Towards the supercharacter theory of the dicyclic group

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Abstract. The dicyclic group of order $4n$ has a presentation

$$T_{4n} = \langle a, b \mid a^{4n} = 1, a^{2n} = b^2, b^{-1}ab = a^{-1} \rangle$$

and is a non-split extension of a cyclic group of order $2n$ by a cyclic group of order 2. In this paper we investigate a few supercharacter theory for T_{4n}.
Keywords: supercharacter, dicyclic group.

1. Introduction

Supercharacter theory was first defined in [1] and [2] to study the irreducible comlex characters of the group $U_n(q)$ of n by n unipotent upper triangular matrices with entries in the Galois field $GF(q)$. But in [4] the authors formally defined the notion of supercharacter theory for an arbitrary finite group and studied supercharacter theory of a famly of finite groups known as algebra groups.

Let G be a finite group and $Irr(G)$ be the set of all its complex irreducible characters, and let $Con(G)$ denote the set of all the conjugacy classes of G. The trivial character of G is denoted by 1_G and its identity element is denoted by 1. In [4] a supercharacter theory for G is defined to be a pair $(\mathcal{X}, \mathcal{K})$ where:

1. $\mathcal{X}$ is a partition of $Irr(G)$,
2. $\mathcal{K}$ is a partition of G and $\{1\} \in \mathcal{K}$,
3. $|\mathcal{X}| = |\mathcal{K}|$,
4. For each $X \in \mathcal{X}$ there is a character $\sum_X$ of G such that $\sum_X(x) = \sum_X(y)$ for all $x, y \in K$, $K \in \mathcal{K}$.

In the above situation $\sum_X$ is called a supercharacter and each member of $\mathcal{K}$ is called a superclass. $Sup(G)$ is the set of all the supercharacter theories for G.

Supercharacter theory of a finite group G may be regarded as a general case of the ordinary character theory. In fact, in a supercharacter theory, supercharacters play the role of irreducible ordinary characters and union of conjugacy classes play the role of the conjugacy classes. In [4] it is shown that $\{1_G\} \in \mathcal{X}$ and if $X \in \mathcal{X}$, then $\sum_X$ is a constant multiple of $\sum_{\chi \in X} \chi(1)\chi$, and that we may assume that $\sum_X = \sum_{\chi \in X} \chi(1)\chi$.

Any finite group G has two extreme supercharacter theories which are denoted by $m(G)$ and $M(G)$ and are called trivial supercharacter theory for G:

$$m(G) = (Con(G), \{\{\chi\} : \chi \in Irr(G)\})$$
$$M(G) = (\{\{1\}, G - \{1\}\}, \{\{1_G\}, Irr(G) - \{1_G\}\})$$

These supercharacter theories are distinct for all groups of order greater than 2. In [3] it is shown that the only finite groups with exactly two supercharacter theories are $\mathbb{Z}_3$, $\mathbb{S}_3$ and $SP_6(2)$.

It is proved in [5] that the set of all supercharacter theories of a group forms a lattice in the following natural way. $Sup(G)$ can be made to a poset by defining $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{X} \leq \mathcal{Y}$ in the sense that every part of $\mathcal{X}$ is a subset of some part of $\mathcal{Y}$. By [5] this definition is equivalent to $(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})$ if $\mathcal{K} \leq \mathcal{L}$. By this definition $m(G)$ is the least and $M(G)$ is the largest element of $Sup(G)$.

The first author to attempt to classify the supercharacter theories of a given finite group was Hendrickson. In [5] the author classified the supercharacter theories of arbitrary cyclic groups. In [10] the authors study the supercharacter theories for extra special p-groups and also Frobenius groups. In [11] the author attempts to find the supercharacter theories for parabolic subgroups. For other families of finite groups, as extensions of the cyclic groups, the supercharacter theories of the dihedral groups were found in [9]. The dihedral group of order $2n$ has presentation:

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle = \langle a \rangle \rtimes \langle b \rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2.$$

The dicyclic group of order $4n$ has presentation

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle \cong \mathbb{Z}_{2n} \cdot \mathbb{Z}_2$$

which is a non-split extension of the cyclic group of order $\mathbb{Z}_{2n}$ by the cyclic group of order 2. Our aim in this paper is to provide some supercharacter theories of T_{4n}.

2. Preliminaries

For a non-trivial supercharacter theory of a group the following is described in [4]. Let G be a finite group and $A \leq \mathbb{A}ut(G)$. Let

$$Irr(G) = \{\chi_1 = 1_G, \chi_2, \cdots, \chi_h\} \text{ and } Con(G) = \{\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \cdots, \mathcal{C}_h\}.$$

Suppose that for each $\alpha \in A$, $\mathcal{C}_i^\alpha = \mathcal{C}_j$, $1 \leq i \leq h$, and $\chi_i^\alpha(g) = \chi_i(g^\alpha)$, for all $g \in G$, and $\alpha \in A$. In this case we have an action of A on both $Irr(G)$

and $Con(G)$ and by a Lemma of Brauer mentioned in [7] the number of conjugacy classes of G fixed by α equals the number of irreducible characters fixed by α, moreover the number of orbits of A on $Con(G)$ equals the number of orbits of A on $Irr(G)$. It is easy to see that the orbits of A on $Irr(G)$ and $Con(G)$ yield a supercharacter theory for G. This supercharacter theory for G is called automorphic. There are groups that all of its supercharacter theories are automorphic.

Theorem 2.1. *Every supercharacter theory of $\mathbb{Z}_p$, p prime, is automorphic. Moreover, for each divisor d of $p-1$, there is a unique supercharacter theory for $\mathbb{Z}_p$ whose non-trivial superclasses all have size d. Therefore the number of supercharacter theories is equal to $\varphi(p-1)$, where φ denotes the Euler totient function.*

According to [4] there is a method of constructing supercharacter theories for a group G of order n which uses the action of a group A of automorphisms of the cyclotomic field $\mathbb{Q}[\xi]$, where ξ is a primitive nth root of unity. It is known that the values of the irreducible characters of G are all lie in the cyclotomic field $\mathbb{Q}(\xi)$. Let $A \leq \mathbb{A}ut(\mathbb{Q}(\xi))$, then for $\sigma \in \mathbb{A}ut(\mathbb{Q}(\xi))$ we have $\sigma(\xi) = \xi^m$, where $(m, n) = 1$. This defines an automorphism of $\mathbb{Q}(\xi)$ and all automorphisms of $\mathbb{Q}(\xi)$ are obtained in this way. Then σ carries the class of $g \in G$ to the class containing g^m. The Brauer Lemma shows that the number of A-orbits on $Irr(G)$ is equal to the number of A-orbits on $Con(G)$. In this case similarly if $\mathcal{X}$ is the set of A-orbits on $Irr(G)$ and $\mathcal{K}$ is the set of unions of A-orbits on $Con(G)$, then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for G. We shall call this supercharacter theory Galois.

At this point we define the supercharacter table. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for a finite group G. Suppose $\mathcal{X} = \{X_1, X_2, \cdots, X_k\}$ is a partition of $Irr(G)$ with corresponding supercharacter $\sum_i = \sum_{\chi \in X_i} \chi(1)\chi$. Let $\mathcal{K} = \{K_1, K_2, \cdots, K_k\}$ be the partition of G into superclasses. In fact, $X_1 = \{1_G\}$, $K_1 = \{1\}$ and K_i's are union of conjugacy classes of G. The supercharacter table of G corresponding to $(\mathcal{X}, \mathcal{K})$ is the following $k \times k$ array.

Table 1

	K_1	K_2	$\cdots$	K_j	$\cdots$	K_k
$\sum_1$	$\sum_1(K_1)$	$\sum_1(K_2)$	$\cdots$	$\sum_1(K_j)$	$\cdots$	$\sum_1(K_k)$
$\sum_2$	$\sum_2(K_1)$	$\sum_2(K_2)$	$\cdots$	$\sum_2(K_j)$	$\cdots$	$\sum_2(K_k)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum_i$	$\sum_i(K_1)$	$\sum_i(K_2)$	$\cdots$	$\sum_i(K_j)$	$\cdots$	$\sum_i(K_k)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum_h$	$\sum_h(K_1)$	$\sum_h(K_2)$	$\cdots$	$\sum_h(K_j)$	$\cdots$	$\sum_h(K_k)$

3. The dicyclic group

In terms of the generators and relations the dicyclic group is defined by

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, n \geq 1.$$

This is a group of order $4n$ and $T_4 \cong \mathbb{Z}_4$, $T_8 \cong Q_8$ the quaternion group of order 8. Since T_{4n} has only one element of order 2, and $\langle a \rangle \trianglelefteq T_{4n}$, therefore $T_{4n} \cong \mathbb{Z}_{2n} \cdot \mathbb{Z}_2$ is a non-split extension of $\mathbb{Z}_{2n}$ by $\mathbb{Z}_2$.

The conjugacy classes and the complex character table of T_{4n} can be found in [8] and are listed below:

$G = T_{4n}$ has $n+3$ conjugacy classes as follows:

$$\begin{aligned}
K_1 &= \{1\} \\
K_2 &= \{a^n\} \\
K_r &= \{a^r, a^{-r}\}, 1 \leq r \leq n-1 \\
K_3' &= \{a^{2j}b \mid 0 \leq j \leq n-1\} \\
K_4' &= \{a^{2j+1}b \mid 0 \leq j \leq n-1\}.
\end{aligned}$$

If n is odd, then $\frac{G}{G'} \cong \mathbb{Z}_4$ and the character table of G is as follows:

Table 2

$\lvert C_G(x) \rvert$	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	$-i$	i
$\underset{1 \leq j \leq n-1}{\psi_j}$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

If n is even, then $\frac{G}{G'} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Table 3

$\lvert C_G(x) \rvert$	$4n$	$4n$	$2n$	4	4
x	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	$(-1)^r$	1	-1
χ_4	1	1	$(-1)^r$	-1	1
$\underset{1 \leq j \leq n-1}{\psi_j}$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

Where $\omega = e^{\frac{2\pi i}{n}}$ in both tables.

Lemma 3.1. *The automorphism group of T_{4n} has order $2n\varphi(2n)$ and its elements are*

$$\mathbb{A}ut(T_{4n}) = \{f_{k,l} \mid f_{k,l}(a) = a^k, f_{k,l}(b) = a^l b, (k, 2n) = 1, 0 \le l < 2n\}.$$

Proof. It is enough to define each automorphism of T_{4n} on a and b. The only elements of order $2n$ in T_{4n} are a^k, where $(k, 2n) = 1$. Therefore if f is an automorphism of T_{4n} we should have $f(a) = a^k$, $(k, 2n) = 1$. But then $f(b)$ cannot be a power of a, and as an element of order 4 we should have $f(b) = a^l b$, $0 \le l < 2n$. Then it is routine to check that f with above definition is infact an automorphism of T_{4n}. Therefore we set $f = f_{k,l}$ and verify that $f_{k,l}f_{k',l'} = f_{kk',kl'+l}$. If Φ_{2n} denotes the groups of units of $\mathbb{Z}_{2n}$ we have $\mathbb{A}ut(T_{4n}) \cong \mathbb{Z}_{2n} \rtimes \Phi_{2n}$. □

Keeping fixed the previous notation we see that orbits of $\mathbb{A}ut(T_{4n})$ on $Con(T_{4n})$ are:

$$K_1, K_2, K_3' \cup K_4', \bigcup_{\substack{k \\ (k,2n)=1}} K_{rk}\,.$$

Orbits of $\mathbb{A}ut(T_{4n})$ on $Irr(T_{4n})$ are:

$$\chi_1, \chi_2 + \chi_4, \chi_3, \sum_{\substack{j \\ (j,2n)=1}} \psi_j \qquad \text{if } n \text{ is odd and,}$$

$$\chi_1, \chi_2, \chi_3 + \chi_4, \sum_{\substack{j \\ (j,2n)=1}} \psi_j \qquad \text{if } n \text{ is even.}$$

Therefore, the above is an automorphic supercharacter theory for T_{4n}. To be precise in the following we give a supercharacter table in case of T_{4p}, where p is an odd prime number. In this case the orbits of $\mathbb{A}ut(T_{4p})$ on $Con(T_{4p})$ are:

$$\begin{aligned} &K_1 = \{1\}, \\ &K_2 = \{a^p\}, \\ &K_3' \cup K_4', \\ &K_5 = \{a^{\pm 1}, a^{\pm 3}, \cdots, a^{\pm(2p-1)}\}, \\ &K_6 = \{a^{\pm 2}, a^{\pm 4}, \cdots, a^{\pm(2p-2)}\}. \end{aligned}$$

Using tables 2 we find the following partition of $Irr(T_{4n})$:

$$\{\chi_1\}, \{\chi_2, \chi_4\}, \{\chi_3\}, \{\psi_j \mid j = 1, 3, \cdots, p-2\}, \{\psi_j \mid j = 2, 4, \cdots, p-1\}$$

Therefore $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for T_{4p} with the following supercharacter table:

Table 4

	K_1	K_2	$K'_3 \cup K'_4$	K_5	K_6
$\sum_1$	1	1	1	1	1
$\sum_2$	2	-2	0	-2	2
$\sum_3$	1	1	-1	1	1
$\sum_4$	$p-1$	$-(p-1)$	0	$2\cos\frac{\pi}{p}$	$-2\cos\frac{\pi}{p}$
$\sum_5$	$p-1$	$p-1$	0	$-2\cos\frac{\pi}{p}$	$2\cos\frac{\pi}{p}$

If $p = 2$, then $T_8 \cong Q_8$ is the quaternion group of order 8 and this case we have:

$$\mathcal{K} = \{K_1 = \{1\}, K_2 = \{a^2\}, K_3 = \{a\}, K_4 = \{class(b), class(ab)\}\}$$
$$\mathcal{X} = \{\{\chi_1\}, \{\chi_2\}, \{\chi_3, \chi_4\}, \{\psi_1\}\}$$

Table 5

	K_1	K_2	K_3	K_4
$\sum_1$	1	1	1	1
$\sum_2$	1	1	1	-1
$\sum_3$	2	2	-2	0
$\sum_4$	2	-2	0	0

Lemma 3.2. *If n is even, then T_{4n} and D_{4n} have the same number of supercharacter theory.*

Proof. If n is even it is easy to see that T_{4n} and D_{4n} have the sane character table. But then groups with the same character tables have the same supercharacter theory. $\square$

Next, we will consider a certain subgroup of $\mathbb{A}ut(T_n)$. Let

$$H = \{f_{1,l} \mid 0 \le l < 2n\} \cong \mathbb{Z}_{2n},$$

where $f_{1,l}(a) = a$, $f_{1,l}(b) = a^l b$, $0 \le l < 2n$. Orbits of H on $Con(T_{4n})$ are:

$$\begin{aligned} &K_1 = \{1\},\\ &K_2 = \{a^n\},\\ &K_r = \{a^r\}, 1 \le r \le n-1,\\ &K'_3 \cup K'_4. \end{aligned}$$

Therefore, there are $n+2$ superclasses, the supercharacter are:

$$\begin{cases} \sum\limits_1 = \chi_1, \sum\limits_2 = \chi_2 + \chi_4, \sum\limits_3 = \chi_3, \sum\limits_j = \psi_j, 1 \le j \le n-1, & \text{if } n \text{ is odd,} \\ \sum\limits_1 = \chi_1, \sum\limits_2 = \chi_2, \sum\limits_3 = \chi_3 + \chi_4, \sum\limits_j = \psi_j, 1 \le j \le n-1, & \text{if } n \text{ is even.} \end{cases}$$

Now, let p be an odd prime and consider the subgroup

$$K = \{f_{k,0} \mid (k, 2p) = 1\} \cong \mathbb{Z}_{p-1},$$

where $f_{k,0}(a) = a^k$, $k \neq p$, $k = 1, 3, \cdots, 2p-1; f_{k,0} = b$.

In this case, regarding the action of K, the superclasses are $K_1 = \{1\}$, $K_2 = \{a^p\}$, $K_3 = class(a)$, $K_4 = class(a^2)$, $K_5 = class(b)$, $K_6 = class(b^3)$.

The supercharacters are:

$$\{\chi_1\}, \{\chi_2\}, \{\chi_3\}, \{\chi_4\}, \sum_{\substack{j \text{ odd} \\ 1 \leq j \leq p-1}} 2\psi_j, \quad \sum_{\substack{j \text{ even} \\ 1 \leq j \leq p-1}} 2\psi_j.$$

Acknowledgements

The author would like to thank the referee for his useful comments.

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Accepted: July 06, 2020