Finite groups with $2pqr^2$ elements of maximal order

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Abstract. It is an interesting and difficult topic to determine the structure of a finite group with a given number of elements of maxiaml order. This topic is related to Thompson's conjecture, that is, if two finite groups have the same order type and one of them is solvable, then the other is solvable. In this article, we continue to this investigation and show that finite groups with $2pqr^2$ elements of maximal order are solvable, where p, q, r are primes and $5 < p < q < r$.

Keywords: finite groups, solvable groups, the order of elements.

1. Introduction

In this paper, all groups considered are finite groups. Let G be a finite group. By $O(x)$ we denote the order of an element $x \in G$, by $\pi(G)$ we denote the set of prime divisors of $|G|$ and by $\pi_e(G)$ we denote the set of orders of elements

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of the group G. k denotes the largest number of $\pi_e(G)$. $M_l(G)$ denotes the set of elements of order l in G, and so $M(G)$ denotes the set of elements with maximal order in G. In the present paper, we also use the following notation: for $p \in \pi(G)$, G_p denotes a Sylow p-subgroup of the group G. In addition, the set of all Sylow p-subgroups of the group G is denoted by $Syl_p(G)$ and $n_p(G) = |Syl_p(G)|$. The Euler function of a positive integer x is denoted by $\varphi(x)$. By $p^n \parallel |G|$ we denote $p^n||G|$ but $p^{n+1} \nmid |G|$. The other symbols and notations are standard in(cf.[6]).

The topic to study solvability of finite groups is one of the most important topic of finite group theory. The topic in this article is related to the following famous conjecture:

Thompson's conjecture. Let G_1, G_2 be finite groups. Assume that $|M_\ell(G_1)|$ $=|M_\ell(G_2)|$ with $\ell = 1, 2, 3, \cdots$. If G_1 is solvable, then G_2 is solvable (see [9, Problem 12.37]).

Up to now, there is no exhaustive answer to the Thompson's Conjecture. It is well known that the properties of elements of maximal or minimal order of a finite group strongly influence the structure of the given group in some scene. Some authors studied if G is solvable while the number of elements of maximal order is specially restricted (see [1],[3],[8]),etc. If G_1 and G_2 satisfies the hypothesis of Thompson's conjecture, then G_1, G_2 have the same number of elements of maximal order. Hence if we proved G is solvable for special $|M(G)|$, then for all groups G_1 and G_2 with $|M_\ell(G_1)| = |M_\ell(G_2)|$ equals this number are solvable. Hence, this study may give useful information to the Thompson's conjecture. The aim of the present paper is to study if a finite group having exactly $2pqr^2$ elements of the maximal order is solvable, we prove the following theorem:

Theorem 1.1. *Let G be a finite group having exactly $2pqr^2$ elements of maximal order, where p, q, r are primes and $5 < p < q < r$, then G is solvable.*

2. Preliminaries

We collect in this section some known results and elementary facts as well as some technical lemmas.

Lemma 2.1. ([12]). *Suppose that G has exactly n cyclic subgroups of order ℓ, then $|M_\ell(G)| = n\varphi(\ell)$. In particular, if n is a number of cyclic subgroups of the group G of order k, then $|M(G)| = n\varphi(k)$.*

Lemma 2.2. ([12]). *Suppose k is the largest order of elements of G. If $|M(G)| = \varphi(k)$, then G is supersolvable.*

Lemma 2.3. ([3]). *There exists a positive number α such that $|G|$ divides $|M(G)|k^\alpha$.*

Lemma 2.4. ([3]). *If there exists a prime divisor p of the number k such that $p(p-1) > |M(G)|$, then the group G contains a unique normal Sylow p-subgroups S and $|S| = p$.*

A finite non-abelian simple group G is called a simple K_n-group if $|\pi(G)| = n$.

Lemma 2.5. ([7]). *Let G be a simple K_3-group. Then G is isomorphic to one of following simple groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$. For convenience, we list the information of all K_3-simple groups in Table 1.*

G	$\lvert G\rvert$	$\lvert Out(G)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2

Table 1:

Lemma 2.6. ([10]). *Let G be a simple K_4-group. Then G is isomorphic to one of following simple groups:*

(1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$;

(2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$;

(3) *$L_2(r)$, where r is a prime number such that $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ for $a \geq 1, b \geq 1, c \geq 1$, and a prime number $v > 3$;*

(4) *$L_2(2^m)$, where the number $m \geq 2$ is such that the number $2^m - 1 = u$ is prime and $2^m + 1 = 3t^b$, moreover, $t > 3$ is a prime number and $b \geq 1$;*

(5) *$L_2(3^m)$, where the number $m \geq 2$ is such that $3^m + 1 = 4t, 3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u$; moreover, the number u and t are odd prime numbers, $b \geq 1$, and $c \geq 1$;*

(6) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^3D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$ *and* ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.

Lemma 2.7. ([10]). *Let G be a simple K_4-group with $3 \notin \pi(G)$. Then $G \cong S_z(8)(2^6 \cdot 5 \cdot\ 7 \cdot 13)$ or $G \cong S_z(32)(2^{10} \cdot 5^2 \cdot\ 31 \cdot 41)$.*

Lemma 2.8. ([5]). *If $|G| = n$ and $(n, 15) = 1$, then G is solvable.*

Lemma 2.9. ([11]). *If $|M(G)| = 2m$, where $(m, 2) = 1$, then one of the following holds:*

(1) *$k = 4$, s^α or $2s^\alpha$, where s is an odd prime number and $\alpha \in N$;*

(2) *If G is non-solvable, then $k = 2s^\alpha$ for some odd prime number s and $2 \| \varphi(k) = (s-1)s^{\alpha-1}$; moreover, each Sylow 2-subgroup of group G contains a maximal subgroup which is elementary abelian;*

(3) *If $k = 14$, $|G| = 2^\alpha \cdot 3 \cdot 7^\beta$, and G is non-solvable, then $G \cong E \times L_2(7)$, where E is an elementary abelian 2-group.*

Lemma 2.10. ([11]). *Suppose that G has n cyclic subgroups A_i of order k, where $1, 2, \cdots, n$. Let $\{A_1, A_2, \cdots, A_d\}$ be a complete system of representatives of the conjugate classes of n cyclic subgroups of order k, and n_i is the length of the conjugate class containing A_i. Then the following statements hold:*

(1) *$n_i = |G : N_G(A_i)|$, $n = \sum_{i=1}^{d} n_i$, and $\pi(n_i) \cup \pi(A_i) = \pi(n_1) \cup \pi(A_1)$, where $i = 1, 2, \cdots, d$;*

(2) *$\pi(C_G(A_i)) = \pi(A_i)$, $|N_G(A_i) : C_G(A_i)| \,\big|\, \varphi(k)$ and $|G| = |G : N_G(A_i)||N_G(A_i) : C_G(A_i)||C_G(A_i)|$, where $i = 1, 2, \cdots, d$;*

(3) *Let $A =< a >$, $|a| = k$. If $i = 1$, $|M(G)| = 2m$, where $(m, 2) = 1$, then G is solvable.*

Lemma 2.11. *Suppose that G is a finite non-solvable group such that $|M(G)| = 2pqr^2 = n \cdot \varphi(k)$. Then $k = 2s^\alpha$, where $\alpha \in N$, $\alpha \leq 3$, and s is an odd prime number. In addition, one of the following statements holds:*

(1) *$\alpha = 3$ and $s = r \in \{2p + 1, 2q + 1\}$;*

(2) *$\alpha = 2$ and either $s = q = 2p+1$, $n = r^2$ or $s = r \in \{2p+1, 2q+1, 2pq+1\}$;*

(3) *$\alpha = 1$ and $s \in \{2p+1, 2q+1, 2r+1, 2pq+1, 2pr+1, 2qr+1, 2r^2+1, 2pqr+1, 2pr^2+1, 2qr^2+1\}$.*

Proof. Since the G is non-solvable, it follows from Lemma 2.8 that $k = 2s^\alpha$, where s is an odd prime number. Then $2pqr^2 = n \cdot \varphi(2s^\alpha)$ by Lemma 2.1. If $\alpha > 3$, then $s^3 | \varphi(s^\alpha)$. Hence, $s^3 | pqr^2$, which leads to a contradiction. If $\alpha = 3$, then $s^2(s-1) | 2pqr^2$, thus only has $s = r$ and $(s-1)/2 \in \{p, q, pq\}$. Further, if $(s-1)/2 = pq$, then $\varphi(k) = 2pqr^2$. By Lemma 2.2, we know that G is supersolvable, which leads to a contradiction. Hence, $s = r \in \{2p + 1, 2q + 1\}$. If $\alpha = 2$, then $\frac{s-1}{2} s | pqr^2$. Consider the following three possible cases:

If $s = p$, then $(s-1)/2 < p$. Therefore $(s-1)/2 = 1$ and $s = p = 3$, which contradicts our assumption.

If $s = q$, then $(s-1)/2 = p$. Hence $s = q = 2p+1$ and $n = r^2$.

If $s = r$, then $(s-1)/2 \in \{p, q, pq\}$.

If $\alpha = 1$, then $(s-1)/2 | pqr^2$. If $s = 2pqr^2 + 1$, then in view of the fact $\varphi(k) = 2pqr^2$. By Lemma 2.2, we know that G is supersolvable, which leads to a contradiction. Hence, $s \in \{2p+1, 2q+1, 2r+1, 2pq+1, 2pr+1, 2qr+1, 2r^2+1, 2pqr+1, 2pr^2+1, 2qr^2+1\}$.

3. Proof of Theorem 1.1

In the following, we will use the notations in Lemma 2.9- 2.11. By assumption, G has n cyclic subgroups of order k. For any $1 \leq i \leq d$, we choose a complete system of representatives A_i of the conjugate classes of these subgroups and set $n_i = |G : N_G(A_i)|$. Suppose that G is non-solvable, then by Lemma 2.10 we have $d \geq 2$. In addition, by Lemma 2.9, one has that s is an odd prime. We will prove the Theorem 1.1 case by case depending on the values of α and s in Lemma 2.11.

Case 1. Let $\alpha = 3$ and $s = r \in \{2p+1, 2q+1\}$.

Subcase 1.1. If $s = r = 2p+1$, then by Lemma 2.11 we have $k = 2(2p+1)^3 = 2r^3$. Let $A_i =< a_i >$ and $O(a_i) = 2r^3$. By Lemma 2.3, we get $|G||2^\alpha pqr^\beta$, where $\alpha, \beta > 0$. By Lemma 2.1 and Lemma 2.10, we can get $n = \sum_{i=1}^{d} n_i = q$, and so $q \nmid n_i$ for any i and $1 \leq i \leq d$. By Lemma 2.10(1), we couclude that $|N_G(A_i) : C_G(A_i)||2pr^2$. Also, since $\pi(C_G(A_i)) = \{2, r\}$, thus G is a $\{2, p, r\}$-group. Therefore, one has that $3 \notin \pi(G)$, contradicting to Lemma 2.5.

Subcase 1.2. If $s = r = 2q+1$, then by Lemma 2.11 we have $k = 2(2q+1)^3 = 2r^3$. Suppose that $A_i =< a_i >$ and $O(a_i) = 2r^3$. By Lemma 2.3, we know that $|G||2^\alpha pqr^\beta$, where $\alpha, \beta > 0$. By Lemma 2.1 and Lemma 2.10(1), we get that $n = \sum_{i=1}^{d} n_i = p$, and hence $p \nmid n_i$ for any i and $1 \leq i \leq d$. By Lemma 2.10(2), we couclude that $|N_G(A_i) : C_G(A_i)||2qr^2$. Since $\pi(C_G(A_i)) = \{2, r\}$, then G is a $\{2, q, r\}$-group. By Lemma 2.5, we deduce that $3 \nmid |G|$, which leads to a contradiction.

Case 2. Let $\alpha = 2$, $s = q = 2p+1$ and $n = r^2$. Then by Lemma 2.11 we have $k = 2(2p+1)^2 = 2q^2$. By Lemma 2.3, we get $|G||2^\alpha pr^2q^\beta$, where $\alpha, \beta > 0$. Therefore, $\pi(G) \subseteq \{2, p, q, r\}$. Since $5 < p < q < r$, then by Lemma 2.5 and Lemma 2.7, we can conclude that there exists no such normal series of G: $H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3-groups or simple K_4-groups. Hence G is solvable, a contradiction.

Case 3. Let $\alpha = 2$ and $s = r \in \{2p+1, 2q+1\}$. Suppose that $s = r = 2p+1$. Let $A_i =< a_i >$ and $O(a_i) = 2r^2$. Since $k = 2r^2$, then by Lemma 2.3, we have that $\pi(G) \subseteq \{2, p, q, r\}$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)||2pr$. Also, we know $\pi(C_G(A_i)) = \{2, r\}$.

Therefore, G is a $\{2, p, r\}$-group. By Lemma 2.5, we conclude that G is solvable, a contradiction. If $q|n_i$ for each positive integer i and $1 \leq i \leq d$, then by Lemma 2.1 and Lemma 2.10(1) we have $n = \sum_{i=1}^{d} n_i = qr$. Consequently, G is a $\{2, p, q, r\}$-group. Similar to the previous case, a contradiction. Replacing q with p in the previous case, we arrive at a contradiction also in the case where $s = r = 2q + 1$.

Case 4. Let $\alpha = 2$ and $s = r = 2pq + 1$. Then by Lemma 2.11 we have $k = 2(2pq + 1)^2 = 2r^2$. Assume that $A_i =< a_i >$ and $O(a_i) = 2r^2$. By Lemma 2.3, we can get $|G||2^{\alpha}pqr^{\beta}$, where $\alpha, \beta > 0$. By hypothesis $n = r$, and by Lemma 2.10 we have $n = \sum_{i=1}^{d} n_i = r$. Hence, $r \nmid n_i$ for any i and $1 \leq i \leq d$. In addition, it follows from Lemma 2.10(2) that $|N_G(A_i) : C_G(A_i)||2pqr$. Therefore, $\pi(G) \subseteq \{2, p, q, r\}$. Since $5 < p < q < r$, then by Lemma 2.5 and Lemma 2.7, we can conclude that there exists no such normal series of G: $H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3-groups or simple K_4-groups. Hence G is solvable, a contradiction.

Case 5. Let $\alpha = 1$ and $s = r \in \{2pqr + 1, 2pr^2 + 1, 2qr^2 + 1\}$. Then by Lemma 2.11 we have $k \in \{2(2pqr + 1), 2(2pr^2 + 1), 2(2qr^2 + 1)\}$.

Subcase 5.1. If $s = 2pqr + 1$, then $k = 2s = 2(2pqr + 1)$. Assume that $A_i =< a_i >$ and $O(a_i) = 2s$. Then $|N_G(A_i) : C_G(A_i)||2pqr$, thus by Lemma 2.10(2) we know that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and so G is solvable by Lemma 2.8, a contradiction.

Subcase 5.2. Let $s \in \{2pr^2 + 1, 2qr^2 + 1\}$. If $s = 2pr^2 + 1$, since $(2pr^2 + 1)2pr^2 > 2pqr^2$, then by Lemma 2.4 we couclude that $G_{2pr^2+1} \trianglelefteq G$ and $|G_{2pr^2+1}| = 2pr^2 + 1$. Hence, $G_{2pr^2+1}(G)$ is cyclic. Since $(2pr^2 + 1)q > k$, we get $(2pr^2 + 1)q \notin \pi_e(G)$. Therefore, the action of the group G_q on G_{2pr^2+1} is the Frobenius action. This yields the divisibility of $q|2pr^2$, which leads to a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2qr^2 + 1$.

Case 6. Let $\alpha = 1$ and $s \in \{2pr+1, 2qr+1\}$. Then by Lemma 2.11 we have $k \in \{2(2pr + 1), 2(2qr + 1)\}$. If $s = 2pr + 1$, then by Lemma 2.3 one has that $\pi(G) \subseteq \{2, p, q, r, s\}$. Assume that $A_i =< a_i >$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)||2pr$. Therefore, G is a $\{2, p, r, s\}$-group. Since $5 < p < r < s$, then by Lemma 2.5 and Lemma 2.7, we deduce that G is solvable. Otherwise, by Lemma 2.10(2) we know that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and hence G is solvable by Lemma 2.8, a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2qr + 1$.

Case 7. Let $\alpha = 1$ and $s = 2pq + 1$. Then by Lemma 2.11 we have $k = 2(2pq + 1)$. Assume that $A_i =< a_i >$ and $O(a_i) = 2s$. By Lemma 2.3, we can get $|G||2^{\alpha}pqr^2s^{\beta}$, where $\alpha, \beta > 0$. Then we know that $\pi(G) \subseteq \{2, p, q, r, s\}$.

Since $5 < p < q < r$ and $5 < s$, then $3 \nmid |G|$ and $5 \nmid |G|$, and so G is solvable by Lemma 2.8, a contradiction.

Case 8. Let $\alpha = 1$ and $s = 2r^2 + 1$. Then by Lemma 2.11 we have $k = 2s$. By Lemma 2.3, we have $\pi(G) \subseteq \{2, p, q, r, s\}$. Let $A_i =< a_i >$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $p \nmid n_i$, then $|N_G(A_i) : C_G(A_i)||2r^2$. Also, we know $\pi(C_G(A_i)) = \{2, s\}$. Then, $\pi(G) \subseteq \{2, q, r, s\}$. Since $5 < q < r < s$, then by Lemma 2.5 and Lemma 2.7, we infer that G does not admit any simple K_3-group or simple K_4-group as its quotient group, which implies G is solvable, a contradiction. If $p|n_i$ for any i and $1 \leq i \leq d$, then $n = \sum_{i=1}^{d} n_i = pq$ by Lemma 2.1 and Lemma 2.10(1). Therefore, there exists a positive integer $1 \leq i \leq d$ such that $q \nmid n_i$. Then G is a $\{2, p, r, s\}$-group. Reasoning as above, we can get a contradiction.

Case 9. Let $\alpha = 1$ and $s \in \{2p + 1, 2q + 1\}$. Then by Lemma 2.11 we have $k = 2s$. If $s = 2p + 1$, assume that $A_i =< a_i >$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $q \nmid n_i$, then $|N_G(A_i) : C_G(A_i)||2p$. On the other hand, by Lemma 2.10(2) we have $\pi(C_G(A_i)) = \{2, s\}$. Therefore, G is a $\{2, p, r, s\}$-group. Since $5 < p < r$ and $5 < s$, thus, by Lemma 2.5 and Lemma 2.7, the group G does not admit any simple K_3-group or simple K_4-group as its quotient group. Hence G is solvable, a contradiction. If $q|n_i$ for any positive integer i and $1 \leq i \leq d$, then by Lemma 2.1 and Lemma 2.10(1) we have $n = \sum_{i=1}^{d} n_i = qr^2$. Therefore, we can get that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r$ and $5 < s$, similar to the previous case, we can get a contradiction. If we again replacing q with p in the presented above, then we arrive at a contradiction also in the case where $s = 2q + 1$.

Case 10. Let $\alpha = 1$ and $s = 2r + 1$. Then by Lemma 2.11 we have $k = 2s$. Let $A_i =< a_i >$ and $O(a_i) = 2s$. If there exists a positive integer i and $1 \leq i \leq d$ such that $p \nmid n_i$, then $|N_G(A_i) : C_G(A_i)||2r$. Then, we have $\pi(G) \subseteq \{2, q, r, s\}$. Since $5 < q < r < s$, then by Lemma 2.5 and Lemma 2.7, one has that G does not admit any simple K_3-group or simple K_4-group as its quotient group. Hence G is solvable, a contradiction. If $p|n_i$ for any i and $1 \leq i \leq d$, then $n = \sum_{i=1}^{d} n_i = pqr$ by Lemma 2.1 and Lemma 2.10(1). Suppose that there exists a positive integer j and $1 \leq j \leq d$ such that $q \nmid n_j$, Therefore, $\pi(G) \subseteq \{2, p, r, s\}$, again a contradiction. Otherwise, we get that $\pi(G) \subseteq \{2, p, q, r, s\}$. Since $5 < p < q < r < s$, then we have $3 \nmid |G|$ and $5 \nmid |G|$. Thus G is solvable by Lemma 2.8, which contradicts our assumption.

This completes the proof of Theorem 1.1.

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References

[1] B. Asadian, N. Ahanjideh, *Structure of the finite groups with* 4*p* *elements of maximal order,* Quasigroups and Related System, 24 (2016), 157-168.

[2] R. Brandle, W.J. Shi, *Finite groups whose elements order are consecutive integers*, J. Algebra, 143 (1991), 388-400.

[3] G.Y. Chen, W.J. Shi, *Finite groups with* 30 *elements of maximal order*, Appl. Categor. Struct., 16 (2008), 239-247.

[4] J.H. Conway, R.T. Curtis, and S.P. Norton, et al., *Atlas of finite groups,* Clarendon Press, Oxford, 1985.

[5] Z.M. Chen, *Inside and outside-*Σ *group and minimal non-*Σ *group*, Southwest Normal University Press, 1988.

[6] D. Gorenstein, *Finite groups*, Harper and Row Press, New York, 1968.

[7] M. Herzog, *On finite simple groups of order divisible by there primes only*, J. Algebra, 120 (1968), 383-388.

[8] Y.Y. Jiang, *A theorem of finite groups with* 18*p* *having maximal order*, Algebra Colloquium, 15 (2008), 317-329.

[9] V.D. Mazurov and E.I. Khukhro, *Unsolved problems in group theorem*, Russian Academy of Sciences, Institute of Mathematics, Novosibirsk, 17 (2010).

[10] W.J. Shi, *On simple* K_4*-groups*, Chinese Science Bull., 36 (1991), 1281-1283.

[11] Y. Xu, J.J. Gao, H.L. Hou, *Finite groups with* 6*pq* *elements of the largest order*, Italian Journal of Pure and Applied Mathematics, 31 (2013), 277-284.

[12] C. Yang, *Finite groups based on the numbers of elements of maximal order*, Ann. Math. (China), 14A(5) (1993), 561-567.

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