Bach tensor on $N(\kappa)$-paracontact metric 3-manifolds

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Abstract. In this paper, we characterize the Bach tensor on $N(\kappa)$-paracontact metric 3-manifold. It is proved that a $N(\kappa)$-paracontact metric 3-manifold with purely transversal Bach tensor is of constant scalar curvature 6κ.

Keywords: $N(\kappa)$-paracontact metric 3-manifold, Cotton tensor, Bach tensor, Scalar curvature.

1. Introduction

The notion of Bach tensor was introduced by Rudolf Bach in [1] when studying so-called conformal relativity. That is, instead of using the Hilbert-Einstein functional, one considers the functional

$$\mathcal{W}(g) = \int_{M^4} |W(g)|^2 dv_g, \tag{1}$$

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for a 4-dimensional compact Riemannian manifolds M, where W denotes the Weyl tensor of type $(1,3)$ defined by

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\} \\ &+ \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2}$$

where R denotes the Riemannian curvature tensor, S is a Ricci tensor and Ricci operator Q is defined by $g(QX,Y) = S(X,Y)$. The Critical points of the functional (1) are characterized by the vanishing of a symmetric trace free $(0,2)$ type tensor $\mathcal{B}$ is usually referred as Bach tensor and the metric is called Bach flat if $\mathcal{B}$ vanishes. On any Riemannian manifold (M,g) of dimension $(2n+1)$, the Bach tensor $\mathcal{B}$ is defined by

$$\begin{aligned} \mathcal{B}(X,Y) &= \frac{1}{2n-2}\sum_{i,j=1}^{2n+1}((\nabla_{e_i}\nabla_{e_j}W)(X,e_i,e_j,Y) \\ &+ \frac{1}{2n-1}\sum_{i,j=1}^{2n+1} S(e_i,e_j)W(X,e_i,e_j,Y), \end{aligned} \tag{3}$$

where $\{e_i\}_{i=1}^{2n+1}$ is a local orthonormal frame on (M,g). A Riemannian manifold M is said to be Einstein if the Ricci tensor S is a constant multiple of the metric tensor g. Now, from (2) and the contraction of Bianchi second identity it follows that $divW = \frac{2n-2}{2n-1}\mathcal{C}$, where $\mathcal{C}$ is the $(0,3)$-type Cotton tensor tensor defined by[2]

$$\begin{aligned} \mathcal{C}(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &- \frac{1}{4n}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}. \end{aligned} \tag{4}$$

Now, making use of Weyl tensor (2) and Cotton tensor (4), the Bach tensor (3) can be expressed as [3]

$$\mathcal{B}(X,Y) = \frac{1}{2n-1}\left[\sum_{i=1}^{2n+1}(\nabla_{e_i}C)(e_i,X)Y + \sum_{i,j=1}^{2n+1} S(e_i,e_j)W(X,e_i,e_j,Y)\right]. \tag{5}$$

In dimension 3, the Weyl tensor W vanishes, and hence the expression of Bach tensor transforms into

$$\mathcal{B}(X,Y) = \sum_{i=1}^{3}(\nabla_{e_i}\mathcal{C})(e_i,X)Y. \tag{6}$$

A Riemannian metric g is called Bach flat if the Bach tensor $\mathcal{B}$ of g vanishes. By (3), it is easy to see that Bach flatness is natural generalization of Einstein and conformal flatness. For more details about Bach tensor, we refer to reader [4, 5, 7, 8] and references therein.

On the other hand, study of nullity distribution on paracontact geometry is one among the most interesting topics in modern paracontact geometry. In 1985, Kaneyuki and Kozai [10] initiated the study of paracontact geometry. The importance of paracontact geometry interplays with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. A systematics study of paracontact metric manifolds was carried out by Zamkovoy [19]. Further, the study was taken up by many authors, for readers we refer the papers [9, 11, 12, 13, 14, 15, 16, 17, 18, 20] and others.

Recently, Ghosh and Sharma in [6] studied Sasakian manifold with purely transversal Bach tensor. In particular, they proved that if a Sasakian manifold admits a purely transversal Bach tensor, then g has constant scalar curvature $\geq 2n(2n+1)$, with equality holds if and only if g is Einstein, and the Ricci tensor of g has a constant norm. Also, they studied $(\kappa,\mu)-$contact manifolds with vanishing Bach tensor in [7]. This works turns our attention to study Bach tensor in the framework of certain class of paracontact metric manifolds, particularly, on 3-dimensional $N(\kappa)-$paracontact metric manifolds (briefy, $N(\kappa)-$paracontact metric 3-manifolds).

The paper is organized as follows: Section 2 is concerned with the basic formulas and properties of $N(\kappa)-$paracontact metric manifolds. Section 2 is concerned with the study of $N(\kappa)-$paracontact metric 3-manifolds with purely transversal Bach tensor. It is proved that a $N(\kappa)$-paracontact metric 3-manifold with purely transversal Bach tensor is of constant scalar curvature.

2. Preliminaries

An almost paracontact structure on a (2n+1)-dimensional smooth manifold M^{2n+1} is a triplet (ϕ,ξ,η), where φ is a $(1,1)$-type tensor field , ξ is a vector field called the Reeb vector field and η is a 1-form such that:

(i) $\varphi^2 X = X - \eta(X)\xi$,

(ii) $\varphi(\xi) = 0, \eta\cdot\varphi = 0, \eta(\xi) = 1$,

(iii) the tensor field φ induces an almost paracomplex structure on each fibre of $\mathcal{D} = ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\varphi$ and $\mathcal{D}^-_\varphi$ of φ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\varphi X,\varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{7}$$

for all $X,Y \in \chi(M)$, is called almost paracontact metric manifold and (φ,ξ,η,g) is said to be an almost paracontact metric structure. An almost paracontact structure is normal [30] if and only if the $(1,2)$-type torsion tensor $N_\varphi = [\varphi,\varphi] - 2d\eta\otimes\xi = 0$, where $[\varphi,\varphi](X,Y) = \varphi^2[X,Y][\varphi,\varphi]$. An almost paracontact structure is called a paracontact structure if $g(X,\varphi Y) = d\eta(X,Y)$

[20]. Any almost paracontact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ admits a φ basis [20], that is, a pseudo-orthonormal basis of vector fields of the form $\xi, E_1, E_2, ..., E_n, \varphi E_1, \varphi E_2, ..., \varphi E_n$, where $\xi, E_1, E_2, ..., E_n$ are space-like vector fields and then, by (2.1) the vector fields $\varphi E_1, \varphi E_2, ..., \varphi E_n$ are time-like. For a three-dimensional almost paracontact metric manifold, any (local) pseudo-orthonormal basis of $ker(\eta)$ determines a φ-basis, up to sign. If e_2, e_3 is a (local) pseudo-orthonormal basis of $ker(\eta)$, with e_3, time-like, so by (7) vector field $\varphi e_2 \in ker(\eta)$ is time-like and orthogonal to e_2. Therefore, $\varphi e_2 = \pm e_3$ and $\xi, e_2, \pm e_3$ is a φ-basis [13]. In a paracontact metric manifold, one can introduce a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ such that $\varphi h + h\varphi = 0$ and $h\xi = 0$ satisfying [12, 16, 19, 20]

$$\nabla_X \xi = -\varphi X + \varphi h X, \tag{8}$$

$$(\nabla_X \eta)Y = g(X - hX, \varphi Y), \tag{9}$$

$$(\nabla_X \varphi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{10}$$

for any $X, Y \in \chi(M)$. Note that, the condition $h = 0$ is equivalent to ξ being Killing vector field and then (φ, ξ, η, g) is said to be K-paracontact structure. A paracontact metric manifold is said to be a (κ, μ)-paracontact metric manifold if the curvature tensor R satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - n(X)Y) + \mu(\eta(Y)hX - n(X)hY), \tag{11}$$

for all vector fields $X, Y \in \chi(M)$ and κ, μ are real constants. In particular, if $\mu = 0$, then the (κ, μ)-paracontact metric manifold reduces to an $N(\kappa)$-paracontact metric manifold. Thus, for an $N(\kappa)$-paracontact metric manifold the Riemannian curvature tensor R satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - n(X)Y). \tag{12}$$

In a 3-dimensional $N(\kappa)$-paracontact metric manifold $(M^3, \varphi, \xi, \eta, g)$, the following relations hold [14]:

$$QX = \left(\frac{r}{2} - \kappa\right) X + \left(3\kappa - \frac{r}{2}\right) \eta(X)\xi, \tag{13}$$

$$S(X,Y) = \left(\frac{r}{2} - \kappa\right) g(X,Y) + \left(3\kappa - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{14}$$

$$S(X,\xi) = 2\kappa\eta(X), \tag{15}$$

$$Q\xi = 2\kappa\xi, \tag{16}$$

for all vector fields $X, Y \in \chi(M)$, where Q and S are the Ricci operator and the Ricci tensor, respectively.

The covariant differentiation of (14) along any arbitrary vector field Z gives

$$\begin{aligned}(\nabla_Z S)(X,Y) &= \frac{dr(Z)}{2}[g(X,Y) - \eta(X)\eta(Y)] \\ &+ \left(3\kappa - \frac{r}{2}\right)[(\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)].\end{aligned}$$

Making use of (9) and replacing X by φX in above relation, we obtain

$$(\nabla_Z S)(\varphi X, Y) = \frac{dr(Z)}{2}[g(\varphi X, Y) - \eta(X)\eta(Y)] + \left(3\kappa - \frac{r}{2}\right)[g(Z - hZ, X)\eta(Y) - \eta(X)\eta(Y)\eta(Z)]. \tag{17}$$

In an orthogonal frame field $\{e_i\}_{i=1}^3$ of tangent space M, setting $X = Z = e_i$ in (17) and taking summation over i and then using a fact that $Tr(h) = 0$, we get

$$\sum_{i=1}^{3}(\nabla_{e_i} S)(\varphi e_i, Y) = \sum_{i=1}^{3} g((\nabla_{e_i} Q)\varphi e_i, Y) = (6\kappa - r)\eta(Y) - \frac{1}{2}dr(\varphi Y). \tag{18}$$

Again, setting $X = Y = e_i$ in (17) and taking summation over i and then using the fact that $Tr(\varphi) = 0, h\xi = 0$, we get

$$\sum_{i=1}^{3}(\nabla_Z S)(\varphi e_i, e_i) = \sum_{i=1}^{3} g((\nabla_Z Q)\varphi e_i, e_i) = 0. \tag{19}$$

Next, replacing X by hX in the relation (17), we obtain

$$(\nabla_Z S)(\varphi hX, Y) = \frac{dr(Z)}{2}g(\varphi hX, Y) + \left(3\kappa - \frac{r}{2}\right) g(Z - hZ, hX)\eta(Y). \tag{20}$$

Taking $X = Z = e_i$ in (20) and taking summation over i and then using a fact that $Tr(h) = Tr(h^2) = 0$, we get

$$\sum_{i=1}^{3}(\nabla_{e_i} S)(\varphi h e_i, Y) = \sum_{i=1}^{3} g((\nabla_{e_i} Q)\varphi h e_i, Y) = \frac{1}{2}dr(\varphi hY). \tag{21}$$

Also, by setting $X = Y = e_i$ and taking summation over i and then using the fact that $Tr(\varphi h) = 0, h\xi = 0$, we obtain

$$\sum_{i=1}^{3}(\nabla_Z S)(\varphi h e_i, e_i) = \sum_{i=1}^{3} g((\nabla_Z Q)\varphi h e_i, e_i) = 0. \tag{22}$$

Now, we recall some results on 3-dimensional $N(\kappa)$-paracontact metric manifolds which will be useful for the next section:

Lemma 2.1 ([14])**.** *An $N(\kappa)$-paracontact metric 3-manifold $(M^3, \varphi, \xi, \eta, g)$ is locally φ-symmetric if and only if the scalar curvature r of g is constant.*

Lemma 2.2 ([18])**.** *On any $N(\kappa)$-paracontact metric 3-manifold $\xi r = 0$.*

3. Bach tensor on $N(\kappa)$-paracontact metric 3-manifolds

In this section, we aim to study $N(\kappa)$-paracontact metric 3-manifold with purely transversal Bach tensor, i.e., $\mathcal{B}(X,\xi)=0$. First, we present the following result:

Lemma 3.1. *For a $N(\kappa)$-paracontact metric 3-manifold M^3, the following relation hold:*

$$\begin{aligned}\sum_{i=1}^{3}(\nabla_{e_i}\mathcal{C})(e_i,Y)\xi &= 3(6\kappa-r)\eta(Y)-\frac{3}{2}dr(\varphi Y)-\frac{1}{2}dr(\varphi hY)\\ &+ \sum_{i=1}^{3}g((\nabla_{e_i}\varphi h)Y,Qe_i)-\sum_{i=1}^{3}g((\nabla_{e_i}\varphi h)e_i,QY)\\ &- \frac{1}{4}g(\nabla_\xi Dr,Y).\end{aligned}\tag{23}$$

Proof. In a 3-dimensional manifold (M^3,g), for $X,Y,Z\in\chi(M)$ the Cotton tensor is given by

$$\begin{aligned}\mathcal{C}(X,Y)Z &= (\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)\\ &- \frac{1}{4}\{dr(X)g(Y,Z)-dr(Y)g(X,Z)\}.\end{aligned}\tag{24}$$

Replacing Z by ξ in (24), we get

$$\begin{aligned}\mathcal{C}(X,Y)\xi &= (\nabla_X S)(Y,\xi)-(\nabla_Y S)(X,\xi)\\ &- \frac{1}{4}\{dr(X)\eta(Y)-dr(Y)\eta(X)\}.\end{aligned}\tag{25}$$

Differentiating covariantly (16) along an arbitrary vector field X, we obtain

$$(\nabla_X Q)\xi = 2\kappa\nabla_X\xi - Q\nabla_X\xi.\tag{26}$$

Using (8) in the above equation, we get

$$(\nabla_X Q)\xi = -2\kappa(\varphi X-\varphi hX)+Q(\varphi X-\varphi hX).\tag{27}$$

Take an inner product of (27) with respect to vector field Y, we obtain

$$\begin{aligned}g((\nabla_X Q)\xi,Y)=(\nabla_X S)(Y,\xi) &= -2\kappa g(\varphi X-\varphi hX,Y)\\ &+ g(\varphi X-\varphi hX,QY).\end{aligned}\tag{28}$$

Making use of (28) in (25) shows that

$$\begin{aligned}\mathcal{C}(X,Y)\xi &= -4\kappa g(\varphi X,Y)+2g(Q\varphi X,Y)-g(Q\varphi hX,Y)+g(Q\varphi hY,X)\\ &- \frac{1}{4}[dr(X)\eta(Y)-dr(Y)\eta(X)].\end{aligned}\tag{29}$$

Taking covariant differentiation of $\mathcal{C}(X,Y)\xi$ along an arbitrary vector field W, we get

$$\begin{aligned}(\nabla_W\mathcal{C})(X,Y)\xi &= \nabla_W\mathcal{C}(X,Y)\xi - \mathcal{C}(\nabla_W X,Y)\xi \\ &- \mathcal{C}(X,\nabla_W Y)\xi - \mathcal{C}(X,Y)\nabla_W\xi. \end{aligned}\tag{30}$$

Using (27) and (29) in (30), we deduce

$$\begin{aligned}(\nabla_W\mathcal{C})(X,Y)\xi &= \mathcal{C}(X,Y)\varphi W - \mathcal{C}(X,Y)\varphi hW - 4\kappa g((\nabla_W\varphi)X,Y) \\ &+ 2[g((\nabla_W Q)\varphi X,Y) + g(Q(\nabla_W\varphi)X,Y)] \\ &- g((\nabla_W Q)\varphi hX,Y) - g(Q(\nabla_W\varphi h)X,Y) \\ &+ g((\nabla_W Q)\varphi hY,X) + g(Q(\nabla_W\varphi h)Y,X) \\ &- \frac{1}{4}[dr(X)(\nabla_W\eta)(Y) - dr(Y)(\nabla_W\eta)(X) \\ &- g(\nabla_W Dr,X)\eta(Y) + g(\nabla_W Dr,Y)\eta(X)]. \end{aligned}$$

From the relation (9), (10) and (24) above equation becomes

$$\begin{aligned}(\nabla_W\mathcal{C})(X,Y)\xi &= (\nabla_X S)(Y,\varphi W) - (\nabla_Y S)(X,\varphi W) - (\nabla_X S)(Y,\varphi hW) \\ &+ (\nabla_Y S)(X,\varphi hW) - 4kg(W-hW,Y)\eta(X) \\ &+ 2g(W-hW,QY)\eta(X) + 2g((\nabla_W Q)\varphi X,Y) \\ &- g((\nabla_W Q)\varphi hX,Y) - g(Q(\nabla_W\varphi h)X,Y) \\ &+ g((\nabla_W Q)\varphi hY,X) + g(Q(\nabla_W\varphi h)Y,X) \\ &+ \frac{1}{4}[g(\nabla_W Dr,X)\eta(Y) - g(\nabla_W Dr,Y)\eta(X)]. \end{aligned}\tag{31}$$

Setting $X = W = e_i$ in (31) and summing over i and then use of the relations (18), (19), (21) and (22) proves our lemma. □

Now, we state and prove the main result of this paper:

Theorem 3.1. *If an $N(\kappa)$-paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor, then the scalar curvature r is constant.*

Proof. In an $N(\kappa)$-paracontact metric 3-manifold with $\kappa > -1$, we have

$$\begin{aligned}(\nabla_X\varphi h)Y &= (1+\kappa)[g(X,Y)\xi - \eta(X)\eta(Y)\xi] - g(hX,Y)\xi \\ &+ (1+\kappa)\eta(Y)[X - \eta(X)\xi] - \eta(Y)hX. \end{aligned}\tag{32}$$

Taking an inner product of (32) with respect to QZ, we get

$$\begin{aligned}g((\nabla_X\varphi h)Y,QZ) &= 2\kappa(1+\kappa)[g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)] \\ &- 2\kappa g(hX,Y)\eta(Z) + (1+\kappa)g(X,QZ)\eta(Y) \\ &- g(hX,QZ)\eta(Y). \end{aligned}\tag{33}$$

Take an orthogonal frame field $\{e_i\}$, $i = 1, 2, 3$ of tangent space M^3, setting $X = Y = e_i$ in (33) and then summing over i, we obtain

$$\sum_{i=1}^{3} g((\nabla_{e_i}\varphi h)e_i, QZ) = 4\kappa(1+\kappa)\eta(Z). \tag{34}$$

Again, take an orthogonal frame field $\{e_i\}$, $i = 1, 2, 3$ of tangent space M^3, setting $X = Z = e_i$ in (33) and then summing over i gives

$$\sum_{i=1}^{3} g((\nabla_{e_i}\varphi h)Y, Qe_i) = (1+\kappa)(r-2\kappa)\eta(Y). \tag{35}$$

Replacing Z by Y, in (34) and subtracting it in (35), gives

$$\sum_{i=1}^{3} g((\nabla_{e_i}\varphi h)Y, Qe_i) - \sum_{i=1}^{3} g((\nabla_{e_i}\varphi h)e_i, QY) = (1+\kappa)(r-6\kappa)\eta(Y). \tag{36}$$

Making use of (36) in (23) yields

$$\begin{aligned} \sum_{i=1}^{3}(\nabla_{e_i}\mathcal{C})(e_i, Y)\xi &= (6\kappa - r)(2-\kappa)\eta(Y) - \frac{3}{2}dr(\varphi Y) \\ &- \frac{1}{2}dr(\varphi hY) - \frac{1}{4}g(\nabla_\xi Dr, Y). \end{aligned} \tag{37}$$

By employing (37) in (6), we obtain

$$g(\nabla_\xi Dr, Y) = 4(6\kappa - r)(2-\kappa)\eta(Y) - 6dr(\varphi Y) - 2dr(\varphi hX). \tag{38}$$

Replacing Y by φY in the above equation and simplification leads the following

$$\nabla_\xi Dr = (6+2h)\varphi Dr. \tag{39}$$

From Lemma 2.2, we get $dr(\xi) = 0$ implies that $\nabla_\xi Dr = 0$. Therefore from above relation we get

$$(6+2h)\varphi Dr = 0, \tag{40}$$

which implies that $\varphi Dr = 0$. If $\varphi Dr = 0$, then $Dr = 0$, which shows that r is constant. This proves the theorem. □

From the Lemma 2.1 and Theorem 3.1, we have the following:

Corollary 3.1. *A 3-dimensional $N(\kappa \neq -1)$-paracontact metric manifold M^3 with purely transversal Bach tensor is locally φ-symmetric.*

Following the proof of the above theorem 3.1, we are in a position to prove

Corollary 3.2. *If an $N(\kappa)$-paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor, then the scalar curvature r is equal to 6κ.*

Proof. Since M^3 is a $N(\kappa)$-paracontact metric 3-manifold with $\kappa > -1$ and has a purely transversal Bach tensor. we have from Theorem 3.1 and (38) that

$$4(6\kappa - r)(2 - \kappa)\eta(Y) = 0. \tag{41}$$

This gives $r = 6\kappa$. □

Theorem 3.2. *An $N(\kappa)$-paracontact metric 3-manifold M^3 with $\kappa > -1$ has purely transversal Bach tensor if and only if the manifold M^3 is of constant scalar curvature $r = 6\kappa$.*

Proof. Let M^3 be an $N(\kappa)$-paracontact metric 3-manifold with $\kappa > -1$ has constant scalar curvature $r = 6\kappa$. Then from (37) we get

$$\sum_{i=1}^{3}(\nabla_{e_i}\mathcal{C})(e_i, Y)\xi = 0$$

which intern gives from (6) that $\mathcal{B}(Y, \xi) = 0$. That is, the manifold M^3 has purely transversal Bach tensor. The converse is obvious from Theorem 3.1 and Corollary 3.2.

□

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