitzian mapping in modular function spaces. Firstly, a new double random iteration processes for a random operator in modular function space is introduced. Then we show that the sequence generated by the iteration convergence strongly to a random fixed point of the ρ-generalized Lipschitzian mapping. Moreover, we study a random operator iteration for a nonexpansive semigroup and a ρ-generalized Lipschitzian mapping in a modular function space. Finally, we present some examples to show the validity of the results.

At this step, we recall some definitions (see [2, 4, 7, 8, 9, 15, 16, 17, 18, 21, 23, 22, 24]).

Definition 1.1. *Let X be an arbitrary vector space over $K = (\mathbb{R}$ or $\mathbb{C})$.*
a) A functional $\rho : X \longrightarrow [0, \infty]$ is called modular if it satisfies the following conditions:
i) $\rho(x) = 0$ if and only if $x = 0$.
ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.
iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$.
If iii) is replaced by:
iii)′ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for α, $\beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$, then the modular ρ is called a convex modular.
b) A modular ρ defines a corresponding modular space, i.e. the space X_ρ given by

$$X_\rho = \{x \in X \;| \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

c) If ρ is a convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0; \quad \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}.$$

Remark 1.1. Notice that ρ is an increasing function. Suppose $0 < a < b$, then property (iii) with $y = 0$, shows that $\rho(ax) = \rho(\frac{a}{b}(bx)) \leq \rho(bx)$.

Definition 1.2. *Let X_ρ be a modular space.*
a) A sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ is said to be:
i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.
c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\in\mathbb{N}} \subset B$ and $\rho(x_n - x) \to 0$ then $x \in B$.
d) A subset $B \subset X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ-diameter of B.
e) A function $f : X_\rho \to X_\rho$ is called ρ-continuous if $\rho(x_n - x) \to 0$, then $\rho(f(x_n) - f(x)) \to 0$.

Let Ω be a nonempty set and Σ a nontrivial σ-algebra of subsets of Ω. Suppose P is a δ-ring of subsets of Σ, such that $E \cap A \in P$ for any $E \in P$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in P$ such that $\Omega = \cup K_n$. By ξ we denote the linear space of all simple functions with supports from P. By M we will denote the space of all measurable functions, i.e. all functions $f : \Omega \longrightarrow R$ such that there exists a sequence $\{g_n\} \in \xi$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f(w)$ for all $w \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.3. *A function $\rho : \xi \times \Sigma \longrightarrow [0, \infty]$ is called a modular function if:*

(1) *$\rho(0, E) = 0$, for any $E \in \Sigma$,*

(2) $\rho(f,E) \leq \rho(g,E)$ *whenever* $|f(w)| \leq |g(w)|$, *for any* $w \in \Omega$, $f,g \in \xi$ *and* $E \in \Sigma$,

(3) $\rho(f,.) : \Sigma \longrightarrow [0,\infty]$ *is a* σ*-subadvitive measure for every* $f \in \xi$,

(4) $\rho(\alpha, A) \longrightarrow 0$ *as* α *decreases to* 0 *for every* $A \in P$, *where* $\rho(\alpha, A) = \rho(\alpha 1_A, A)$,

(5) *if there exists* $\alpha > 0$ *such that* $\rho(\alpha, A) = 0$ *then* $\rho(\beta, A) = 0$ *for every* $\beta > 0$,

(6) *for any* $\alpha > 0$, $\rho(\alpha, .)$ *is order continuous on* P, *that is* $\rho(\alpha, A_n) \longrightarrow 0$ *if* $\{A_n\} \in P$ *and decreases to* ϕ.

Definition 1.4. *Let* ρ *be a modular function . A modular function space is the vector space* $L_\rho(\Omega, \Sigma)$, *or briefly* L_ρ, *defined by*

$$L_\rho = \{f \in M; \rho(\alpha f) \to 0 \quad as \ \ \alpha \to 0\}.$$

Let C be a nonempty subset of a modular space X_ρ, $T : C \longrightarrow C$ be a mapping then $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T.

Lemma 1.1 ([25])**.** *Assume* $\{a_n\}$ *is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence of real numbers such that*

(I) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.

(II) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Now, we consider a generalized Lipschitzian mappings in modular space as below (also see [5]).

Definition 1.5. *Let* X_ρ *be a modular space and* C *be a nonempty subset of* X_ρ. *A mapping* $T : C \to C$ *is said to be* ρ*-generalized Lipschitzian mapping if there exist* $r, l \in \mathbb{R}^+$ *with* $r > l$ *and*

$$\begin{aligned} \rho(r(T^n x - T^n y)) \leq\ & a_n \rho(l(x - y)) \\ & + b_n[\rho(l(x - T^n x)) + \rho(l(T^n y - y))] \\ & + c_n[\rho(l(x - T^n y)) + \rho(l(T^n x - y))], \end{aligned} \tag{1}$$

for each $x, y \in C$ *and* $n \geq 1$, *where* a_n, b_n *and* c_n *are nonnegative constants such that there exists an integer* n_0 *such that* $b_n + c_n < 1$, *for all* $n > n_0$.

Definition 1.6. *Let X_ρ be a modular space and C be a nonempty subset of X_ρ. A mapping $T : C \to C$ is said to be uniformly ρ-generalized Lipschitzian mapping, if there exist $r, l \in \mathbb{R}^+$ with $r > l$ and*

$$\begin{aligned} \rho(r(Tx - Ty)) \leq a\rho(l(x - y)) \\ + b[\rho(l(x - Tx)) + \rho(l(Ty - y))] \\ + c[\rho(l(x - Ty)) + \rho(l(Tx - y))], \end{aligned} \tag{2}$$

for each $x, y \in C$ where a, b and c are nonnegative constants and $b + c < 1$.

2. Random iterations for ρ-generalized Lipschitzian mappings

In 2002, Moore [12], considered the double sequence iteration

$$x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_n T_k x_{k,n},$$

where $T_k x = (1 - a_k)z + a_k Tx$ for all $x \in C$ and C is a bounded, closed, convex and nonempty subset of a real Hilbert space H. Also $\{\alpha_n\}_{n\geq 1}$ and $\{a_k\}_{k\geq 0}$ are sequences in $(0, 1)$. Moore proved the double sequence $\{x_{k,n}\}$ is strongly convergent to a fixed point of T in C.

Definition 2.1 ([12]). *Let X_ρ be a modular space where ρ is convex and satisfing the Δ_2-condition and $\mathbb{N}$ denotes the set of all natural numbers. We consider function $f : \mathbb{N} \times \mathbb{N} \to X_\rho$ defined by $f(n, m) = x_{n,m} \in X_\rho$. A double sequence $\{x_{n,m}\}$ is said to be strongly ρ-convergent to z if given any $\epsilon > 0$ there exist $N, M > 0$ such that $\rho(x_{n,m} - z) < \epsilon$ for all $n \geq N$, $m \geq M$. If for all $n, r \geq N$ and $m, t \geq M$, we have that $\rho(x_{n,r} - x_{m,t}) < \epsilon$, then the double sequence is said to be ρ-Cauchy. Furthermore, if for each fixed n, $\rho(x_{n,m} - z_n) \longrightarrow 0$ as $m \to \infty$ and $\rho(z_n - z) \longrightarrow 0$ as $n \to \infty$, then $\rho(x_{n,m} - z) \longrightarrow 0$ as $n, m \to \infty$.*

In the sequel, we define random operator in modular space.

Definition 2.2. *Let (Ω, Σ) be measurable space and L_ρ a modular function space. Suppose C is a nonempty subset of L_ρ,*

(1) *a mapping $T : \Omega \times C \longrightarrow C$ is called a random operator if $T(., x) : \Omega \longrightarrow C$ is measurable for each fixed $x \in C$,*

(2) *a measurable function $f : \Omega \longrightarrow C$ is called a random fixed point of the random operator $T : \Omega \times C \longrightarrow C$ if $T(t, f(t)) = f(t)$, for all $t \in \Omega$,*

(3) *a random operator $T : \Omega \times C \longrightarrow C$ is called continuous if for fixed $t \in \Omega$ $T(t, .) : C \longrightarrow C$ is continuous.*

We need the following theorem for the main results.

Theorem 2.1. *Let X_ρ be a ρ-complete modular space, where ρ is convex and satisfy the Δ_2-condition. Suppose C is a nonempty, convex and ρ-closed subset of X_ρ, $T : C \to C$ is a uniformly ρ-generalized Lipschitzian mapping satisfy inequality (2), such that $\frac{a+b+c}{1-(b+c)} < 1$. Then there exists a unique fixed point $z \in C$ such that $Tz = z$.*

Proof. Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l}$ and choose $r > 2l$, therefore $\alpha l < r$. Let x be an arbitrary point of C and construct sequence $\{Tx_n\}$ as follows, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. For each integer $n \geq 1$, we have,

$$\begin{aligned}\rho(r(Tx_{n+1} - Tx_n)) \leq\ & a\rho(l(x_{n+1} - x_n)) \\ & + b[\rho(l(Tx_{n+1} - x_{n+1})) + \rho(l(Tx_n - x_n))] \\ & + c[\rho(l(Tx_{n+1} - x_n)) + \rho(l(Tx_n - x_{n+1}))],\end{aligned}$$

on the other hand,

$$\begin{aligned}\rho(l(Tx_{n+1} - x_n)) &\leq \rho(\alpha l(Tx_{n+1} - x_{n+1})) + \rho(r(x_{n+1} - x_n)) \\ &\leq \rho(r(Tx_{n+1} - Tx_n)) + \rho(r(x_n - x_{n+1})).\end{aligned}$$

It follows that,

$$\rho(r(Tx_{n+1} - Tx_n)) \leq \frac{a+b+c}{1-(b+c)}\rho(r(x_{n+1} - x_n)) = k\rho(r(x_{n+1} - x_n)),$$

where $k = \frac{a+b+c}{1-(b+c)}$. Continuing this processes, we have $\rho(r(Tx_{n+1} - Tx_n)) \leq k^n\rho(r(Tx - x))$, which implies that $\lim_{n\longrightarrow\infty} \rho(r(Tx_{n+1} - Tx_n)) = 0$. If $l < r_0 < 2l$ and since ρ is an increasing function, then $\rho(r_0(Tx_{n+1} - Tx_n)) \leq \rho(r(Tx_{n+1} - Tx_n))$, by Δ_2- condition $\lim_{n\longrightarrow\infty} \rho(r_0(Tx_{n+1} - Tx_n)) = 0$ for $l < r_0 < 2l$. Thus, for any $r > l$ $\lim_{n\longrightarrow\infty} \rho(r(Tx_{n+1} - Tx_n)) = 0$.

We show that $\{Tx_n\}$ is ρ-Cauchy sequence. If not, then there exists an $\varepsilon > 0$ and two sequences of integers $\{n(s)\}$, $\{m(s)\}$, with, $n(s) > m(s) \geq s$, such that

$$\rho(r(Tx_{n(s)} - Tx_{m(s)})) \geq \varepsilon \quad \text{for} \quad s = 1, 2, \ldots. \tag{3}$$

We assume,

$$\rho(r(x_{n(s)} - Tx_{m(s)})) < \varepsilon. \tag{4}$$

In order to show this, suppose $n(s)$ is the smallest number exceeding $m(s)$ for which (3) is hold and

$$\sum_s = \{n \in \mathbb{N} | \exists m(s) \in \mathbb{N}; \rho(r(Tx_n - Tx_{m(s)})) \geq \varepsilon \ \text{ and } \ n > m(s) \geq s\}.$$

Obviously $\sum_s \neq \phi$ and since $\sum_s \subset \mathbb{N}$, then by well ordering principle, the minimum element of $\sum_s$ is denoted by $n(s)$, and clearly (4) is hold. Now

$$\begin{aligned}\rho(r(Tx_{m(s)} - Tx_{n(s)})) \leq\ & a\rho(l(x_{m(s)} - x_{n(s)})) \\ & + b[\rho(l(Tx_{m(s)} - x_{m(s)})) + \rho(l(Tx_{n(s)} - x_{n(s)}))] \\ & + c[\rho(l(Tx_{m(s)} - x_{n(s)})) + \rho(l(Tx_{n(s)} - x_{m(s)}))],\end{aligned}$$

moreover

$$\rho(l(x_{m(s)} - x_{n(s)})) \leq \rho(\alpha l(Tx_{m(s)} - x_{m(s)})) + \rho(r(Tx_{m(s)} - x_{n(s)})),$$

and

$$\rho(l(Tx_{n(s)} - x_{m(s)})) \leq \rho(r(Tx_{n(s)} - Tx_{m(s)}) + \rho(\alpha l(Tx_{m(s)} - x_{m(s)})),$$

therefore

$$\begin{aligned}\rho(r(Tx_{m(s)} - Tx_{n(s)})) \leq & \frac{a+c}{1-c}\rho(r(Tx_{m(s)} - x_{n(s)})) \\ & + \frac{a+b+c}{1-c}\rho(\alpha l(Tx_{m(s)} - x_{m(s)})) \\ & + \frac{b}{1-c}\rho(l(Tx_{n(s)} - x_{n(s)})).\end{aligned}$$

Since $\frac{a+b+c}{1-(b+c)} < 1$ then $\frac{a+c}{1-c} < 1$, also by using Δ_2-condition as $s \longrightarrow \infty$, we have, $\varepsilon \leq \frac{a+c}{1-c}\varepsilon$, which is a contradiction, therefore $\{Tx_n\}$ is ρ-Cauchy sequence. Since X_ρ is ρ-complete and C, ρ-closed, then there exist a $w \in C$ such that $\rho(Tx_n - w) \longrightarrow 0$. Since X_ρ is ρ-complete and C, ρ-closed, then there exists a $w \in C$ such that $\rho(Tx_n - w) \longrightarrow 0$. Now, we prove w is a fixed point of T.

$$\begin{aligned}\rho(r(Tx_{n+1} - Tw)) \leq & a\rho(l(x_{n+1} - w)) \\ & + b[\rho(l(Tx_{n+1} - x_{n+1})) + \rho(l(Tw - w))] \\ & + c[\rho(l(Tx_{n+1} - w)) + \rho(l(Tw - x_{n+1}))],\end{aligned}$$

therefore as $n \longrightarrow \infty$,

$$\rho(r(Tw - w)) \leq (b+c)\rho(l(Tw - w)) \leq (b+c)\rho(r(Tw - w)),$$

since $b + c < 1$ then $\rho(r(Tw - w)) = 0$ and $Tw = w$. For uniqueness, let z and w be two arbitrary fixed points of T. Then

$$\begin{aligned}\rho(r(Tz - Tw)) \leq & a\rho(l(z - w)) \\ & + b[\rho(l(Tz - z)) + \rho(l(Tw - w))] \\ & + c[\rho(l(Tz - w)) + \rho(l(Tw - z))],\end{aligned}$$

since $\rho(r(z - w)) \leq (a + 2c)\rho(l(z - w))$ then $\rho(r(z - w)) = 0$ or $z = w$. □

Let C be a nonempty subset of a modular function L_ρ and $T : \Omega \times C \longrightarrow C$ be a uniformly ρ-generalized Lipschitizian random operator, that is,

$$\begin{aligned}\rho(r(T(t,u) - T(t,v))) \leq & a\rho(\frac{l}{2}(u - v)) \\ & + b[\rho(\frac{l}{2}(u - T(t,u))) + \rho(\frac{l}{2}(T(t,v) - v))] \\ & + c[\rho(\frac{l}{2}(u - T(t,v))) + \rho(\frac{l}{2}(v - T(t,u)))],\end{aligned} \tag{5}$$

for each $u, v \in C$ and $t \in \Omega$, also $r > l$. Let $x_{0,0} : \Omega \longrightarrow C$ be an arbitrary measurable mapping, consider the following iterative sequence $\{x_{k,n}(t)\}$ defined by,

$$\begin{cases} y_{k,n}(t) = (1-\beta_n)x_{k,n}(t) + \beta_n T_k(t, x_{k,n}(t)), \\ x_{k,n+1}(t) = (1-\alpha_n)x_{k,n}(t) + \alpha_n T_k(t, y_{k,n}(t)), \end{cases} \tag{6}$$

for $t \in \Omega$, where $0 < \alpha_n, \beta_n < 1$. For an arbitrary but fixed $z \in C$, define $T_k : \Omega \times C \to C$ by $T_k(t, x(t)) = (1-\lambda_k)z(t) + \lambda_k T(t, x(t))$, for all $x \in C$, $t \in \Omega$ and $\lambda_k \in (0, 1)$.

Since C is convex, it follows that $x_{k,n}$ is a mapping from Ω to C, for all $n \geq 0$ and $k \geq 0$.

In the sequel, we can prove that the sequence $\{x_{k,n}(t)\}$ generated by (6), is strongly convergent in a modular function L_ρ.

Theorem 2.2. *Let L_ρ be a ρ-complete modular function space where ρ is convex and satisfies the Δ_2-condition. Suppose C is a nonempty, ρ-closed and convex subset of L_ρ and*

(I) *T be a continuous random operator and for all $t \in \Omega$, $T(t, .) : C \to C$ satisfies inequality (5), where $\frac{a+b+c}{1-(b+c)} < 1$, also $r > 2l$.*

(II) $\sum_{n\geq 0} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

(III) $\lim_{k\to\infty} \lambda_k = 1$ *and for all $x \in C$ and $r \in \mathbb{R}^+$, $\rho(r(x(t) - z(t))) \leq m < \infty$.*

If $F(T) \neq \emptyset$, then the sequence $\{x_{k,n}(t)\}_{k\geq 0, n\geq 0}$, generated by (6), is strongly convergent to a random fixed point w of T in C.

Proof. For each $k \geq 0$ and for all $t \in \Omega$, we have,

$$\begin{aligned} \rho(r(T_k(t, u(t)) - T_k(t, v(t)))) &= \rho(r\lambda_k(T(t, u(t)) - T(t, v(t)))) \\ &\leq a\lambda_k \rho(\frac{l}{2}(u(t) - v(t))) \\ &+ b\lambda_k[\rho(\frac{l}{2}(u(t) - T(t, u(t)))) \\ &+ \rho(\frac{l}{2}(T(t, v(t)) - v(t)))] \\ &+ c\lambda_k[\rho(\frac{l}{2}(u(t) - T(t, v(t)))) \\ &+ \rho(\frac{l}{2}(T(t, u(t)) - v(t)))]. \end{aligned}$$

Since $T(t, u(t)) = \frac{1}{\lambda_k}[T_k(t, u(t)) - (1-\lambda_k)z(t)]$,

$$\rho(\frac{l}{2}(T(t, u(t)) - u(t))) \leq \rho(\frac{l}{\lambda_k}(T_k(t, u(t)) - u(t))) + \rho(l\frac{1-\lambda_k}{\lambda_k}(u(t) - z(t))).$$

Also, since $\frac{1}{2} < \lambda_k < 1$ then $\frac{l}{\lambda_k} < 2l$ and $\frac{1-\lambda_k}{\lambda_k} < 1$. It follows that,

$$\rho(\frac{l}{2}(T(t,u(t)) - u(t))) \leq \rho(l_0(T_k(t,u(t)) - u(t))) + \frac{1-\lambda_k}{\lambda_k}m,$$

where $l_0 = 2l$. Therefore, ,

$$\begin{aligned}\rho(r(T_k(t,u(t)) - T_k(t,v(t)))) \leq\ & a\rho(l_0(u(t) - v(t))) \\ & + b[\rho(l_0(T_k(t,u(t)) - u(t))) \\ & + \rho(l_0(T_k(t,v(t)) - v(t)))] \\ & + c[\rho(l_0(T_k(t,u(t)) - v(t))) \\ & + \rho(l_0(T_k(t,v(t)) - u(t)))] \\ & + 2\frac{1-\lambda_k}{\lambda_k}m(b+c).\end{aligned}$$

By Theorem 2.1, $\lim_{k\to\infty} \lambda_k = 1$, shows that, for each $k \geq 0$, T_k has a unique fixed point w_k, say, in C. Since C is ρ-closed, it follows that w_k is a mapping from Ω to C. For each $k \geq 0$, we show that $\rho(x_{k,n}(t) - w_k(t)) \longrightarrow 0$ as $n \longrightarrow \infty$. By equality (6),

$$\begin{aligned}\rho(r(x_{k,n+1}(t)-w_k(t)))\leq\ & (1-\alpha_n)\rho(r(x_{k,n}(t)-w_k(t))) \\ & +\alpha_n\rho(r(T_k(t,y_{k,n}(t))-w_k(t))),\end{aligned}$$

on the other hand,

$$\begin{aligned}\rho(r(T_k(t,y_{k,n}(t)) - w_k(t))) \leq\ & a\rho(l_0(y_{k,n}(t) - w_k(t))) \\ & + b\rho(l_0(T_k(t,y_{k,n}(t)) - y_{k,n}(t))) \\ & + c[\rho(l_0(y_{k,n}(t) - w_k(t))) \\ & + \rho(l_0(T_k(t,y_{k,n}(t)) - w_k(t)))] \\ & + 2\frac{1-\lambda_k}{\lambda_k}m(b+c),\end{aligned}$$

moreover,

$$\rho(l_0(T_k(t,y_{k,n}(t))-y_{k,n}(t))) \leq \rho(\alpha l_0(T_k(t,y_{k,n}(t))-w_k(t)))+\rho(r(y_{k,n}(t)-w_k(t))),$$

where $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l_0}$.

Case 1. Let $r > 2l_0$ then $\alpha l_0 < r$, we have,

$$\rho(r(T_k(t,y_{k,n}(t)) - w_k(t))) \leq q\rho(r(y_{k,n}(t) - w_k(t))) + 2\frac{1-\lambda_k}{\lambda_k}m,$$

where $q = \frac{a+b+c}{1-(b+c)}$. Therefore,

$$\begin{aligned}\rho(r(x_{k,n+1}(t) - w_k(t))) \leq\ & (1-\alpha_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ & + \alpha_n q\rho(r(y_{k,n}(t) - w_k(t))) \\ & + \alpha_n\Theta_k,\end{aligned} \tag{7}$$

where $\Theta_k = 2\frac{1-\lambda_k}{\lambda_k} m$. By the same argument,

$$\rho(r(y_{k,n}(t) - w_k(t))) \leq (1-\beta_n)\rho(r(x_{k,n}(t) - w_k(t))) + \beta_n q\rho(r(x_{k,n}(t) - w_k(t))) + \beta_n\Theta_k. \tag{8}$$

By inequalities (7) and (8),

$$\begin{aligned} \rho(r(x_{k,n+1}(t) - w_k(t))) \leq{}& (1-\alpha_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n q(1-\beta_n)\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n q^2\beta_n\rho(r(x_{k,n}(t) - w_k(t))) \\ &+ \alpha_n(q\beta_n + 1)\Theta_k. \end{aligned} \tag{9}$$

By Lemma 1.1, since $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{k\to\infty}\Theta_k = 0$, then $\rho(r(x_{k,n}(t) - w_k(t))) \longrightarrow 0$ and by Δ_2-condition $\rho(x_{k,n}(t) - w_k(t)) \longrightarrow 0$ as $n \longrightarrow \infty$.

Case 2. Let $l_0<r_0<2l_0$, since ρ is an increasing function, then $\rho(r_0(x_{k,n}(t) - w_k(t))) \leq \rho(r(x_{k,n}(t) - w_k(t)))$, by Δ_2-condition $\lim_{n\longrightarrow\infty}(r_0(x_{k,n}(t) - w_k(t))) = 0$, for $l < r_0 < 2l_0$. Thus, for any $r > l_0$, $\lim_{n\longrightarrow\infty}\rho(r(x_{k,n}(t) - w_k(t))) = 0$. Since $w_k(t) = (1-\lambda_k)z(t) + \lambda_k T(t, w_k(t))$ then

$$\rho(r(w_k(t) - T(t, w_k(t)))) = \rho(r(\frac{1-\lambda_k}{\lambda_k}(z(t) - w_k(t)))) \leq \frac{1-\lambda_k}{\lambda_k} m.$$

Since $\lambda_k \longrightarrow 1$ then $\rho(w_k(t) - T(t, w_k(t))) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore, $\{w_k\}$ is an approximate random fixed point sequence for T.

We show that the sequence $\{w_k\}$ is ρ-Cauchy. For $0 < n \leq m$, we have

$$\begin{aligned} \rho(l(w_m(t) - w_n(t))) \leq{}& (\lambda_m - \lambda_n)\rho(\beta l(T(t, w_n(t)) - z(t))) \\ &+ \rho(r(T(t, w_m(t)) - T(t, w_n(t)))), \end{aligned}$$

where $\beta \in \mathbb{R}^+$ be the conjugate of $\frac{r}{l}$. On the other hand

$$\begin{aligned} \rho(r(T(t, w_n(t)) - T(t, w_m(t)))) \leq{}& (a+2c)\rho(l(w_n(t) - w_m(t))) \\ &+ (b+c)\rho(l(T(t, w_n(t)) - w_n(t))) \\ &+ (b+c)\rho(l(T(t, w_m(t)) - w_m(t))). \end{aligned}$$

Therefore,

$$\begin{aligned} \rho(l(w_n(t) - w_m(t))) \leq{}& \frac{(\lambda_m - \lambda_n)}{1-(a+2c)} m \\ &+ \frac{b+c}{1-(a+2c)}\rho(r(T(t, w_m(t)) - w_m(t))) \\ &+ \frac{b+c}{1-(a+2c)}\rho(r(T(t, w_n(t)) - w_n(t))). \end{aligned}$$

By $\lambda_k \longrightarrow 1$, $\{w_k\}$, is a ρ-Cauchy sequences. Since $\{w_k\} \in C$ and L_ρ is ρ-complete, there exists $w \in C$ such that $\rho(w_k(t) - w(t)) \longrightarrow 0$ as $n \longrightarrow \infty$.

Notice that T is a ρ-continuous mapping and $\rho(T(t, w_k(t)) - T(t, w(t))) \longrightarrow 0$ as $n \longrightarrow \infty$. Since

$$\rho(\frac{1}{3}(T(t, w(t)) - w(t))) \leq \rho(T(t, w(t)) - T(t, w_k(t))) + \rho(T(t, w_k(t)) - w_k(t)) + \rho(w_k(t) - w(t)),$$

we have $\rho(T(t, w(t)) - w(t)) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $w : \Omega \longrightarrow C$ is the limit of a sequence of measurable mapping, is also measurable, and $w : \Omega \longrightarrow C$ is a random fixed point of the random operator T. Since

$$\rho(\frac{1}{3}(T(t, x_{k,n}(t)) - x_{k,n}(t))) \leq \rho(T(t, x_{k,n}(t)) - T(t, w_k(t))) + \rho(T(t, w_k(t)) - w_k(t)) + \rho(w_k(t) - x_{k,n}(t)),$$

then $\rho(T(t, x_{k,n}(t)) - x_{k,n}(t)) \longrightarrow 0$. □

Now, we can prove Theorem 2.3 (by the same technique of Theorem 2.2).

Theorem 2.3. *Suppose all the conditions of Theorem 2.2 are hold. We consider the following sequence* $\{x_{k,n}(t)\}$,

$$(10) \qquad \begin{cases} y_{k,n}(t) = \beta_n x_{k,n}(t) + (1 - \beta_n) T_k(t, x_{k,n}(t)), \\ x_{k,n+1}(t) = \alpha_n x_{k,n}(t) + (1 - \alpha_n) T_k(t, y_{k,n}(t)). \end{cases}$$

If $F(T) \neq \emptyset$, *then the sequence* $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$, *generated by* (10), *is strongly convergent to a random fixed point* w *of* T *in* C.

Theorem 2.4. *Under the conditions of Theorem 2.2, we consider the following sequence* $\{x_{k,n}(t)\}_{k \geq 0, n \geq 0}$,

$$(11) \qquad \begin{cases} x^1_{k,n}(t) = (1 - \beta_{n,0}) x_{k,n}(t) + \beta_{n,0} T_k(t, x_{k,n}(t)), \\ x^2_{k,n}(t) = (1 - \beta_{n,1}) x_{k,n}(t) + \beta_{n,1} T_k(t, x^1_{k,n}(t)), \\ \vdots \\ x^m_{k,n}(t) = (1 - \beta_{n,m-1}) x_{k,n}(t) + \beta_{n,m-1} T_k(t, x^{m-1}_{k,n}(t)), \\ x_{k,n+1}(t) = (1 - \beta_{n,m}) x_{k,n}(t) + \beta_{n,m} T_k(t, x^m_{k,n}(t)), \end{cases}$$

where $\{\beta_{n,i}\}$, $i = 0, \ldots, m$ *be real sequences in* $(0, 1)$. *Assume the following conditions are satisfied:*

(I) $\sum_{n \geq 0} \beta_{n,m} = \infty$ *and* $\lim_{n \longrightarrow \infty} \beta_{n,m} = 0$.

(II) $0 < \liminf_{n \longrightarrow \infty} \beta_{n,i} \leq \limsup_{n \longrightarrow \infty} \beta_{n,i} < 1$, *for all* $i = 0, \ldots, m - 1$.

(III) $\lim_{k \to \infty} \lambda_k = 1$.

(IV) *For all* $x \in C$ *and* $r \in \mathbb{R}^+$, $\rho(r(x(t) - z(t))) \leq m < \infty$.

(V) $F(T) \neq \emptyset$.

Then the double sequence $\{x_{k,n}(t)\}_{k\geq 0, n\geq 0}$ *generated by (11), is strongly convergent to a random fixed point* w *of* T *in* C.

Proof. By the same technique of Theorem 2.2,

$$\rho(r(x_{k,n+1}(t) - w_k(t))) \leq \mu_n \rho(r(x_{k,n}(t) - w_k(t))),$$

where

$$\begin{aligned} \mu_n = &(1-\beta_{n,m}) + \beta_{n,m} q[(1-\beta_{n,m-1}) \\ &+ \beta_{n,m-1} q[(1-\beta_{n,m-2}) + \beta_{n,m-2} q[\ldots[\ldots[(1-\beta_{n,1}) \\ &+ \beta_{n,1} q[(1-\beta_{n,0}) + \beta_{n,0} q]]\ldots] \\ &+ \beta_{n,m}(1 + q\beta_{n,m-1} + \ldots + \beta_{n,m-1}\beta_{n,m-2}\ldots\beta_{n,0} q^m)\Theta_k. \end{aligned}$$

Since $\lim_{n\to\infty} \beta_{n,m} = 0$, then $\rho(x_{k,n}(t) - w_k(t)) \longrightarrow 0$ as $n \longrightarrow \infty$. □

3. Random iteration sequences for ρ-nonexpansive semigroup

In 1992, Khamsi [8] studied the theory of semigroup in modular function space in the context of Musielak-Orlicz space and investigated its applications to differential equations. Since then it has an intensive development, see [1], [10] and [11]. In order to study this space, we recall the definition of ρ-nonexpansive semigroup in modular function space (see [8]) .

Let L^φ be the Musielak-Orlicz space and C a nonempty subset of L^φ, and $I : C \to C$ denotes the identity mapping. A one-parameter family $\zeta = \{T(t) : 0 \leq t < \infty\}$ from self-mappings of a nonempty, closed and convex subset C of L^φ is said to be a ρ-nonexpansive semigroup on C if the following conditions are satisfied:

(I) $T(0)x = x$, for all $x \in C$.

(II) $T(s+t)x = T(s)T(t)x$, for all $x \in C$ and $s, t \geq 0$.

(III) For each $x \in C$, the mapping $t \longrightarrow T(t)x$ is continuous on $[0, \infty)$.

(IV) $\rho(T(t)x - T(t)y) \leq \rho(x - y)$, for all $x, y \in C$.

We denote by $Fix(\zeta)$ the set of all common fixed point of ζ; that is, $Fix(\zeta) = \{x \in C : T(s)x = x, \forall s > 0\}$. Then $Fix(\zeta)$ is nonempty if C is bounded.

In this section, the convergence of the modified new iteration processes is presented. This processes is defined for each arbitrary measurable mapping, $x_1 : \Omega \longrightarrow C$, by

$$\begin{cases} u_n(t) = \frac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j(t, x_n(t)), \\ y_n(t) = \beta_n x_n(t) + (1-\beta_n)\frac{1}{w_n}\int_0^{w_n} K(s) u_n(t) ds, \\ x_{n+1}(t) = \alpha_n x_n(t) + (1-\alpha_n) T^n(t, y_n(t)), \end{cases} \tag{12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$.

Theorem 3.1. *Let L^φ be a ρ-complete Musielak-Orlicz space where ρ is convex and satisfies the Δ_2-condition. Suppose C is a nonempty, ρ-closed and convex subset of L^φ and T is a continuous random operator and for all $t \in \Omega$, $T(t,.) : C \to C$, satisfies inequality (1), $k_n = \frac{a_n+b_n+c_n}{1-(b_n+c_n)}$ and $\{k_n\} \subset [1,\infty)$ such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Let $\zeta = \{K(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on C and $\{w_n\}$ a positive real divergent sequence. Also $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$. If $F = F(T) \cap F(\zeta) \neq \emptyset$, then the sequence $\{x_n(t)\}_{n\geq 0}$, generated by (12), is strongly convergent to a random fixed point $v \in F$.*

Proof. For $v \in F$ and $1 \leq j \leq n$,

$$\rho(r(T^j(t,x_n(t))-v(t))) \leq \frac{a_j+b_j+c_j}{1-(b_j+c_j)}\rho(r(x_n(t)-v(t))) \leq k_j\rho(r(x_n(t)-v(t))),$$

hence,

$$\begin{aligned}\rho(r(u_n(t)-v(t))) &\leq \frac{1}{n+1}[\rho(r(x_n(t)-v(t))) + \sum_{j=1}^{n} k_j\rho(r(x_n(t)-v(t)))]\\ &\leq \eta_n\rho(r(x_n(t)-v(t))),\end{aligned}$$

where $\eta_n = \max\{k_j, 1 \leq j \leq n\}$. By process (12);

$$\begin{aligned}\rho(r(x_{n+1}(t)-v(t))) \leq{}& \alpha_n\rho(r(x_n(t)-v(t))) + (1-\alpha_n)\rho(r(T^n(t,y_n(t))-v(t)))\\ \leq{}&\alpha_n\rho(r(x_n(t)-v(t)))\\ &+ (1-\alpha_n)k_n[\beta_n\rho(r(x_n(t)-v(t)))\\ &+ (1-\beta_n)\eta_n\rho(r(x_n(t)-v(t))),\end{aligned}$$

therefore

$$\rho(r(x_{n+1}(t)-v(t))) \leq \{\alpha_n+(1-\alpha_n)k_n[\beta_n+(1-\beta_n)\eta_n]\}\rho(r(x_n(t)-v(t))). \tag{13}$$

By Lemma 1.1 and inequality (13), $\lim_{n\to\infty} \rho(x_n(t)-v(t)) = 0$. Since C is ρ-closed, it follows that v is a mapping from Ω into C. Since $v : \Omega \longrightarrow C$, is the pointwise limit of the sequence $\{x_n\}$ of measurable functions from Ω into C is also measurable and so, $T(.,v(.)) : \Omega \longrightarrow C$ is measurable. By $\rho(\frac{1}{2}(x_n(t) - T(t,x_n(t))) \leq \rho(x_n(t)-v(t)) + \rho(v(t)-T(t,x_n(t)))$, we have $\lim_{n\longrightarrow\infty} \rho(x_n(t) - T(t,x_n(t))) = 0$. □

Now we can prove Theorem 3.2 (by the same argument of Theorem 3.1).

Theorem 3.2. *Suppose all the conditions of Theorem 3.1 are hold. Consider the following sequence* $\{x_n(t)\}_{n\geq 0}$,

$$(14)\qquad \begin{cases} u_n(t) = \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j(t, x_n(t)),\\ x_n^1(t) = (1-\beta_{n,0})x_n(t) + \beta_{n,0}\dfrac{1}{w_n}\displaystyle\int_0^{w_n} K(s)u_n(t)ds,\\ x_n^2(t) = (1-\beta_{n,1})x_n(t) + \beta_{n,1}T^n(t, x_n^1(t)),\\ \vdots\\ x_n^m(t) = (1-\beta_{n,m-1})x_n(t) + \beta_{n,m-1}T^n(t, x_n^{m-1}(t)),\\ x_{n+1}(t) = (1-\beta_{n,m})x_n(t) + \beta_{n,m}T^n(t, x_n^m(t)). \end{cases}$$

Let

(I) $\sum_{n\geq 0}\beta_{n,m} = \infty$ *and* $\lim_{n\longrightarrow\infty}\beta_{n,m} = 0$.

(II) $0 < \liminf_{n\longrightarrow\infty}\beta_{n,i} \leq \limsup_{n\longrightarrow\infty}\beta_{n,i} < 1$, *for all* $i = 0, \ldots, m-1$.

(III) $F = F(T)\cap F(\zeta) \neq \emptyset$.

Then the sequence $\{x_n(t)\}_{n\geq 0}$, *generated by (14), is strongly convergent to a random fixed point* $v \in F$.

4. Numerical examples

In this section, some numerical examples (using Matlab software) are presented. First, we show that a generalized Lipschitzian mapping is not necessarily a Lipschitzian mapping.

Example 4.1. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. For each $x \in C$, we define

$$(15)\qquad Tx = \begin{cases} \dfrac{rx}{1+x}, & \text{if } x \in [0, \frac{1}{4}],\\ 0, & \text{if } x \in (\frac{1}{4}, \infty), \end{cases}$$

where $0 < r < \frac{1}{4}$. Then $T : C \longrightarrow C$ is not continuous at $x = \frac{1}{4}$ and hence T is not a Lipschitzian mapping. Set $C_1 = [0, \frac{1}{4}]$ and $C_2 = (\frac{1}{4}, \infty)$. In order to prove $T(t, .) : C \longrightarrow C$ is a generalized Lipschitzian mapping, we need the following steps:

For all $x, y \in C_1$ and $n \geq 1$,

$$|Tx - Ty| = |\frac{rx}{1+x} - \frac{ry}{1+y}| = |\frac{rx(1+y) - ry(1+x)}{(1+x)(1+y)}| \leq r|x-y|,$$

and $|T^2x - T^2y| = |\frac{rTx}{1+Tx} - \frac{rTy}{1+Ty}| \leq r|Tx - Ty| \leq r^2|x-y|$. By induction, for all $n \geq 1$,

$$|T^nx - T^ny| \leq r^n|x-y| + r^n(|x - T^nx| + |y - T^ny|) + r^n(|x - T^ny| + |y - T^nx|).$$

For all $x, y \in C_2$ and $n \geq 1$, $|T^n x - T^n y| = 0 \leq |x - y|$.

For $x \in C_1$ and $y \in C_2$, $|Tx - Ty| = |\frac{rx}{1+x} - 0|$. By induction, for $n \in \mathbb{N}$, $T^{n+1}x = \frac{rT^n x}{1+T^n x} \leq rT^n x \leq r^{n+1}x$ and $|T^n x - T^n y| = |T^n x - 0| \leq r^n|x-y| + r^n(|x - T^n x| + |y - T^n y|) + r^n(|x - T^n y| + |y - T^n x|)$. Therefore, $T : C \longrightarrow C$ is a generalized Lipschitzian mapping.

Theorem 4.1. *Let L^φ, be the Musielak-Orlicz space and $I = [0, b'] \subset \mathbb{R}$. Suppose that ρ is convex and satisfies the Δ_2-condition. Since topologies generated by $\|.\|_\rho$ and ρ are equivalent, then we consider Banach space $(L^\varphi, \|.\|_\rho)$ also $C(I, L^\varphi)$ denotes the space of all $\|.\|_\rho$-continuous function from I to L^φ with the modular $\|x\|_\sigma = \sup_{t\in I} \|x(t)\|_\rho$ also $C(I, L^\varphi)$ is a real vector space (see [2]). Suppose A is a nonempty, convex and ρ-closed subset of L^φ, $T(t, .) : A \to A$ is a uniformly ρ-generalized Lipschitzian mapping satisfies inequality (2), such that $k = \frac{a+b+c}{1-(b+c)} < 1$. For each $u, v \in A$ and $t \in I$. Let $U_1 : I \longrightarrow A$ be an arbitrary measurable mapping, consider the following iterative sequence $\{U_n(t)\}$ for $n = 2, 4, 6, \ldots$, defined by*

$$\begin{cases} U_n(t) = \dfrac{1}{2n} U_1(t) + \Lambda(t, U_{n-1}(t)) \displaystyle\int_0^t \omega(t, s)\Phi(s, U_{n-1}(s))ds, \\ Y_n(t) = \beta_n U_{n-1}(t) + (1 - \beta_n)\Phi(t, U_n(t)), \\ X_n(t) = \alpha_n U_{n-1}(t) + (1 - \alpha_n)\Lambda(t, Y_n(t)), \\ U_{n+1}(t) = T(t, X_n(t)), \end{cases} \tag{16}$$

for $t \in I$, where $0 < \alpha_n, \beta_n < 1$. Assume the following conditions are satisfied:

(1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

(2) *Φ is function from $I \times A$ into A such that $\Phi(t, .) : A \longrightarrow A$ is a $\|.\|_\rho$-continuous and $t \longrightarrow \Phi(t, x)$ is measurable for every $x \in A$. Also, for $t \in I$, $x \longrightarrow \Phi(t, x)$ is nondecreasing on A. For all $t \in I$ and $x, y \in A$ there exists $q \geq 0$ such that $\|\Phi(t, x) - \Phi(t, y)\|_\rho \leq q\|x - y\|_\rho$. Also, there exists function $\beta \in L^1(I)$ and a nondecreasing continuous function $\gamma : [0, \infty) \longrightarrow (0, \infty)$ such that $\|\Phi(t, x)\|_\rho \leq \beta(t)\gamma(\|x\|_\rho)$, for all $t \in I$ and $x \in A$.*

(3) *Λ is function from $I \times A$ into A such that $\Lambda(t, .) : A \longrightarrow A$ is $\|.\|_\rho$-continuous and there exists a $h \geq 0$ such that $\|\Lambda(t, x) - \Lambda(t, y)\|_\rho \leq h\|x - y\|_\rho$, for all $t \in I$ and $x, y \in A$. Also for $x \in A$, $t \longrightarrow \Lambda(t, x)$ is nondecreasing on I and for $t \in I$, $x \longrightarrow \Lambda(t, x)$ is nondecreasing on A.*

(4) *ω is function from $I \times I$ into $\mathbb{R}^+$. For each $t \in I$, $\omega(t, s)$ is measurable on $[0, t]$. Also $\overline{\omega(t)} = esssup|\omega(t, s)|$ is bounded on $[0, b']$ and $r = \sup|\overline{\omega(t)}|$. The map $\omega(., s) : t \longrightarrow \omega(t, s)$ is continuous from I to $L^\infty(I)$. Also for $s \in I$, $t \longrightarrow \omega(t, s)$ is nondecreasing on I.*

Let $F = F(T) \cap F(\Lambda) \cap F(\Phi) \neq \emptyset$, and there exists a constant $d \geq 0$ such that for all $t \in I$ and $u \in C(I, A)$; $\|u(t)\|_\rho \leq d$ and $\int_0^t \beta(s)ds < \frac{d}{(hd+h')rb'} \int_0^t \frac{1}{\gamma(d)}ds$, where $h' := \sup\{\|\Lambda(t,x)\|_\rho, t \in I, x \in A\}$. For $z \in F$, since

$$\begin{aligned}
\|\Lambda(t, U_{n-1}(t)) \int_0^t \omega(t,s)\Phi(s, U_{n-1}(s))ds\|_\rho &= \|(\Lambda(t, U_{n-1}(t)) - \Lambda(t,0) \\
&\quad + \Lambda(t,0)) \int_0^t \omega(t,s)\Phi(s, U_{n-1}(s))ds\|_\rho \\
&\leq (h\|U_{n-1}(t)\|_\rho + h')r \int_0^t \beta(s)\gamma(\|U_{n-1}(s)\|_\rho)ds \\
&\leq (dh + h')r \int_0^t \beta(s)\gamma(d)ds \\
&\leq (dh + h')r \int_0^{b'} \frac{d\gamma(d)}{(hd + h')rb'\gamma(d)}ds \\
&\leq d,
\end{aligned}$$

and

$$\begin{aligned}
\|U_{n+1}(t) - z(t)\|_\rho &\leq k\|X_n(t) - z(t)\|_\rho \\
&\leq \alpha_n\|U_{n-1}(t) - z(t)\|_\rho + kh(1-\alpha_n)\|Y_n(t) - z(t)\|_\rho \\
&\leq \alpha_n\|U_{n-1}(t) - z(t)\|\rho \\
&\quad + kh(1-\alpha_n)\beta_n\|U_{n-1}(t) - z(t)\|\rho \\
&\quad + khq(1-\alpha_n)(1-\beta_n)\frac{1}{2n}\|U_1(t) - z(t)\|_\rho \\
&\quad + dkhq(1-\alpha_n)(1-\beta_n),
\end{aligned}$$

then the sequence $\{U_n(t)\}_{n\geq 1}$, generated by (16), is strongly convergent to a random fixed point $v \in F$.

Example 4.2. Let $\{U_n\}$ be the sequence defined by (16), T be mapping defined by (15) and $r = \frac{1}{5}$. Also $\alpha_n = \frac{1}{2n}$ and $\beta_n = \frac{n}{2n+1}$. Suppose that $\omega(t,s) = \frac{1}{2}$, $\Lambda(t,x) = \frac{x}{2}$ and $\Phi(t,x) = \frac{x}{2}$. We have

$$\begin{cases}
U_n(t) = \dfrac{1}{2n}U_1(t) + \dfrac{U_{n-1}(t)}{8}\displaystyle\int_0^t U_{n-1}(s)ds, \\
Y_n(t) = \dfrac{n}{2n+1}U_{n-1}(t) + \dfrac{n+1}{2n+1}\dfrac{U_n(t)}{2}, \\
X_n(t) = \dfrac{1}{2n}U_{n-1}(t) + \dfrac{2n-1}{2n}\dfrac{Y_n(t)}{2} \\
U_{n+1}(t) = \dfrac{X_n(t)}{5(1+X_n(t))}.
\end{cases} \tag{17}$$

Let $t = \frac{1}{4}$ and $U_1(\frac{1}{4}) = \frac{1}{5}$, the sequence $\{U_n(\frac{1}{4})\}_{n\geq 1}$, is strongly convergent to 0.

n	$U_n(\frac{1}{4})$	n	$U_n(\frac{1}{4})$	n	$U_n(\frac{1}{4})$
1	0.2	14	0.00714285868	27	0.00010176298
2	0.20125	15	0.0001898664	28	0.00357142889
3	0.01579818388	16	0.00625000112	29	0.00009445820
4	0.02500779945	17	0.00016592096	30	0.00333333361
5	0.00160418577	18	0.00555555641	31	0.00008813187
6	0.01666674708	19	0.00014734113	32	0.00312500024
7	0.00050465598	20	0.00500000067	33	0.00008259973
8	0.01250000795	21	0.00013250412	34	0.00294117668
9	0.0051201096	22	0.00454545509	35	0.00007772107
10	0.01000000357	23	0.00012038213	36	0.00277777796
11	0.00026713396	24	0.00416666711	37	0.00007338656
12	0.00833333556	25	0.00011029226	38	0.00263157911
13	0.00022190673	26	0.00384615422	39	0.00006950997

Example 4.3. Let $X_\rho = \mathbb{R}$ be the set of real numbers, $C = [0, \infty)$ and $\{x_{k,n}\}$ be the sequence defined by (6), where

(1) $Tx = \frac{x}{70000}$ and $z = 0$.

(1) Suppose $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{n}{20n+1}$ and $\lambda_k = \frac{7k}{7k+100}$.

We have $F(T) = \{0\}$ and

$$\begin{cases} y_{k,n} = \dfrac{19n+1}{20n+1} x_{k,n} + \dfrac{n}{20n+1} \left(\dfrac{7k}{7k+100} \right) \dfrac{x_{k,n}}{70000}, \\ x_{k,n+1} = \dfrac{20n-1}{20n} x_{k,n} + \dfrac{1}{20n} \left(\dfrac{7k}{7k+100} \right) \dfrac{y_{k,n}}{70000}. \end{cases}$$

If $x_{1,1} = 0.001$, so

n	$x_{1,n}$	n	$x_{1,n}$
1	0.001	11	0.000512
2	0.0009	12	0.000484
3	0.000832	13	0.000457
4	0.000777	14	0.000433
5	0.000728	15	0.00041
6	0.000684	16	0.000388
7	0.000644	17	0.000367
8	0.000607	18	0.000348
9	0.000573	19	0.000329
10	0.000541	20	0.000312

Example 4.4. Let $X_\rho = \mathbb{R}$ be the set of real numbers, $C = [0, \infty)$ and $\{x_{k,n}\}$ be the sequence defined by (10), where $Tx = \frac{x}{70000}$ and $z = 0$. Also $\alpha_n = \frac{1}{20n}$, $\beta_n = \frac{n}{20n+1}$ and $\lambda_k = \frac{7k}{7k+100}$. We have,

$$\begin{cases} y_{k,n} = \dfrac{n}{20n+1} x_{k,n} + \dfrac{19n+1}{20n+1} (\dfrac{7k}{7k+100}) \dfrac{z_{k,n}}{70000}, \\ x_{k,n+1} = \dfrac{1}{20n} x_{k,n} + \dfrac{20n-1}{20n} (\dfrac{7k}{7k+100}) \dfrac{y_{k,n}}{70000}. \end{cases}$$

If $x_{1,1} = 0.001$, so

n	$x_{1,n}$	n	$x_{1,n}$
1	0.001	11	2.691×10^{-23}
2	5×10^{-5}	12	1.223×10^{-25}
3	1.25×10^{-6}	13	5.097×10^{-28}
4	2.083×10^{-8}	14	1.96×10^{-30}
5	2.604×10^{-10}	15	7.001×10^{-33}
6	2.64×10^{-12}	16	2.333×10^{-35}
7	2.17×10^{-14}	17	7.293×10^{-38}
8	1.55×10^{-16}	18	2.145×10^{-40}
9	9.688×10^{-19}	19	5.959×10^{-43}
10	8.382×10^{-21}	20	1.568×10^{-45}

Corollary 4.1. *Examples 4.3 and 4.4 show that the rate of convergence of sequence* $\{x_{k,n}\}$ *generated by* (10)*, is faster than the rate of convergence of sequence* $\{x_{k,n}\}$ *generated by* (6)*.*

Example 4.5. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. Consider:

(1) $Tx = \frac{x}{1000}$.

(2) $K(s) = e^{-s}$ and $t_n = n$.

(3) $\alpha_n = \frac{1}{17n}$, $\beta_n = \frac{n}{15n+1}$.

Let $\{x_n\}$ be the sequence defined by (12). So,

$$\begin{cases} u_n = \dfrac{1}{n+1} \displaystyle\sum_{j=0}^{n} \dfrac{x_n}{1000^j}, \\ y_n = \dfrac{n}{15n+1} x_n + \dfrac{14n+1}{15n+1} (\dfrac{1-e^{-n}}{n}) u_n, \\ x_{n+1} = \dfrac{1}{17n} x_n + \dfrac{1}{1000^n} \dfrac{17n-1}{17n} y_n, \end{cases}$$

and $Fix(\zeta) \cap Fix(T) = \{0\}$. If $x_1 = 1$, then

n	x_n	n	x_n	n	x_n
1	1	11	1.374×10^{-19}	21	1.017×10^{-43}
2	0.05916	12	7.351×10^{-22}	22	2.849×10^{-46}
3	0.00174	13	3.603×10^{-24}	23	7.618×10^{-49}
4	0.00003	14	1.63×10^{-26}	24	1.948×10^{-51}
5	5.017×10^{-7}	15	6.851×10^{-29}	25	4.775×10^{-54}
6	5.902×10^{-9}	16	2.686×10^{-31}	26	1.123×10^{-56}
7	5.787×10^{-11}	17	9.878×10^{-34}	27	2.542×10^{-59}
8	4.863×10^{-13}	18	3.418×10^{-36}	28	5.538×10^{-62}
9	3.575×10^{-15}	19	1.117×10^{-38}	29	1.163×10^{-64}
10	2.337×10^{-17}	20	3.458×10^{-41}	30	2.36×10^{-67}

Example 4.6. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$, where $Tx = \frac{x}{800}$. Also $\alpha_n = \frac{1}{30n}$ and $\beta_n = \frac{29n+1}{30n+1}$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} u_n = \dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} \dfrac{x_n}{800^j}, \\ y_n = \dfrac{29n+1}{30n+1}x_n + \dfrac{n}{30n+1}u_n, \\ x_{n+1} = \dfrac{1}{30n}x_n + \dfrac{1}{800^n}\dfrac{30n-1}{30n}y_n. \end{cases}$$

If $x_1 = 1$, so

n	x_n	n	x_n
1	1	11	4.833×10^{-22}
2	0.0345	12	1.464×10^{-24}
3	5.754×10^{-4}	13	4.068×10^{-27}
4	6.393×10^{-6}	14	1.043×10^{-29}
5	5.327×10^{-8}	15	2.484×10^{-32}
6	3.551×10^{-10}	16	5.52×10^{-35}
7	1.973×10^{-12}	17	1.15×10^{-37}
8	9.396×10^{-15}	18	2.254×10^{-40}
9	3.915×10^{-17}	19	4.175×10^{-43}
10	1.45×10^{-19}	20	7.325×10^{-46}

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