Biharmonic classification of infinite graphs: Bi-networks and Poisson networks

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Abstract. In an infinite graph, by using potential-theoretic methods based on the discrete Laplace operator, the properties of biharmonic functions are studied; the notion of bipotentials leads to a classification theory of the infinite graphs. Depending on the solvability of the discrete Poisson equation, we introduce the concept of Poisson networks and investigate some special properties of such infinite graphs.

Keywords: infinite graphs, bisuperharmonic functions, Poisson networks.

1. Introduction

On infinite graphs with associated transition functions (as in electrical networks or in random walks), by using the Laplacian operator a general theory of harmonic functions and potentials can be developed [1],[2],[6] and [7]. This leads to a classification of the infinite graphs as hyperbolic and parabolic (as the transient and the recurrent walks in probability). Here we consider two different classifications of infinite graphs: the first, of the graphs X with two associated sets of transition functions, leading to the study of biharmonic functions on X; and the second, of the graphs X on which the discrete version of the Poisson equation $\Delta u = f$ can be solved.

2. Function theory on infinite graphs: a summary

In an infinite graph X with a countable number of vertices and a countable number of edges, two vertices x and y are said to be neighbours (written $x \sim y$) if there is an edge joining x and y. We say that there is a self loop at x if $x \sim x$. If each vertex has only a finite number of neighbours, X is said to be locally finite. A path $\{x = x_0, x_1, x_2, ..., x_n = y\}$ connecting the vertices x, y is a collection of distinct vertices x_i such that $x_{i+1} \sim x_i$ for $0 \leq i \leq n-1$; if there is a path connecting any two vertices in X, then X is said to be connected. The set $\{t(x, y)\}$ of transition functions consists of real valued functions $t(x, y) \geq 0$ for any pair of vertices x, y. If $t(x, y) = t(y, x)$ for any pair of vertices x, y then we say that the transition function is symmetric. However in keeping with the example of a (not necessarily reversible) random walk, we work here with non-symmetric transition function. Now, for the definition of an infinite network: An infinite graph X is said to be an infinite network $\{X,t\}$ if X is connected, locally finite, without self loops and provided with a transition function $t(x,y)$.

We introduce now the discrete Laplace operator on a subset of the network X. All the functions considered here are real-valued. If a vertex x and all its neighbours are in A, then x is said to be an interior vertex of A. The set $\overset{\circ}{A}$ consists of all interior vertices of A. Denote $\partial A = A \setminus \overset{\circ}{A}$. For a real valued function $u(x)$ on A and for $z \in \overset{\circ}{A}$, the Laplacian of $u(x)$ at z is $\Delta u(z) = \sum_{y\sim z} t(z,y)[u(y)-u(z)]$. Note that $\sum_{y\sim z} t(z,y)[u(y)-u(z)] = \sum_{y\in X} t(z,y)[u(y)-u(z)]$ since $t(z,y)>0$ if and only if $y \sim z$. The function $u(x)$ on A is said to be harmonic at $z \in \overset{\circ}{A}$ if and only if $\Delta u(z) = 0$. The function $u(x)$ on A is said to be harmonic (respectively superharmonic) on A if $\Delta u(z) = 0$ (respectively $\Delta u(z) \leq 0$) for every $z \in \overset{\circ}{A}$.

Discrete Dirichlet solution: In an infinite network $\{X,t\}$, let F be a subset and $E \subset \overset{\circ}{F}$. Suppose $f(x)$ is a real valued function on $F \setminus E$, such that $v \leq f \leq u$ on $F \setminus E$ where v,u are defined on $F, v \leq u$ on F $\Delta u \leq 0$ and $\Delta v \geq 0$ at each vertex in E. Then there exists a function $\varphi(x)$ which is uniquely determined by $f(x)$ if E is a finite set such that $\Delta\varphi(z) = 0$ if $z \in E$ and $\varphi(x) = f(x)$ if $x \in F \setminus E$.

We illustrate now with two examples of the presence of the discrete Dirichlet problem in electrical networks and random walks.

1. In a finite connected electrical network X, if a and b are two nodes, then the effective resistance $r(a,b)$ between a and b is the voltage when a unit current enters a and leaves b. What is the value of $r(a,b)$?

   The problem is to find the quantity $r(a,b) = [\psi(a)-\psi(b)]$ where $-\Delta\psi(a) = 1, -\Delta\psi(b) = -1$ and $\Delta\psi(x) = 0$ if $x \neq a,b$. For this, let us start with the Dirichlet solution $\varphi(x)$ for which $\varphi(a) = 1, \varphi(b) = 0$ and $\Delta\varphi(x) = 0$ if $x \neq a,b$ since the transition function is symmetric in an electrical network, $\sum_{x\in X} -\Delta\varphi(x) = 0$. Hence we should have $-\Delta\varphi(a) = \alpha$ and $-\Delta\varphi(b) = -\alpha$ for some α. Now by the Maximum principle, $0 \leq \varphi \leq 1$ on X. Hence $\alpha \geq 0$. But $\alpha = 0$ would mean that $\varphi = 0$ by the maximum principle. Hence $\alpha > 0$. Write $\psi(x) = \frac{1}{\alpha}\varphi(x)$ to obtain the unique function $\psi(x)$ such that $-\Delta\psi(a) = 1, -\Delta\psi(b) = -1$ and $\Delta\psi(x) = 0$ for $x \neq a,b$. Hence $r(a,b) = \psi(a) - \psi(b) = \frac{1}{\alpha}\varphi(a) - 0 = \frac{1}{\alpha} = \frac{1}{-\Delta\varphi(a)}$.

2. In a random walk X, let A and B be two disjoint sets. The problem is to find the probability of the walker starting at some state reaches A before arriving in B.

   To solve this, suppose $\varphi(x)$ denotes the probability of the walker starting at the state x reaches A before arriving in B. Then clearly $\varphi(x) = 1$ if

$x \in A$ and $\varphi(x) = 0$ if $x \in B$. Suppose now $x \notin A \cup B$. Then $\varphi(x) = \sum_{y \sim x} p(x,y)\varphi(y)$; for the walker goes from x to a neighbour state y with the probability $p(x,y)$ and then from y reaches A first with probability $\varphi(y)$. Since $\sum_{y \sim x} p(x,y) = 1$ we find $\sum_{y \sim x} p(x,y)[\varphi(y) - \varphi(x)]$ that is, $-\Delta\varphi(x) = 0$ for any $x \notin A \cup B$. Thus the probability of the walk starting at x reaches A first is the Dirichlet solution $\varphi(x)$ such that $\varphi(x) = 1$ if $x \in A, \varphi(x) = 0$ if $x \in B$ and $\Delta\varphi(x) = 0$ if $x \notin A \cup B$.

The following is the discrete version of an important theorem in $\mathbb{R}^n$.

Poisson kernel of a finite set: Let E be a finite set in a network for any $z \in \partial E$, let $P_E(z,x)$ be the function on E such that $P_E(z,x)$ is harmonic at every vertex x in $\overset{\circ}{E}$, $P_E(z,z) = 1$ and $P_E(z,y) = 0$ for every $y \in \partial E \setminus \{z\}$. We say that $P_E(z,x)$ is the Poisson Kernel associated with the finite set E having the pole at $z \in \partial E$. Suppose now $f(\xi)$ is a function defined on ∂E. Then $h(x) = \sum_{\xi \in \partial E} P_E(\xi, x) f(\xi)$ is defined on E, harmonic at every vertex in $\overset{\circ}{E}$ and $h(\xi) = f(\xi)$ for every ξ in ∂E.

Poisson equation: Given a function $u(x)$ on a network $\{X,t\}$, it is routine to calculate the Laplacian $\Delta u(x)$. The converse is the problem of the discrete Poisson equation: Given a function $f(x)$ on X, is it possible to find a solution $u(x)$ to the Poisson equation $\Delta u(x) = f(x)$? Actually we can find a solution only if certain restrictions are placed on X or on $f(x)$.

In the continuous case, if $f(x)$ is a locally (Lebesgue) integrable function on $\mathbb{R}^n, n \geq 2$, then Brelot[4] had proved that a solution $u(x)$ could be found on $\mathbb{R}^n$ satisfying $\Delta u(x) = f(x)$ in the sense of distributions, by using a Runge-type approximation result. Of course, if $f(x)$ is continuously differentiable there always exists locally a solution to the equation $\Delta u(x) = f(x)$ in the classical sense.

In the case of a finite network $\{X,t\}$ with n vertices $x_1, x_2, ..., x_n$ there exists a vector $(y_1, y_2, y_3, ..., y_n)$ such that $\Delta u(x_i) = f(x_i)$ has a solution if and only if $\sum_{i=1}^{n} y_i f(x_i) = 0$, see [3, Theorem 3.5]. This is proved by using perron frobenius theorem in matrix theory. If the network is symmetric as in the case of an electrical network, then $y_i = 1$ for all i.

In the case of an infinite network, if X is a tree (that is there is no closed path $[x_0, x_1, ..., x_n = x_0]$ in X when $n \geq 3$) without terminal vertices, then for any function $f(x)$ there always exists a solution to the equation $\Delta u(x) = f(x)$, see [2, Theorem 5.1.4]. However in the general case, we are able to prove that for an arbitrary function $f(x), \Delta u(x) = f(x)$ has a solution only with some restrictions on X.

3. Binetworks

Let X be an infinite graph and (X,t) an infinite network with the Green function $G(x,y) = G_y(x)$. Let $\{t^*(x,y)\}$ be another set of transition functions on the graph X such that (X, t^*) is an infinite network with Green function $G^*(x, y)$. Let Δ and Δ^* be the Laplacians on (X, t) and (X, t^*) respectively. We refer to $\{X, t, t^*\}$ as a binetwork.

Definition 3.1. *Let $\mathcal{B}$ be the class of pairs of functions (f, g) where $-\Delta f = g$ on X.*

1. *If $-\Delta^* g = 0$, then (f, g) is said to be a biharmonic pair; and f is referred to as a biharmonic function on X.*
2. *If $-\Delta^* g \geq 0$, then (f, g) is said to be a bisuperharmonic pair; and f is referred to as a bisuperharmonic function on X.*
3. *If f is a $\Delta-$ potential and g is a Δ^*- potential, then (f, g) is said to be a bipotential pair; and f is referred to as a bipotential on X.*
4. *If f is a $\Delta-$ potential and g is a Δ^*- harmonic function, then (f, g) is said to be a biharmonic potential pair.*

Lemma 3.1. *Let s be a non-negative bisuperharmonic function that is $\Delta-$superharmonic. Let $-\Delta s = u$. Suppose f is a real-valued function, $0 \leq f \leq u$. Then there exists a $\Delta-$ potential q such that $-\Delta q = f$ and $q \leq s$.*

Proof. Since s is non-negative $\Delta-$superharmonic and $-\Delta s = u$, then s is the sum of a potential and a harmonic function $h \geq 0$. Hence

$$\begin{aligned} s(x) &= \sum_{y \in X} G_y(x)[-\Delta s(x)] + h(x) \\ &\geq \sum_{y \in X} G_y(x)[-\Delta s(x)] \\ &= \sum_{y \in X} G_y(x) u(x) \\ &\geq \sum_{y \in X} G_y(x) f(x) = q(x). \end{aligned}$$

Consequently $q(x)$ is a $\Delta-$ potential, $q(x) \leq s(x)$ and $-\Delta q(x) = f(x)$. □

Proposition 3.1. *A non-negative bisuperharmonic function that is $\Delta-$superharmonic is the unique sum of a bipotential and a biharmonic potential up to an additive non-negative $\Delta-$harmonic function.*

Proof. Let $s \geq 0$ be bisuperharmonic, that is $-\Delta s = u$ where $u \geq 0$ is Δ^* superharmonic. Now $u = p + v$, where p is a Δ^*- potential and $v \geq 0$ is a Δ^*- harmonic function. Then by Lemma 3.1, there exist (q, p) and (p_1, v) a bipotential pair and a biharmonic potential pair so that $-\Delta(q + p_1) = p + v = u$. Hence there exists a $\Delta-$harmonic function h such that $s = q + p_1 + h$. Since $s \geq 0, -h \leq q + p_1$; since $q + p_1$ is a $\Delta-$potential and h is $\Delta-$harmonic, we have

$-h \leq 0$. Thus the decomposition of s as stated in the proposition.
For the uniqueness, suppose $p' + p'_1 + h'$ is another such decomposition. Since $p + p_1 + h = p' + p'_1 + h'$, we have by the uniqueness of Riesz decomposition $h = h'$. Then $-\Delta p - \Delta p_1 = -\Delta p' - \Delta p'_1$. Here again $-\Delta p$ and $-\Delta p'$ are Δ^*-potentials and $-\Delta p_1$ and $-\Delta p'_1$ are Δ^*-harmonic, so that $-\Delta p = -\Delta p'$ and $-\Delta p_1 = -\Delta p'_1$. Hence $p_1 = p'_1+$ (a harmonic function). Since p_1, p'_1 are $\Delta-$potentials we conclude that $p_1 = p'_1$. Thus the uniqueness of decomposition of the function s. □

Lemma 3.2. *If $\{(f_n, g_n)\}$ is a sequence from $\mathcal{B}$ and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in X$, then $g(x) = \lim_{n\to\infty} g_n(x)$ exists and $(f, g) \in \mathcal{B}$. In this case, we write $(f, g) = \lim_{n\to\infty}(f_n, g_n)$.*

Proof. For any x,

$$g_n(x) = -\Delta f_n(x) = \sum t^*(x, y)[f_n(x) - f_n(y)] \to \sum t^*(x, y)[f(x) - f(y)],$$

when $n \to \infty$. Hence, $\lim_{n\to\infty} g_n(x) = -\Delta f(x)$. □

Proposition 3.2. *Let $\{b_n\}$ be a sequence of biharmonic (respectively bisuperharmonic) functions defined on X. If $b(x) = \lim_{n\to\infty} b_n(x)$ is finite for each $x \in X$ then $b(x)$ is biharmonic on X.*

Proof. Let $\{(b_n, v_n)\}$ be a sequence of biharmonic pairs, that is $-\Delta b_n = v_n$ is Δ^*- harmonic. Suppose $\lim_{n\to\infty} b_n(x) = b(x)$ for each vertex $x \in X$. Then by the above Lemma, $\lim_{n\to\infty} v_n(x) = v(x)$ exists. Since each $v_n(x)$ is Δ^*-harmonic, so is $v(x)$; since $-\Delta b_n = v_n$, taking limits we have $-\Delta b = v$. Hence $(b, v) = \lim_{n\to\infty}(b_n, v_n)$ is a biharmonic pair, that is b is a biharmonic function. Similar proof for bisuperharmonic functions also. □

Definition 3.2. *The infinite graph X with two sets of transition functions $\{t(x, y)\}$ and $\{t^*(x, y)\}$ is said to be a bipotential binetwork if there exists a positive bipotential on $\{X, t, t^*\}$.*

Proposition 3.3. *In a bipotential binetwork X if $p(x)$ is any Δ^*- potential with finite harmonic support, then there exists a unique $\Delta-$ potential $q(x)$ on X such that (q, p) is a bipotential.*

Proof. Let X be a bipotential binetwork; that is there exists a positive bipotential (u, v) on X. Now, if p is a Δ^*- potential with finite harmonic support, then for some $\alpha > 0$, $\alpha p(x) \leq v(x)$ on X by the Domination Principle. Now $u(x) = \sum_{y\in X} G(x, y)v(y) \geq \alpha \sum_{y\in X} G(x, y)p(y)$. Hence $q(x) = \sum_{y\in X} G(x, y)p(y)$ is a $\Delta-$ potential so that (q, p) is a bipotential. For the uniqueness, suppose (q_1, p) is another bipotential then $-\Delta q = -\Delta q_1 = p$ so that $h = q - q_1$ is a $\Delta-$ harmonic function. Since q and q_1 are potentials, $h = 0$. □

Corollary 3.1. *In a bipotential binetwork $\{X, t, t^*\}$, for any given vertex y in X, there exists a unique bipotential $Q_y(x) = Q(x, y)$ such that $-\Delta Q_y(x) = G^*_y(x)$.*

In a bipotential binetwork $\{X, t, t^*\}$, the function $Q(x, y) = Q_y(x)$ is refered to as the biharmonic Green potential with support at the vertex y. We know that a non-negative superharmonic function is the increasing limit of a sequence of potentials. We prove here that a non-negative bisuperharmonic function is the limit of an increasing sequence of bipotentials up to an additive non-negative harmonic function.

Proposition 3.4. *Let $s = (s_1, s_2)$ be a non-negative bisuperharmonic pair. Then up to an additive non-negative harmonic function s is the limit of an increasing sequence of bipotentials. That is, $s = (h, 0) + \lim_{n\to\infty}(q_n, p_n)$ where h is a non-negative harmonic function and $\{(q_n, p_n)\}$ is an increasing sequence of bipotential pairs.*

Proof. Since $s_1 \geq 0$ and $-\Delta s_1 = s_2 \geq 0$, then s_1 is $\Delta-$ superharmonic; hence it is the sum of a $\Delta-$ potential p and a non-negative $\Delta-$ harmonic function h, so that $-\Delta p = -\Delta s_1 = s_2$. Hence, $p(x) = \sum_{y\in X} G_y(x)[-\Delta p(y)] = \sum_{y\in X} G_y(x)s_2(y)$. Now s_2 is the increasing limit of a sequence of Δ^*- potentials $\{p_n\}$. Since $p_n \leq s_2, q_n(x) = \sum_{y\in X} G_y(x)p_n(y)$ is a $\Delta-$potential so that (q_n, p_n) is a bipotential pair; since $\{p_n\}$ is increasing, so is $\{q_n\}$. Also $q_n(x) \leq \sum_{y\in X} G_y(x)s_2(y) = p(x)$. Hence, $\lim_{n\to\infty} q_n(x) = q(x)$ is $\Delta-$ superharmonic; actually $q(x)$ is a $\Delta-$ potential since $q_n(x) \leq p(x)$ for each n. Note also that $q(x) = \lim_{n\to\infty} \sum_{y\in X} G_y(x)p_n(y) = \sum_{y\in X} G_y(x)s_2(y) = p(x)$, a $\Delta-$ potential. Thus, $s_1(x) = p(x) + h(x) = q(x) + h(x)$ where $q(x)$ is the limit of the increasing sequence of bipotentials q_n. Hence (s_1, s_2) can be represented as $(s_1, s_2) = (h, 0) + \lim_{n\to\infty}(q_n, p_n)$. □

Corollary 3.2. *In a binetwork $\{X, t, t^*\}$, if there exists a positive biharmonic potential, then there exist bipotentials on X.*

Proof. Let u be a positive biharmonic potential, hence u is not harmonic. Now by the Proposition 3.4, u is the increasing limit of bipotentials up to an additive harmonic function. Hence there exist bipotentials on X. □

Example 3.1. An example of a binetwork in which bipotentials exist but no positive biharmonic potential. Let $X = \{0, 1, 2, ...\}$ be the infinite graph in which the transition weights are given by $t(k-1, k) = k(k+1), k \geq 1$ on the edge $k-1, k$ and $t^* = t$. Thus $\{X, t, t^*\}$ is a binetwork with symmetric conductance. The network $\{X, t\}$ is hyperbolic. If $G(x, y) = G_y(x)$ is the symmetric Green function of $\{X, t\}$, we have in particular $G_0(k) = \frac{1}{k+1}, k \geq 0$. Further, in the binetwork $\{X, t, t^*\}$

1. Any non-negative harmonic function is constant, For if $h \geq 0$ is harmonic on X, $h(0) = a$ then $0 = \Delta h(0) = t(0, 1)[h(1) - h(0)] \Rightarrow h(1) = a$. Then $0 = \Delta h(1) = t(1, 0)[h(0) - h(1)] + t(1, 2)[h(2) - h(1)] \Rightarrow h(2) = a$. This procedure leads to the conclusion $h(n) = a$ for all n.

2. There is no positive biharmonic potential on X.

   For suppose p is a potential and $h > 0$ a harmonic function such that $\Delta p = -h$ on X. Since all positive harmonic functions are constant, we can take $\Delta p = -1$ This means $p(n) = \sum_k [-\Delta p(k)] G_k(n) = \sum_k G_k(n)$ for all $n \geq 0$, hence $p(0) = \sum_k G_k(0) = \sum_k G_0(k)$ because of the symmetry of conductance. Now $G_0(k) = \frac{1}{k+1}$ which shows that $p(0) = \infty$, a contradiction. Hence there cannot be any non-zero biharmonic potential on X.

3. However there exist bipotentials on X. For, let $G_0(k)$ be the Green potential with harmonic support at $\{0\}$. Then $s(n) = \sum_k G_0(k) G_k(n)$ is a potential if $s(n)$ is finite at one vertex. Now $s(0) = \sum_k G_0(k) G_k(0) = \sum_k G_0(k) G_0(k) = \sum \frac{1}{(k+1)^2} < \infty$. Hence $s(n)$ is a potential such that $-\Delta s(n) = G_0(n)$ so that $s(n)$ is a bipotential on X.

Remark 3.1. In the continuous case, the above example is an analogue of the Euclidean spaces $\mathbb{R}^n, n \geq 5$.

Example 3.2. An example of a hyperbolic network which is not a bipotential network: Consider $X = \{1, 2, ...\}$ in which the symmetric transition functions are given by $t(k, k+1) = t(k+1, k) = \frac{\sqrt{k(k+1)}}{\sqrt{k+1} - \sqrt{k}}$ for $k \geq 1$. Consider the function $u(k) = \frac{1}{\sqrt{k}}$ for $k \geq 1$. Then, $\Delta u(1) = \frac{\sqrt{2}}{\sqrt{2}-1}(\frac{1}{\sqrt{2}} - 1) = -1$ and $\Delta u(k) = \frac{\sqrt{k(k+1)}}{\sqrt{k+1}-\sqrt{k}}(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}) + \frac{\sqrt{k(k-1)}}{\sqrt{k}-\sqrt{k-1}}(\frac{1}{\sqrt{k-1}} - \frac{1}{\sqrt{k}}) = -1 + 1 = 0$ Thus, $u(k) > 0$ is superharmonic on X with harmonic support at $k = 1$. Since $u(k)$ tends to 0 when k tends to ∞, $u(k)$ is a potential. Consequently if $G_l(k) = G(k, l)$ is the symmetric Green function on X, then $G_1(k) = u(k) = \frac{1}{\sqrt{k}}$. Now suppose X is a bipotential network, then there exists a potential $p > 0$ on X such that $p(k) = \sum_l G(k, l) G_1(l)$ so that $p(1)$ should be finite. But $p(1) = \sum_l G(1, l) G_1(l) = \sum_l [G_1(l)]^2 = \sum_l \frac{1}{l} = \infty$, contradicting the assertion that $p(1)$ should be finite. Hence $\{X, t\}$ is hyperbolic but not a bipotential network.

Remark 3.2. In the continuous case, the above example is an analogue of the Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^4$.

Restricting the natural order for the following of real-valued functions on a network $\{X, t\}$, we shall now introduce a specific order for the family of real-valued functions defined on the network $\{X, t\}$. This specific order is useful in the study of functions on a binetwork $\{X, t, t^*\}$.

Definition 3.3. *For two real-valued functions f, g defined on a network $\{X, t\}$ write $f \succ g$ if and only if $f \geq g$ and $-\Delta f \geq -\Delta g$. We refer to $\succ$ as the specific order defined on the functions on X.*

Proposition 3.5. *In a network $\{X, t\}$, $f \succ g$ if and only if there exists a $\Delta-$superharmonic function $s \geq 0$ such that $f = g + s$.*

Proof. If there exists a $\Delta-$superharmonic function $s \geq 0$ such that $f = g + s$, then $f \geq g$ and $-\Delta f = -\Delta g - \Delta s$. Since $-\Delta s \geq 0$ we have $-\Delta f \geq -\Delta g$ and $f \geq g$.

Conversely suppose $f \geq g$ and $-\Delta f \geq -\Delta g$. Then $-\Delta(f - g) \geq 0$. That is $f - g = s$ is a superharmonic function. Which is non-negative. □

Proposition 3.6. *A bisuperharmonic function $u \succ 0$ is the unique sum of a bipotential and a biharmonic function $h \succ 0$.*

Proof. $-\Delta u = s \geq 0$ is superharmonic. We have proved (Proposition 3.1) that u is the unique sum of a bipotential q and a biharmonic potential H and a non-negative harmonic function h. Since $H + h \succ 0$ is biharmonic, we have proved the proposition. □

Proposition 3.7. *A biharmonic function $h \succ 0$ is the unique sum of a biharmonic potential and a non-negative harmonic function.*

Proof. Since $h \succ 0$, then h is a non-negative superharmonic function so that $h = (a\ potential)p + (a\ harmonic\ function\ u \geq 0)$. Now, $-\Delta p = -\Delta h$; since h is biharmonic, Δh is harmonic so that $-\Delta p$ is a non-negative harmonic function. Hence p is a biharmonic potential. □

Proposition 3.8. *A bisuperharmonic function $s \succ 0$ is a bipotential if and only if for any bisubharmonic function $u \prec s$ we have $u \prec 0$.*

Proof. Let s be a bipotential and the bisubharmonic function $u \prec s$. Since $-\Delta u \leq -\Delta s$ where $-\Delta u$ is subharmonic and $-\Delta s$ is a potential, we have $-\Delta u \leq 0$. That is u is subharmonic. Since $u \leq s$ where s is a potential, we have $u \leq 0$. Hence $u \prec 0$.

Conversely, let $s \succ 0$ be bisuperharmonic such that for any bisubharmonic $u \prec s$ we have $u \prec 0$. Then by Proposition3.6, s is the unique sum of a bipotential and a biharmonic function $h \succ 0$. But $s \succ h$ so that by the present assumption $h \prec 0$. Consequently $h = 0$ which implies that s is a bipotential. □

Consequences:

1. Let $s \succ 0$ be bisuperharmonic. Then s is the unique sum of a bipotential q and a biharmonic function $H \succ 0$. Then H is the greatest biharmonic minorant of s for the order $\succ$. For, let u be biharmonic and $u \prec s = q + H$. Then $u - H \prec q$. Since q is a bipotential, we have $u - H \prec 0$, that is $u \prec H$.

2. The bisuperharmonic function $s \succ 0$ is a bipotential if and only if its greatest biharmonic minorant is 0.

Recall the following known result: Let $\{p_n\}$ be a sequence of potentials such that $\sum p_n(x)$ is finite at one vertex, then $\sum p_n(x)$ is a potential.

Proof. Since $\sum p_n(x)$ is finite at one vertex, the sum represents a non-negative superharmonic function which is finite at every vertex. Let now u be a subharmonic function such that $u(x) \leq \sum_{n=1}^{\infty} p_n(x)$. Then $u(x) - \sum_{n=2}^{\infty} p_n(x) \leq p_1(x)$. Here the left side is subharmonic and the right side is a potential, hence $u(x) \leq \sum_2^{\infty} p_n(x)$. Proceeding in the same way, we conclude that for any given $\epsilon > 0$, $u(x) \leq \sum_m^{\infty} p_n(x) < \epsilon$ for some m. This leads to the conclusion $u \leq 0$ on X, so that $\sum_{n=1}^{\infty} p_n(x)$ is a potential on X. □

Proposition 3.9. *Let $\{s_n\}$ be a sequence of bipotentials. Suppose $\sum_{n=1}^{\infty} s_n(x)$ is finite at one vertex. Then $\sum_{n=1}^{\infty} s_n(x)$ is a bipotential.*

Proof. Since each $s_n(x)$ is a potential, $s(x) = \sum_{n=1}^{\infty} s_n(x)$ is a potential. Write $p_n(x) = \sum_{k=1}^{n} s_k(x)$. Then

$$\begin{aligned} -\Delta s(x) &= -\Delta[\lim_{n\to\infty} p_n(x)] \\ &= \lim_{n\to\infty}[-\Delta p_n(x)] \\ &= \lim_{n\to\infty}[\sum_{k=1}^{n} -\Delta s_k(x)] \\ &= \sum_{k=1}^{\infty}[-\Delta s_k(x)]. \end{aligned}$$

Since each $-\Delta s_k(x)$ is a potential and $-\Delta s(x)$ is finite. We conclude that $-\Delta s(x)$ is also a potential. Then $s(x)$ and $-\Delta s(x)$ are potentials. That is $s(x) = \sum_{n=1}^{\infty} s_n(x)$ is a bipotential. □

The cone of bipotentials and the cone of biharmonic potentials are lattices for the specific order $\prec$.

Proposition 3.10. *Let $q_1 \succ 0$, $q_2 \succ 0$ be bipotentials. Then $q_1 \curlyvee q_2$ and $q_1 \curlywedge q_2$ are bipotentials.*

Proof. Write $q_i(x) = \sum_{y\in X} G_y(x)p_i(y)$ where $p_i(x)$ are two potentials. Then, for the natural order

$$p_3(x) = \inf(p_1(x), p_2(x))$$

is a potential and $s(x) = \sum_{y\in X} G_y(x)p_3(y) \succ 0$ is a bipotential. Note that, $s \prec q_1$ and $s \prec q_2$. Now, suppose u is a bipotential such that $u \prec q_1$ and $u \prec q_2$. Then, $-\Delta u \leq \inf(-\Delta q_1, -\Delta q_2) = p_3 = -\Delta s$; also since $-\Delta u \leq -\Delta s, (u - s)$ is a subharmonic function v; since u is a potential and $v \leq u$, then $v \leq 0$. Thus, $u \leq s$. Consequently, $u \prec s$, showing that $s = q_1 \curlywedge q_2$. On the other hand, let $p_4(x)$ is the infimum for the natural order of the family of superharmonic functions $u(x) \geq \sup(p_1(x), p_2(x))$. Then $p_4(x)$ is a non-negative superharmonic function; in fact, $p_4(x)$ is a potential since it is majorised by the potential $p_1(x) + p_2(x)$. Clearly $q_1 \curlyvee q_2(x) = \sum_{y\in X} G_y(x)p_4(y)$ is a bipotential. □

Remark. The extreme elements for the cone of bipotentials being proportional to $Q_z(x) = \sum_{y\in X} G_y(x)G_z(y), z \in X$, a real-valued function $s(x)$ is a bipotential if and only if it is of the form $s(x) = \sum_z a(z)Q_z(x)$ for suitable non-negative constants $\{a(z)\}$.

Proposition 3.11. *Let $B_1 \succ 0, B_2 \succ 0$ be biharmonic potentials. Then $B_1 \curlyvee B_2$ and $B_1 \curlywedge B_2$ are biharmonic potentials.*

Proof. Write $B_i(x) = \sum_{y\in X} G_y(x)H_i(y)$ where $H_i(x)$ are two non-negative harmonic functions. Then for the natural order let $H_3(x)$ be the greatest harmonic minorant of the superharmonic function $\inf(H_1, H_2)$; and $H_4(x)$ be the least harmonic majorant of the subharmonic function $sup(H_1, H_2)$. Clearly, $B_1 \curlywedge B_2(x) = \sum_{y\in X} G_y(x)H_3(y)$ and $B_1 \curlyvee B_2(x) = \sum_{y\in X} G_y(x)H_4(y)$ are biharmonic potentials. □

4. Poisson networks

Definition 4.1. *An infinite network X is called a Poisson network if for any non-negative subharmonic function u, there exists a solution $v(x)$ to the Poisson equation $\Delta v(x) = u(x)$ for $x \in X$.*

Example 4.1. Let the network X be an infinite tree without any terminal vertices. Then, for any real-valued function $f(x)$ on X, there exists a function $u(x)$ such that $\Delta u(x) = f(x)$ (see, [2][Theorem 5.1.4]). Hence, any infinite tree X, hyperbolic or parabolic, without terminal vertices is a Poisson network.

In an infinite network, let us denote by $\mathfrak{F}$ the class of real-valued functions $f(x)$ such that $-\Delta g(x) = f(x)$ has a solution.

Remark 4.1. Let X be a Poisson network:

1. If h is any harmonic function, there exists $s(x)$ such that $-\Delta s(x) = h(x)$

   Proof. Note that, $h = h^+ - h^-$ where h^+ and h^- are non-negative subharmonic functions. This implies there exist v_1, v_2 such that $\Delta v_1 = h^+$ and $\Delta v_2 = h^-$. Then $\Delta(v_1 - v_2) = h$. Take $s = v_2 - v_1$ so that $-\Delta s(x) = h(x)$. □

2. Let s be a superharmonic function majorized by a harmonic function h. Then $s \in \mathfrak{F}$.

   Proof. Let $u = h - s$. Then $u \geq 0$ is subharmonic so that for some function $f, \Delta f = u$. Now h being harmonic, $\Delta g = h$ has a solution. Take $v = f - g$ so that $-\Delta v = s$. That is $s \in \mathfrak{F}$. □

Consequently, for any $y \in X$, if $G_y(x)$ is the Green potential with harmonic support $\{y\}$, then $-\Delta v(x) = G_y(x)$ has a solution.

Note. If $v(x)$ is non-negative in the above equation $-\Delta v(x) = G_y(x)$ for some vertex $\{y\}$ then X is a bipotential network.

Recall that a superharmonic function $s(x)$ on an infinite network is said to be admissible if there exists a subharmonic minorant for s outside a finite set.

Definition 4.2. *A hyperbolic Poisson network X is said to be singular, if for some vertex $y, -\Delta v(x) = G_y(x)$ and $v(x)$ is not admissible.*

Note. Poisson network is a bipotential network if and only if it is non-singular.

Proposition 4.1. *In a singular Poisson network, if for a superharmonic function $s > 0$ we have $-\Delta v(x) = s(x)$, then the superharmonic function v is not admissible.*

Proof. Since X is singular Poisson, for some vertex $y, -\Delta u(x) = G_y(x)$ and $u(x)$ is not admissible. Now by the Domination Principle (see Theorem 3.3.6 [2]), for some $\alpha > 0, \alpha s(x) \geq G_y(x)$. That is $-\Delta[\alpha v(x)] \geq -\Delta u(x)$, hence $u(x) - \alpha v(x) = f(x)$ is subharmonic. Now if v were admissible, then $\alpha v(x)$ will have a subharmonic minorant outside a finite set; Consequently $u(x)$ has a subharmonic minorant outside a finite set. That is $u(x)$ is admissible, contradicting the assumption. □

Homogeneous trees: let T be a homogeneous tree, that is each vertex has only $(q + 1)$ neighbours, $q \geq 2$. Then as mentioned earlier, since there are no terminal vertices, T is a Poisson network. For a vertex $e \in T$, we have as in Cartier [5] the Green potential $G(x, e) = G_e(x)$.

Lemma 4.1. *In a hyperbolic network (X, t), let $h(x)$ be harmonic outside a finite set. Then $h = H + p_1 - p_2$ outside a finite set, where H is harmonic on X and p_1, p_2 are potentials on X with finite harmonic support.*

Proof. Let h be defined on $X \setminus \overset{\circ}{E}$ where E is a finite set. Let v be the Dirichlet solution on E such that $\Delta v = 0$ every vertex in $\overset{\circ}{E}$ and $v = h$ on ∂E. Then the function u on X extending v on E by h on $X \setminus E$ is such that $\Delta u = 0$ on X except for a finite number of vertices $\{a_i\}$ on ∂E. Let $\Delta u(a_i) = \alpha_i \neq 0$. Then $H(x) = u(x) + \sum_i \alpha_i G_{a_i}(x)$ is harmonic on X. Take $p_1(x) = \sum_i (-\alpha_i) G_{a_i}(x)$ if $\alpha_i < 0$ and $p_2(x) = \sum_i \alpha_i G_{a_i}(x)$ if $\alpha_i > 0$. Then $p_1(x)$ and $p_2(x)$ are bounded potentials with finite harmonic support. Consequently outside the finite set $E, h(x) = H(x) + p_1(x) - p_2(x)$. □

Proposition 4.2. *Let (X, t) be a non-singular Poisson network. Let $b(x)$ be biharmonic outside a finite set. Then $b = B + q_1 - q_2 + p_1 - p_2$ outside a finite set, where B is biharmonic on X; q_1, q_2 are bipotentials with finite biharmonic support on X; and p_1, p_2 are potentials with finite harmonic support on X.*

Proof. The function $h(x) = -\Delta b(x)$ is harmonic outside a finite set. Hence $h = H' + p'_1 - p'_2$ outside a finite set, as in the Lemma 4.1. Since X is Poisson, if $-\Delta u = H', -\Delta q_1 = p'_1$ and $-\Delta q_2 = p'_2$ then u is biharmonic on X and q_1, q_2 are bipotentials with finite biharmonic support on X. Since $-\Delta b = h = -\Delta(u + q_1 - q_2)$ outside a finite set, $b = u + q_1 - q_2 + v$ outside a finite set where v is harmonic so that $v = H + p_1 - p_2$ as in the Lemma4.1. Write $B = u + H$ to get the decomposition $b = B + q_1 - q_2 + p_1 - p_2$ outside a finite set. □

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