# A note on Einstein-like $\epsilon$-$LP$-Sasakian manifolds

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**Abstract.** The present paper deals with the study on Einstein-like $\epsilon$-Lorentzian para-Sasakian manifolds. A necessary and sufficient condition for an $\epsilon$-Lorentzian para-Sasakian manifold to be Einstein-like is obtained in terms of its curvature tensor. We also obtain the scalar curvature of an Einstein-like $\epsilon$-Lorentzian para-Sasakian manifold. A necessary and sufficient condition for an $\epsilon$-Lorentzian almost para-contact metric hypersurface of an indefinite locally Riemannian product manifold to be $\epsilon$-Lorentzian para-Sasakian is derived as well as it is shown that the $\epsilon$-Lorentzian para-Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein-like.
**Keywords:** $\epsilon-LP$-Sasakian manifold, Einstein-like manifold, $\epsilon-LP$-Sasakian hypersurfaces.

## 1. Introduction

Semi-Riemannian geometry has great importance in the field of general relativity [8] and different areas of physics. I. Sato [11] introduced a structure $(\phi, \xi, \eta)$ satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold in 1976, which is now renowned as an almost para-contact structure. This structure is an analogue of the almost contact structure [5, 10] as well as closely related to the almost product structure. Almost contact manifold is always odd dimen-

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sional but almost para-contact manifold could be even-dimensional as well. T. Adati and K. Motsumoto [1] defined and studied $p$-Sasakian manifold and special $p$-Sasakian manifolds in 1977, which are considered as a special kind of an almost contact Riemannian manifold. Further in 1989, Matsumoto [7] used the structure vector field $-\xi$ instead of $\xi$ in an almost para-contact manifold and associated a Lorentzian metric with the resulting structure, called it a Lorentzian almost para-contact manifold. Such structures are also studied by several authors ([2], [4], [13]). The concept of $\epsilon$-Sasakian manifold is introduced by A. Bejancu and K. L. Duggal in 1993 [3]. M. M. Tripathi, Kilic, Perktas and Keles have studied $\epsilon$- almost para-contact manifolds and in particular, $\epsilon$-para-Sasakian manifolds. U. C. De and A. Sarkar [6] introduced $\epsilon$- Kenmotsu manifold in 2009 and studied some curvature properties on the manifold. R. Prasad and V. Srivastava [9] have studied $\epsilon$-Lorentzian para-Sasakian manifold and also shown its existence by an example in 2012. In 2017, Haseeb, Prakash and Siddiqi [14] have studied quarter-symmetric metric connection in an $\epsilon$-Lorentzian para-Sasakian manifolds.

R. Sharma [12] introduced and studied Einstein-like para-Sasakian manifolds in 1982. Motivated by his study, in this paper we introduce and study Einstein-like $\epsilon - LP$-Sasakian manifold. The paper is organized as follows: In section 2, we give some preliminaries about $\epsilon - LP$-Sasakian manifolds. Section 3 contains definition of an Einstein-like para-Sasakian manifold and some basic properties. We also find some results on scalar curvature and a necessary and sufficient condition for $\epsilon - LP$-Sasakian manifolds to be Einstein-like in terms of its curvature tensor is obtain. In section 4, we obtain a necessary and sufficient condition for $\epsilon$-Lorentzian almost para-contact metric hypersurface of an indefinite locally Riemannian product manifold to be $\epsilon - LP$-Sasakian manifold. Finally, we prove that $\epsilon - LP$-Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature is always Einstein-like.

## 2. Preliminaries

A structure on a differential manifold $M$ of dimension $n$ is called an $\epsilon$-Lorentzian almost para-contact metric structure if it admits a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, a 1-form $\eta$ and semi-Riemannian metric $g$ which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{1}$$

$$g(X,\xi) = \epsilon\eta(X) \Rightarrow g(\xi,\xi) = -\epsilon, \tag{2}$$

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{4}$$

for all vector fields $X, Y \in \chi(M)$, where $\epsilon$ is 1 or $-1$ according to the vector field $\xi$ being timelike or spacelike.

If an $\epsilon$-Lorentzian almost para-contact metric structure satisfies

$$(\nabla_X \phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{5}$$

where $\nabla$ denotes the Levi-Civita connection with respect to $g$, then $M$ is called an $\epsilon - LP$-Sasakian manifold.

In an $\epsilon - LP$-Sasakian manifold, we have [9]

$$\nabla_X \xi = \epsilon \phi X, \tag{6}$$

$$(\nabla_X \eta)Y = g(\phi X, Y) = \Phi(X, Y). \tag{7}$$

Moreover, the curvature tensor $R$ and Ricci tensor $S$ satisfy the following equations:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$

$$S(X,\xi) = (n-1)\eta(X). \tag{9}$$

**Example 2.1.** Let a 3-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where $(x, y, z)$ are standard coordinates of $R^3$. Let $e_1, e_2$ and $e_3$ be vector fields on $M_3$ given by

$$e_1 = \frac{1}{2}e^{2z}\frac{\partial}{\partial y}, \quad e_2 = \frac{1}{2}e^{2z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \text{ and } e_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent vector fields at each point of $M_3$ and form a basis of $T_pM_3$ at each point $p$ of $M_3$. Define a semi-Riemannian metric $g$ on $M_3$ as $g(e_i, e_i) = 1$, for $1 \leq i \leq 2$, $g(e_3, e_3) = -\epsilon$ and $g(e_i, e_j) = 0$, for $i \neq j$ and $1 \leq i$, $j \leq 3$.

Let $\eta$ be a 1-form on $M_3$ defined as $\epsilon\eta(U) = g(U, e_3) = g(U, \xi)$, for all $U \in \chi(M_3)$ and let $\phi$ be a $(1, 1)$ tensor field on $M_3$ defined as

$$\phi(e_1) = -\epsilon e_1, \quad \phi(e_2) = -\epsilon e_2, \quad \phi(e_3) = 0.$$

By applying linearity of $\phi$ and $g$, we have

$$\eta(\xi) = -1, \quad \phi^2(X) = X + \eta(X)\xi,$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y) \quad \text{for all} \quad X, Y \in \chi(M_3).$$

Let $\nabla$ be a Levi-Civita connection with respect to the semi-Riemannian metric $g$. Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1,$$

The Riemannian connection $\nabla$ of the metric $g$ is given by

$$\begin{aligned} 2g(\nabla_U V, W) &= Ug(V, W) + Vg(W, U) - Wg(U, V) \\ &\quad -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]), \end{aligned}$$

which is known as Koszul's formula, we can easily calculate

$$\nabla_{e_1}e_1 = -\epsilon e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -\epsilon e_3, \quad \nabla_{e_2}e_3 = -e_2,$$
$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.$$

From the above it follows that the manifold satisfies $\nabla_X\xi = \epsilon\phi X$, for $\xi = e_3$ and $(\nabla_X\phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi$. Hence the manifold is an $\epsilon - LP$-Sasakian manifold.

## 3. Einstein-like $\epsilon-LP$-Sasakian manifolds

We begin with the following definition analogous to Einstein-like para-Sasakian manifolds (Sharma, 1982).

**Definition 3.1.** *An $\epsilon$-Lorentzian almost para contact metric manifold is said to be Einstein-like if its Ricci tensor $S$ satisfies*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + cg(\phi X,Y), \tag{10}$$

*for some real constants $a, b$ and $c$.*

**Proposition 3.1.** *In an Einstein-like $\epsilon$-Lorentzian almost para contact metric manifold, we have*

$$S(\phi X,Y) = ag(\phi X,Y) + cg(\phi X,\phi Y), \tag{11}$$

$$S(X,\xi) = \epsilon a\eta(X) - b\eta(X). \tag{12}$$

Moreover, if the manifold is $\epsilon - LP$-Sasakian manifold, the following conditions also hold

$$\epsilon a - b = n - 1, \tag{13}$$

$$r = na - \epsilon b + c(\text{trace}(\phi)). \tag{14}$$

**Proof.** The equations (11) and (12) are obvious from (10) .

In an $\epsilon - LP$-Sasakian manifold, it follows that $S(X,\xi) = (n-1)\eta(X)$, which in view of (12) implies (13).

Now, let $\{e_1, e_2, ......., e_{n-1}, \xi\}$ be a local orthonormal basis. Then from (10), we have

$$r = \sum_{i=1}^{n}\{a\epsilon_i g(e_i,e_i) + b\epsilon_i\eta(e_i)\eta(e_i) + c\epsilon_i g(\phi e_i,e_i)\},$$

where $\epsilon_i = g(e_i,e_i)$, which gives (14). □

**Theorem 3.1.** *For an Einstein-like $\epsilon - LP$-Sasakian manifold, the scalar curvature $r$ satisfies the following differential equation*

$$c\xi r + 2br = -2\epsilon(n-1)[c^2 - b(n+b)].$$

**Proof.** From (10), it follows that the Ricci operator $Q$ satisfies

$$g(QX,Y) = ag(X,Y) + cg(\phi X,Y) + b\epsilon\eta(X)g(\xi,Y),$$

this implies

$$QX = aX + c\phi X + \epsilon b\eta(X)\xi.$$

Differentiating, we find

$$\begin{aligned}(\nabla_Y Q)X + Q(\nabla_Y X) &= a\nabla_Y X + c(\nabla_Y\phi)X + c\phi(\nabla_Y X)\\ &\quad +\epsilon b(\nabla_Y\eta)(X)\xi + \epsilon b\eta(\nabla_Y X)\xi + \epsilon b\eta(X)\nabla_Y\xi.\end{aligned}$$

Using $(5), (6)$ and $(7)$ in the above equation, we get

$$\begin{aligned}(\nabla_Y Q)X &= c[g(X,Y)\xi + \epsilon\eta(X)Y + 2\epsilon\eta(X)\eta(Y)\xi]\\ &\quad +b[\epsilon g(\phi X,Y)\xi + \eta(X)\phi Y].\end{aligned}$$

Now, let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis. Multiplying both side by $\epsilon_i$ in the above equation and contracting with respect to $Y$, we have

$$(\text{div}Q)X = \{b(\text{trace}(\phi)) + \epsilon c(n-1)\}\eta(X). \tag{15}$$

From (13) and (14), we get

$$r = c(\text{trace}(\phi)) + \epsilon(n-1)(n+b). \tag{16}$$

Using $Xr = 2(\text{div}Q)X$ and (16) in (15), we obtain that in an Einstein-like $\epsilon -$ $LP$-Sasakian manifold, the scalar curvature $r$ satisfies the following differential equation

$$c\xi r + 2br = -2\epsilon(n-1)[c^2 - b(n+b)]. \tag{17}$$

□

**Theorem 3.2.** *In an Einstein-like $\epsilon - LP$-Sasakian manifold, if trace($\phi$) is constant then*

$$b(\textit{trace}(\phi)) = -\epsilon(n-1)c. \tag{18}$$

**Proof.** Using $Xr = 2(\text{div}Q)X$ in (15), we get

$$dr = 2\{b(\text{trace}(\phi)) + \epsilon c(n-1)\}\eta. \tag{19}$$

Since trace($\phi$) is constant, then from (14), it follows that $r$ is constant. Hence (19) gives (18). □

From now, on in this section the trace($\phi$) will be assumed to be constant.

**Theorem 3.3.** *An $\epsilon-LP$-Sasakian manifold with constant trace($\phi$) is Einstein-like if and only if the $(0,2)$-tensor field $C_1^1(\phi R)$ is a linear combination of $g$, $\Phi$ and $\eta\otimes\eta$ formed with constant coefficients.*

**Proof.** Let $R$ be the curvature tensor in an $\epsilon-LP$-Sasakian manifold, we have

$$R(X,Y)\phi Z=\nabla_X\nabla_Y\phi Z-\nabla_Y\nabla_X\phi Z-\nabla_{[X,Y]}\phi Z.$$

Using equations (5), (6) and (7) in the above equation, we have

$$\begin{aligned}R(X,Y)\phi Z=\ &\phi R(X,Y)Z-\epsilon[\Phi(Y,Z)X-\Phi(X,Z)Y]\\&+\epsilon[g(Y,Z)\phi X-g(X,Z)\phi Y]\\&+2\epsilon[\eta(Y)\Phi(X,Z)\xi-\eta(X)\Phi(Y,Z)\xi]\\&+2[\eta(Y)\eta(Z)\phi X-\eta(X)\eta(Z)\phi Y].\end{aligned}\tag{20}$$

From (20), we have

$$\begin{aligned}&{}'R(X,Y,\phi Z,W)-{}'R(X,Y,Z,\phi W)\\&=\epsilon[\Phi(X,W)g(Y,Z)-\Phi(Y,W)g(X,Z)]\\&+\epsilon[\Phi(X,Z)g(Y,W)-g(X,W)\Phi(Y,Z)]\\&+2\eta(Z)[\eta(Y)\Phi(X,W)-\eta(X)\Phi(Y,W)]\\&+2\eta(W)[\eta(Y)\Phi(X,Z)-\eta(X)\Phi(Y,Z)].\end{aligned}\tag{21}$$

where ${}'R(X,Y,Z,W)=g(R(X,Y)Z,W)$.

From (21), we have

$$\begin{aligned}&{}'R(X,Y,\phi Z,\phi W)-{}'R(X,Y,Z,W)\\&=\epsilon[g(\phi X,\phi W)g(\phi Y,\phi Z)-g(\phi X,\phi Z)g(\phi Y,\phi W)]\\&+\epsilon[\Phi(X,Z)\Phi(Y,W)-\Phi(X,W)\Phi(Y,Z)]\\&+\eta(Z)[\eta(Y)g(X,W)-\eta(X)g(Y,W)]\\&+\eta(W)[\eta(X)g(Y,Z)-\eta(Y)g(X,Z)].\end{aligned}\tag{22}$$

From (22), we have

$${}'R(X,Y,\phi Z,\phi W)={}'R(\phi X,\phi Y,Z,W).\tag{23}$$

Using (22) and (23), we have

$$\begin{aligned}{}'R(\phi X,\phi Y,\phi Z,\phi W)=\ &{}'R(X,Y,Z,W)\\&+\eta(Z)[\eta(Y)g(X,W)-\eta(X)g(Y,W)]\\&+\eta(W)[\eta(X)g(Y,Z)-\eta(Y)g(X,Z)].\end{aligned}\tag{24}$$

From (24) , we have

$$S(\phi Y,\phi Z)=S(Y,Z)+(n-1)\eta(Y)\eta(Z)\qquad\text{and}\tag{25}$$

$$S(Y,\xi)=(n-1)\eta(Y). \tag{26}$$

Taking $Z=\phi Z$, we have

$$S(\phi Y,Z)=S(Y,\phi Z). \tag{27}$$

From (20), we have

$$\begin{aligned} S(Y,\phi Z) &= C_1^1(\phi R)(Y,Z)-\epsilon(n-2)\Phi(Y,Z)\\ &\quad+[\epsilon g(Y,Z)+2\eta(Y)\eta(Z)]\mathrm{trace}(\phi). \end{aligned} \tag{28}$$

Now, from (27),we have in $\epsilon-LP$-Sasakian manifold $S(X,\phi Y)=S(\phi X,Y)$ and also it can be verified that $C_1^1(\phi R)(Y,Z)=C_1^1(\phi R)(Z,Y)$; therefore from this observation relation (28) is consistent.

Now, if the manifold $M$ is Einstein-like then from (11) and (18), the equation (28) reduces to

$$bC_1^1(\phi R)=c(b+n-1)g+\epsilon c[b+2(n-1)]\eta\otimes\eta+(a+\epsilon(n-2))\Phi, \tag{29}$$

which shows that $C_1^1(\phi R)$ is a linear combination of $g,\Phi$ and $\eta\otimes\eta$ formed with constant coefficients. The converse is easy to follow. □

**Corollary 3.1.** *In an Einstein-like $\epsilon-LP$-Sasakian manifold with constant trace($\phi$), the $(0,2)$- tensor field $C_1^1(\phi R)$ is parallel along the vector field $\xi$.*

**Proof.** From (5) and (7) , we have$\nabla_\xi\eta=0$, and $\nabla_\xi\phi=0$.

Now, in an $\epsilon-LP$-Sasakian manifold

$$\begin{aligned} (\nabla_\xi\Phi)(X,Y) &= \nabla_\xi\Phi(X,Y)-\Phi(\nabla_\xi X,Y)-\Phi(X,\nabla_\xi Y)\\ &= g(\nabla_\xi\phi X-\phi\nabla_\xi X,Y)\\ &= g((\nabla_\xi\phi)X,Y) \end{aligned}$$

this implies $\nabla_\xi\Phi=0$ for $\epsilon-LP$-Sasakian manifold.

Since in an $\epsilon-LP$-Sasakian manifold $\nabla_\xi\Phi=0$ and $\nabla_\xi\eta=0$, therefore from (29), we conclude that $C_1^1(\phi R)$ is parallel along the vector field $\xi$. □

**Theorem 3.4.** *In an Einstein-like $\epsilon-LP$-Sasakian manifold, we have*

$$L_\xi S=2a\epsilon\Phi+2c(\epsilon g+\eta\otimes\eta). \tag{30}$$

**Proof.** In a $\epsilon-LP$-Sasakian manifold, we obtain

$$\begin{aligned} (L_\xi\eta)X &= \xi\eta(X)-\eta[\xi,X]\\ &= \xi\eta(X)-\eta(\nabla_\xi X-\nabla_X\xi)\\ &= (\nabla_\xi\eta)X+\eta(\nabla_X\xi)\\ &= 0 \end{aligned} \tag{31}$$

and

$$(L_\xi g)(Y,Z) = \xi g(Y,Z) - g([\xi,Y],Z) - g(Y,[\xi,Z]) = 2g(\epsilon\phi Y,Z) = 2\epsilon\Phi(Y,Z). \tag{32}$$

$$(L_\xi\Phi)(Y,Z) = L_\xi g(\phi Y,Z) - \Phi(L_\xi Y,Z) - \Phi(Y,L_\xi Z) = (L_\xi g)(\phi Y,Z) + g((L_\xi\phi)Y,Z) = 2\epsilon\Phi(\phi Y,Z) = 2\epsilon g(Y,Z) + 2\eta(Y)\eta(Z) \tag{33}$$

Thus, we have $L_\xi\eta = 0$, $L_\xi g = 2\epsilon\Phi$, $L_\xi\phi = 0$ and $L_\xi\Phi = 2(\epsilon g + \eta\otimes\eta)$.

Now, taking Lie derivative of $S$ in the direction of $\xi$ in (10) and using above results, we obtain (30). □

**Theorem 3.5.** *In an Einstein-like $\epsilon - LP$-Sasakian manifold with constant trace$(\phi)$, we have*

$$L_\xi(C_1^1(\phi R)) = \frac{2\epsilon c}{b}(b+n-1)\Phi + 2\epsilon(a+\epsilon(n-2))(g+\epsilon\eta\otimes\eta) \tag{34}$$

**Proof.** Taking Lie derivative of $C_1^1(\phi R)$ in the direction of $\xi$ in (29) and using $(31)-(33)$, we obtain (34). Hence the proof. □

## 4. $\epsilon - LP$-Sasakian hypersurfaces

Let $\widetilde{M}_{n+1}$ be a real $(n+1)$ dimensional manifold. Suppose $\widetilde{M}_{n+1}$ is endowed with an almost product structure $J$ and a semi-Riemannian metric $\widetilde{g}$ satisfying

$$\widetilde{g}(J\widetilde{X},J\widetilde{Y}) = \widetilde{g}(\widetilde{X},\widetilde{Y}), \tag{35}$$

for all vector fields $\widetilde{X},\widetilde{Y}$ in $\widetilde{M}_{n+1}$. Then we say that $\widetilde{M}_{n+1}$ is an indefinite almost product Riemannian manifold. Moreover, if on $\widetilde{M}_{n+1}$, we have

$$(\widetilde{\nabla}_{\widetilde{X}}J)\widetilde{Y} = 0, \tag{36}$$

for all $\widetilde{X},\widetilde{Y} \in \Gamma(T\widetilde{M}_{n+1})$, where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to $\widetilde{g}$, we say that $\widetilde{M}_{n+1}$ is an indefinite locally Riemannian product manifold.

Now, let $M_n$ be an orientable non-degenerate hypersurface of $\widetilde{M}_{n+1}$. Suppose that $N$ is the unit normal vector field and $\xi$ is a vector field on $M_n$ such that

$$\widetilde{g}(N,N) = -\epsilon, \tag{37}$$

$$JN = -\xi. \tag{38}$$

Let

$$JX = \phi X + \eta(X)N, \qquad \text{where } X\in\chi(M_n). \tag{39}$$

**Proposition 4.1.** *The set* $(\phi, \xi, \eta, g)$ *is an* $\epsilon$*-Lorentzian almost para-contact metric structure, where* $g$ *is induced metric on* $M_n$*.*

**Proof.** We have, from (38) and (39), $X = J^2X = \phi^2X + \eta(\phi X)N - \eta(X)\xi$.

Equating tangential and normal parts, we get $\phi^2X = X + \eta(X)\xi$ and $\eta(\phi X) = 0$, respectively.

Also, from (38) and (39), we have $N = J^2N = -J\xi = -\phi\xi - \eta(\xi)N$,

Equating tangential and normal parts, we get $\phi\xi = 0$ and $\eta(\xi) = -1$, respectively. Finally, we have $g(X, Y) = \widetilde{g}(JX, JY)$, where $X, Y \in M_n$, which in view of (39) gives (1). Hence the set $(\phi, \xi, \eta, g)$ is an $\epsilon$-Lorentzian almost para-contact metric structure on $M_n$. □

The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \epsilon g(AX, Y)N, \tag{40}$$

$$\widetilde{\nabla}_X N = -AX, \tag{41}$$

where $\nabla$ is the Levi-Civita connection with respect to the Riemannian metric $g$ induced by $\widetilde{g}$ on $M_n$ and $A$ is the shape operator of $M_n$.

**Proposition 4.2.** *The* $\epsilon$*-Lorentzian almost para-contact metric structure on* $M_n$ *satisfies*

$$(\nabla_X \phi)Y = \eta(Y)AX + \epsilon g(AX, Y)\xi, \tag{42}$$

$$(\nabla_X \eta)Y = \epsilon g(AX, \phi Y), \tag{43}$$

$$\nabla_X \xi = \phi AX. \tag{44}$$

**Proof.** Using the equations $(38), (39), (40)$ and $(41)$ in $(\widetilde{\nabla}_X J)Y = 0$, we get $(\nabla_X \phi)Y - \eta(Y)AX - \epsilon g(AX, Y)\xi + ((\nabla_X \eta)Y)N - \epsilon g(AX, \phi Y)N = 0$.

Equating tangential and normal parts, we get (42) and (43), respectively. Equation (42) and (43) implies (44). □

Now, we obtain the following theorems of characterization for $\epsilon{-}LP$-Sasakian hypersurfaces.

**Theorem 4.1.** *Let* $M_n$ *be an orientable hypersurfaces of an indefinite locally Riemannian product manifold. Then* $M_n$ *is an* $\epsilon - LP$*-Sasakian manifold if and only if the shape operator is given by*

$$A = \epsilon I + \epsilon\eta \otimes \xi. \tag{45}$$

**Proof.** Let $M_n$ be a $\epsilon - LP$-Sasakian manifold. Using (6) and (44) , we get

$$AX = \epsilon X + \epsilon\eta(X)\xi - \eta(AX)\xi. \tag{46}$$

In particular, we have $A\xi = -\eta(A\xi)\xi$. Thus, we have

$$\eta(AX) = \epsilon g(\xi, AX) = \epsilon g(A\xi, X) = \epsilon g(-\eta(A\xi)\xi, X) = -\eta(A\xi)\eta(X). \tag{47}$$

Using this in (46), we get

$$A = \epsilon I + (\epsilon + \eta(A\xi))\eta \otimes \xi. \tag{48}$$

Using (42) in (48), we have

$$(\nabla_X \phi)Y = \epsilon\eta(Y)\phi X + 2\epsilon\eta(X)\eta(Y)\xi + 2\eta(A\xi)\eta(X)\eta(Y)\xi + g(X,Y)\xi.$$

Using equation (5), we have $\eta(A\xi)\eta(X)\eta(Y)\xi = 0$. From above, we have $\eta(A\xi) = 0$, which when used in (48) yields (45). Conversely, using (45) in (42) we see that $M_n$ is $\epsilon-LP$-Sasakian manifold. □

Now, assume that the indefinite almost product Riemannian manifold $\widetilde{M}_{n+1}$ is of almost constant curvature so that its curvature tensor $\widetilde{R}$ is given (Yano, 1965) by

$$\begin{aligned} {}'\widetilde{R}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) &= k\{\widetilde{g}(\widetilde{Y},\widetilde{Z})\widetilde{g}(\widetilde{X},\widetilde{W}) - \widetilde{g}(\widetilde{X},\widetilde{W})\widetilde{g}(\widetilde{Y},\widetilde{Z}) \\ &\quad + \widetilde{g}(J\widetilde{Y},\widetilde{Z})\widetilde{g}(J\widetilde{X},\widetilde{W}) - \widetilde{g}(J\widetilde{X},\widetilde{Z})\widetilde{g}(J\widetilde{Y},\widetilde{W})\}) \end{aligned} \tag{49}$$

for all vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z}$ and $\widetilde{W}$ on $\widetilde{M}_{n+1}$. If $M_n$ be a $\epsilon - LP$-Sasakian hypersurface, then in view of (45), (49) and the Gauss equation, we have

$$\begin{aligned} {}'R(X,Y,Z,W) &= (k-\epsilon)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + (k)\{g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)\} \\ &\quad - g(Y,Z)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(Z) \\ &\quad + g(X,Z)\eta(Y)\eta(W) + g(Y,W)\eta(X)\eta(Z). \end{aligned} \tag{50}$$

After calculating from (50), we have

$$R(X,Y)\xi = [\epsilon(k-\epsilon)+1]\{\eta(Y)X - \eta(X)Y\}. \tag{51}$$

Comparing the resulting expression with (8), we find that $k = \epsilon$. With this value of $k$, from (50), we obtain $S(Y,Z) = c(\text{trace}(\phi))\Phi(Y,Z) - (n-1)\eta(Y)\eta(Z)$. Thus, we have

**Theorem 4.2.** *An $\epsilon - LP$-Sasakian hypersurface of an indefinite locally Riemannian product manifold of almost constant curvature ($k = \epsilon$) is always Einstein-like.*

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