Resolving sets in graphs

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Abstract. Let G be a connected graph. A vertex x of a connected graph G is said to *resolve two vertices* u and v of G if $d_G(x,u) \neq d_G(x,v)$. For an ordered set $W = \{x_1,\ldots,x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector* $r_G(v/W) = \big(d_G(v,x_1), d_G(v,x_2), \ldots, d_G(v,x_k)\big)$ is called the *representation* of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The metric dimension of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $\dim(G)$ is called a *basis*. In this paper, we characterize the resolving sets in the join, corona and lexicographic product of two graphs and determine the resolving number of these graphs.

Keywords: resolving set, resolving number, join, corona, lexicographic product.

1. Introduction

All graphs considered in this study are finite, simple, and undirected connected graphs, that is, without loops and multiple edges. For some basic concepts in Graph Theory, we refer readers to [2].

Let $G = \big(V(G), E(G)\big)$ be a connected graph. The *open neighborhood* $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element u of $N_G(v)$ is called a *neighbor* of v. The *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. Thus, the degree of v is given by $deg_G(v) = |N_G(v)|$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = \bigcup_{v\in S} N_G[v]$.

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A vertex x of a connected graph G is said to *resolve two vertices* u and v of G if $d_G(x,u) \neq d_G(x,v)$. For an ordered set $W = \{x_1, \ldots, x_k\} \subseteq V(G)$ and a vertex v in G, the *k-vector* $r_G(v/W) = \big(d_G(v,x_1), d_G(v,x_2), \ldots, d_G(v,x_k)\big)$ is called the *representation* of v with respect to W. The set W is a *resolving set* for G if and only if no two vertices of G have the same representation with respect to W. The metric dimension of G, denoted by $\dim(G)$, is the minimum cardinality over all resolving sets of G. A resolving set of cardinality $\dim(G)$ is called a *basis*.

Canoy and Malacas [1] defined a *locating set* as a set $S \subseteq V(G)$ of G if for every two distinct vertices u and v of $V(G) \setminus S$, $N_G(u) \cap S \neq N_G(v) \cap S$. The *locating number* of G, denoted by $\ln(G)$, is the smallest cardinality of a locating set of G. A locating set of G of cardinality $\ln(G)$ is referred to as an $\ln$*-set* of G.

The problem of uniquely recognizing the possible position of an intruder such as fault in a computer network and spoiled device was the principal motivation in introducing the concept of metric dimension in graphs.

Slater [6] brought in the notion of locating sets and its minimum cardinality as locating number. The same concept was also introduced by Harary and Melter [2] but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. However, in recent studies, locating sets and resolving sets are defined differently. In 2013, Bailey et al. [3] defined a resolving set as a set of vertices S in a graph G such that for any two vertices u, v, there exists $x \in S$ such that the distances $d(u,x) \neq d(v,x)$. Meanwhile in the same year, Canoy and Malacas [1], defined a locating set as a subset S of $V(G)$ in a connected graph G satisfying that $N_G(u) \cap S \neq N_G(v) \cap S$, for all $u, v \in V(G) \setminus S$, $u \neq v$.

Let G be a connected graph. A set $S \subseteq V(G)$ is a *strictly locating set* of G if it is a locating set of G and $N_G(u) \cap S \neq S$, for all $u \in V(G) \setminus S$. The *strictly locating number* of G, denoted by $sln(G)$, is the smallest cardinality of a strictly locating set of G. A strictly locating set of G of cardinality $sln(G)$ is referred to as *sln-set* of G.

A connected graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G, $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$.

This study aims to define and characterize the resolving sets in the join, corona and lexicographic product of two graphs and determine their corresponding resolving number.

2. Preliminary results

Remark 2.1. For any connected graph G of order n, $1 \leq \dim(G) \leq n-1$.

Example 2.1. $\dim(P_n) = 1$ and $\dim(K_n) = n-1$ for $n \geq 2$.

Theorem 2.1 ([5])**.** *Let G be a connected graph of order $n \geq 2$, then* (*i*) *dim*$(G) = 1$ *if and only if* $G = P_n$*;*

(*ii*) *dim*$(G) = n-1$ *if and only if* $G = K_n$*;*

(*iii*) *for* $G = C_n, n \geq 3$*, dim*$(G) = 2$*.*

Theorem 2.2. Let G be a connected graph of order $n = 4$, then $\dim(G) = 2$ if and only if $G \neq K_4$ and $G \neq P_4$.

Proof. Suppose that $\dim(G) = 2$, then by Theorem 2.1(i) and (ii), $G \neq K_4$ and $G \neq P_4$.

For the converse, suppose that $G \neq K_4$ and $G \neq P_4$. Since $n = 4, \dim(G) \geq 2$ by Theorem 2.1(i) and (ii). Choose $x, y \in V(G)$ such that $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$ and let $p \in V(G) \setminus \{x, y, z\}$. Consider the following cases:
Case 1. Suppose that $p \notin N_G(x) \cap N_G(y)$.

Since G is connected, $p \in N_G(z)$, that is, $G = K_{1,3}$. By Theorem 2.1(iv), $\dim(G) = 2$.
Case 2. Suppose that $p \in N_G(x) \cap N_G(y)$.

Then $G = C_4$ or $G = K_1 + P_3$. Hence, by Theorem 2.1(iii), $\dim(C_4) = 2$ and $W = \{x, p\}$ is a minimum resolving set of $K_1 + P_3$. Thus, $\dim(G) = 2$.
Case 3. Suppose that $p \in N_G(x) \setminus N_G(y)$ or $p \in N_G(y) \setminus N_G(x)$, say $p \in N_G(x) \setminus N_G(y)$. Let $W = \{p, x\}$, then W is a resolving set of G. Then $G \cong K_1 + (K_1 \cup K_2)$ since $G \neq P_4$. By Theorem 2.1(iv), $\dim(G) = 2$.

Therefore, in all cases, $\dim(G) = 2$. □

Theorem 2.3. *Let* G *be a connected graph of order* $n = 5$*, then dim*$(G) = 2$ *if and only if there exist distinct vertices* x *and* y *of* G *satisfying one of the following properties:*

(*i*) $|N_G(x) \cap N_G(y)| = 0$ *and* $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 1$*;*

(*ii*) $|N_G(x) \cap N_G(y)| = 1$ *and* $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$ *or* $|N_G(x) \setminus \{y\}| = 2$ *and* $|N_G(y) \setminus \{x\}| = 1$ *or* $|N_G(y) \setminus \{x\}| = 2$ *and* $|N_G(x) \setminus \{y\}| = 1$.

Proof. Suppose that $\dim(G) = 2$, then there exist distinct vertices x and y of G such that $W = \{x, y\}$ is a minimum resolving set of G. Suppose $|N_G(x) \cap N_G(y)| > 1$. Then there exist distinct vertices $u, v \in N_G(x) \cap N_G(y)$. This implies that $r_G(u/W) = r_G(v/W)$, a contradiction to the assumption. Hence, $|N_G(x) \cap N_G(y)| \leq 1$. Suppose $|N_G(x) \cap N_G(y)| = 0$. Since W is a resolving set, $|N_G(x) \setminus \{y\}| \leq 1$. Suppose $|N_G(x) \setminus \{y\}| = 0$, then $|N_G(y) \setminus \{x\}| = 1$ since W is a resolving set. This implies that there exist at least two vertices say u_1, u_2 such that $u_1, u_2 \notin N_G(x) \cup N_G(y)$. Thus, $r_G(u_1/W) = r_G(u_2/W)$, contrary to our assumption that W is a resolving set. Thus, $|N_G(x) \setminus \{y\}| = 1$. Similarly, $|N_G(y) \setminus \{x\}| = 1$. Hence, ($i$) holds.

Suppose that $|N_G(x) \cap N_G(y)| = 1$. Let $p \in N_G(x) \cap N_G(y)$ and let $v, z \in V(G) \setminus \{x, y, p\}$, then $v, z \notin N_G(x) \cap N_G(y)$. Since the subsets of W are $\emptyset, \{x, y\}, \{x\}$, and $\{y\}$ and since $N_G(p) \cap W$ is $\{x, y\}$, the remaining two sets $N_G(v) \cap W$ and $N_G(z) \cap W$ are $\{x\}$ and $\{y\}$ or $\{x\}$ and $\emptyset$ or $\{y\}$ and $\emptyset$, respectively. Thus, $|N_G(x) \setminus \{y\}| = |N_G(y) \setminus \{x\}| = 2$ or $|N_G(x) \setminus \{y\}| = 2$ and $|N_G(y) \setminus \{x\}| = 1$ or $|N_G(y) \setminus \{x\}| = 2$ and $|N_G(x) \setminus \{y\}| = 1$. Therefore, (ii) holds.

For the converse, suppose there exist distinct vertices $x, y \in V(G)$ satisfying (i) or (ii). Let $W = \{x, y\}$, then W is a minimum resolving set of G. Therefore, $\dim(G) = 2$. □

Theorem 2.4. *If W is a locating set of G, then it is a resolving set.*

Proof. Let $W = \{w_1, w_2, \ldots, w_n\}$ be a locating set of G and $x, y \in V(G) \setminus W$ with $x \neq y$, then

$$r_G(x/W) = \big(d_G(x, w_1), d_G(x, w_2), \ldots, d_G(x, w_n)\big) \quad \text{and}$$
$$r_G(y/W) = \big(d_G(y, w_1), d_G(y, w_2), \ldots, d_G(y, w_n)\big).$$

Since W is a locating set, $N_G(x) \cap W \neq N_G(y) \cap W$. Thus, there exists $w_i \in W$ such that $w_i \in N_G(x) \setminus N_G(y)$ or $w_i \in N_G(y) \setminus N_G(x)$. Hence, $d_G(x, w_i) = 1$ but $d_G(y, w_i) \neq 1$ or $d_G(y, w_i) = 1$ but $d_G(x, w_i) \neq 1$. Hence, $r_G(x/W) \neq r_G(y/W)$. Therefore, W is a resolving set of G. □

Remark 2.2. (i) Let G be a connected graph and $W \subseteq V(G)$, then for any two distinct vertices $x, y \in V(G) \setminus W$ with $N_G(x) \cap W \neq N_G(y) \cap W$, we have $r_G(x/W) \neq r_G(y/W)$.

(ii) The converse of Theorem 2.4 is not true.

To see this, consider the graph G in Figure 1. Let $W = \{w_1, w_2, w_3, w_4\}$, then $r_G(x/W) = (1, 1, 1, 2)$, $r_G(y/W) = (1, 1, 1, 4)$ and $r_G(z/W) = (2, 2, 2, 1)$. Thus, W is a resolving set but it is not a locating set since $N_G(x) \cap W = N_G(y) \cap W$.

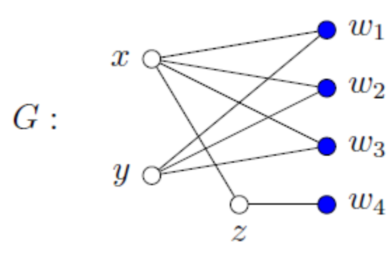

Figure 1: A resolving set $W = \{w_1, w_2, w_3, w_4\}$ of a graph G but not a locating set of G

3. Resolving sets in the join of graphs

Remark 3.1. Let G and H be non-trivial connected graphs and $W \subseteq V(G+H)$. If $W \subseteq V(G)$ or $W \subseteq V(H)$, then W is not a resolving set of $G+H$.

Theorem 3.1. *Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving set of $G+H$ if and only if W is a locating set of $G+H$.*

Proof. If W is a locating set of $G+H$, then it is a resolving set by Theorem 2.4.

Conversely, suppose W is a resolving set of $G+H$. Let $W_G = W \cap V(G)$ and $W_H = W \cap V(H)$, then $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$. If $W_G = \emptyset$, then $W = W_H$. Hence, $W \subseteq V(H)$. By Remark 2.2, W is not a resolving set of $G+H$, a contradiction. Thus, $W_G \neq \emptyset$. Similarly, $W_H \neq \emptyset$. Let $W_G = \{w_1, w_2, \ldots, w_n\}$ and $W_H = \{w_{n+1}, w_{n+2}, \ldots, w_m\}$. Let $x, y \in V(G) \setminus W_G$ with $x \neq y$. We claim that $N_G(x) \cap W_G \neq N_G(y) \cap W_G$. Suppose $N_G(x) \cap W_G = N_G(y) \cap W_G = S$. Let $S = \{w_{j_1}, w_{j_2}, \ldots, w_{j_k}\}$, $j_i \in \{1, 2, \ldots, n\}$, for all $i = 1, 2, \ldots, k$, then $d_G(x, s) = d_G(y, s) = 1$, for all $s \in S$. Hence, $d_{G+H}(x, w) = d_{G+H}(y, w) = 1$, for all $w \in S \cup W_H$ and $d_{G+H}(x, v) = d_{G+H}(y, v) = 2$, for all $v \in W_G \setminus S$. It follows that $r_{G+H}(x/W) = r_{G+H}(y/W)$, a contradiction, since W is a resolving set of $G+H$. Thus, W_G is a locating set of G. Similarly, W_H is a locating set of H. Suppose there exists $u \in V(G) \setminus W_G$ and $v \in V(H) \setminus W_H$ such that $N_G(u) \cap W_G = W_G$ and $N_H(v) \cap W_H = W_H$. Thus, W_G or W_H is a strictly locating set. □

Theorem 3.2 ([1])**.** *Let G and H be non-trivial connected graphs. A set $S \subseteq V(G+H)$ is a locating set in $G+H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are locating sets in G and H, respectively, where S_1 or S_2 is a strictly locating set.*

The next result follows immediately from Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let G and H be non-trivial connected graphs. A set $W \subseteq V(G+H)$ is a resolving set of $G+H$ if and only if $W = W_G \cup W_H$ where $W_G \subseteq V(G)$ and $W_H \subseteq V(H)$ are locating sets of G and H, respectively, where W_G or W_H is a strictly locating set.*

Corollary 3.1 ([1])**.** *Let G and H be connected non-trivial graphs, then*

$$\ln(G+H) = \min\{sln(H) + ln(G), sln(G) + \ln(H)\}.$$

Corollary 3.2 ([1])**.** *Let G be a non-trivial connected graph and let K_n be a complete graph of order $n \geq 2$, then $\ln(G+K_n) = sln(G) + n - 1$.*

The next results follow immediately from Corollary 3.1, Corollary 3.2 and Theorem 2.4.

Corollary 3.3. *Let G and H be connected non-trivial graphs, then*

$$dim(G+H) = \min\{sln(H) + \ln(G), sln(G) + \ln(H)\}.$$

Corollary 3.4. *Let G be a non-trivial connected graph and let K_n be a complete graph of order $n \geq 2$, then $dim(G + K_n) = sln(G) + n - 1$.*

Theorem 3.4. *Let H be a non-trivial connected graph, then $W \subseteq V(H + K_1)$ is a resolving set of $H + K_1$ if and only if W is a locating set of $H + K_1$.*

Proof. Suppose that W is a locating set of $H + K_1$, then by Theorem 2.4, W is a resolving set of $H + K_1$.

For the converse, suppose that W is a resolving set of $H + K_1$ and let $K_1 = \langle v \rangle$. If $v \notin W$, then $W \subseteq V(H)$ is a locating set of H. Suppose there exists $u \in V(H) \setminus W$ such that $N_H(u) \cap W = W$, then $N_{H+K_1}(u) \cap W = W = N_{H+K_1}(v) \cap W$. Hence, W is a strictly locating set of H. Next, suppose that $W = \{v\} \cup W_H$, where $W_H = V(H) \cap W$, then $W_H \neq \emptyset$. Let $W_H = \{w_1, w_2, \ldots, w_n\}$. Let $w, z \in V(H) \setminus W_H$ with $w \neq z$. We claim that

$$N_H(z) \cap W_H \neq N_H(w) \cap W_H.$$

Suppose $(N_H(z) \cap W_H) \cup \{v\} = N_{H+K_1}(z) \cap W = N_{H+K_1}(w) \cap W = (N_H(w) \cap W_H) \cup \{v\} = S$. Let $S = \{v, w_{j_1}, w_{j_2}, \ldots, w_{j_k}\}$, $j_i \in \{1, 2, \ldots, n\}$, for all $i = 1, 2, \ldots, k$, then $d_G(w, s) = d_G(z, s) = 1$, for all $s \in S$. Hence, $d_{H+K_1}(w, y) = d_{H+K_1}(z, y) = 1$, for all $y \in S \cup W_H$ and $d_{H+K_1}(w, x) = d_{G+H}(z, x) = 2$, for all $x \in W \setminus S$. It follows that $r_{H+K_1}(w/W) = r_{H+K_1}(z/W)$, a contradiction since W is a resolving set of $H + K_1$. Hence, W_H is a locating set of H. □

Theorem 3.5 ([1])**.** *Let H be a non-trivial connected graph and let $K_1 = \langle v \rangle$, then $S \subseteq V(H + K_1)$ is a locating set in $H + K_1$ if and only if either $v \notin S$ and S is a strictly locating set in H or $S = \{v\} \cup S_1$, where S_1 is a locating set in H.*

Corollary 3.5 ([1])**.** *Let H be a non-trivial connected graph, then* $\ln(H+K_1) = sln(H)$.

The next results follow immediately from Theorem 2.4, Corollary 3.5 and Theorem 3.5.

Theorem 3.6. *Let H be a non-trivial connected graph and let $K_1 = \langle v \rangle$, then $W \subseteq V(H + K_1)$ is a resolving set of $H + K_1$ if and only if either $v \notin W$ and W is a strictly locating set of H or $W = \{v\} \cup W_H$, where W_H is a locating set of H.*

Corollary 3.6. *Let H be a non-trivial connected graph, then $dim(H + K_1) = sln(H)$.*

4. Resolving sets in the corona of graphs

Remark 4.1. Let $v \in V(G)$. For every $x, y \in V(H^v)$, $d_{G\circ H}(x,w) = d_{G\circ H)}(y,w)$, and $d_{G\circ H}(x,w) \neq d_{G\circ H}(v,w)$ for every $w \in V(G\circ H) \setminus V(H^v)$.

Remark 4.2. Let G and H be non-trivial connected graphs, $W \subseteq V(G\circ H)$ and $A_v = V(H^v) \cap W$ where $v \in V(G)$. For each $x \in V(H^v) \setminus A_v$ and $z \in A_v$,

$$d_{G\circ H}(x,z) = \begin{cases} 1, & \text{if } z \in N_{H^v}(x) \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Let G and H be non-trivial connected graphs, then $W \subseteq V(G \circ H)$ is a resolving set of $G\circ H$ if and only if $W \cap V(H^v) \neq \emptyset$, for all $v \in V(G)$ and $W = A \cup B$, where $A \subseteq V(G)$ and*

$$B = \cup\{B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v\}.$$

Proof. Let W be a resolving set of $G\circ H$. Suppose there exists $u \in V(G)$ such that $W \cap V(H^u) = \emptyset$. By Remark 4.1, $d_{G\circ H}(p,x) = d_{G\circ H}(q,x)$ for every distinct vertices $p, q \in V(H^u)$ and for each $x \in V(G\circ H) \setminus V(H^u)$. Thus, $r_{G\circ H}(p/W) = r_{G\circ H}(q/W)$, a contradiction. Hence, $W \cap V(H^v) \neq \emptyset$ for each $v \in V(G)$. Let $A = V(G) \cap W$, $v \in V(G)$ and $B_v = W \cap V(H^v)$. Let $x, y \in V(H^v) \setminus B_v$ with $x \neq y$, then $x, y \notin W$. Suppose $N_{H^v}(x) \cap B_v = N_{H^v}(y) \cap B_v$, then $d_{G\circ H}(x,p) = 1 = d_{G\circ H}(y,p)$ for each $p \in N_{H^v}(x) \cap B_v$ and $d_{G\circ H}(x,q) = 2 = d_{G\circ H}(y,q)$ for each $q \in B_v \setminus (N_{H^v}(x) \cap B_v)$ by Remark 4.2. By Remark 4.1, it follows that $r_{G\circ H}(x/W) = r_{G\circ H}(y/W)$, a contradiction since W is a resolving set of $G\circ H$. Hence, $N_{H^v}(x) \cap B^v \neq N_{H^v}(y) \cap B_v$. Hence, B_v is a locating set of H^v. Now, let

$$B = \bigcup\{B_v : v \in V(G) \text{ and } B_v \text{ is a locating set of } H^v\},$$

then $W = A \cup B$.

For the converse, suppose $W = A \cup B$, where A and B are the sets possesing the properties described. Let $x, y \in V(G\circ H)\setminus W$ with $x \neq y$, and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$. Suppose $u = v$. Since B_v is a locating set of H^v, $d_{G\circ H)}(x,p) \neq d_{G\circ H}(y,p)$ for some $p \in B_v$ by Remark 4.2 when $x, y \in V(H^v) \setminus B_v$. If $v \notin A$ and $x = v$ or $y \notin A$, then $d_{G\circ H}(x,w) \neq d_{G\circ H}(y,w)$, for all $w \in A$ by Remark 4.1. Thus, $r_{G\circ H}(x/W) \neq r_{G\circ H}(y/W)$.

Suppose now that $u \neq v$. Since $B_u \neq \emptyset$ and $B_v \neq \emptyset$,

$$d_{G\circ H}(x,p) \neq d_{G\circ H}(y,p), \text{ for all } p \in B_u \text{ or } p \in B_v.$$

Hence, $r_{G\circ H}(x/W) \neq r_{G\circ H}(y/W)$. Therefore, W is a resolving set of $G\circ H$. □

Corollary 4.1. *Let G and H be non-trivial connected graphs, where $|V(G)| = m$, then $dim(G\circ H) = m \cdot \ln(H)$.*

Proof. Let W be a minimum resolving set of $G \circ H$, then $W = A \cup B$ where A and B are the sets described in Theorem 4.1. Hence,

$$\begin{aligned}\dim(G \circ H) = |W| &= |A| + |B| \\ &\geq |A| + |V(G)| \cdot \ln(H) \\ &= |A| + m \cdot \ln(H) \\ &\geq m \cdot \ln(H).\end{aligned}$$

Next, let F be a minimum locating set of H. For each $v \in V(G)$, pick $F_v \subseteq V(H^v)$ with $\langle F_v \rangle \cong \langle F \rangle$, then $S = \bigcup_{v \in V(G)} F_v$ is a resolving set of $G \circ H$ by Theorem 4.1. Hence,

$$\dim(G \circ H) \leq |S| = |\bigcup_{v \in V(G)} F_v| = m \cdot |F_v| = m \cdot |F| = m \cdot \ln(H).$$

Therefore, $\dim(G \circ H) = m \cdot \ln(H)$. □

5. Resolving sets in the lexicographic product of graphs

Remark 5.1. Let G and H be non-trivial connected graphs and $W = \bigcup_{x \in S} \big(\{x\} \times T_x \big)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

(i) If $(x, p) \in V\big(G[H]\big) \backslash W$, then $r_{G[H]}\big((x, p)/W\big) = (p_1, p_2, \ldots, p_r, c_1, c_2, \ldots, c_t)$.

where $p_i = d_{G[H]}\big((x, p), (x, q_i)\big),\ q_i \in T_x,\ r = |T_x|$ and
$c_j = d_{G[H]}\big((x, p), (x_j, y_j)\big),\ x_j \in S \setminus \{x\},\ y_j \in T_{x_j},\ t = |S| - 1.$

Thus,

$$p_i = \begin{cases} 1, & \text{if } q_i \in N_H(p) \cap T_x \\ 2, & \text{otherwise} \end{cases}$$

and $c_j = d_G(x, x_j)$ for each $x_j \in S \setminus \{x\}$ and for any $y_j \in T_{x_j}$.

(ii) If $(x, p), (x, q) \in V\big(G[H]\big) \setminus W,\ p \neq q$ and $N_H(p) \cap T_x = N_H(q) \cap T_x$, then $r_{G[H]}\big((x, p)/W\big) = r_{G[H]}\big((x, q)/W\big)$.

(iii) If $(x, p), (y, q) \in V\big(G[H]\big) \setminus W$, where $x, y \in E(G)$ such that $N_H(p) \cap T_x = T_x$ and $N_H(q) \cap T_y = T_y$, then $r_{G[H]}\big((x, p)/W\big) = r_{G[H]}\big((y, q)/W\big)$.

Theorem 5.1 ([4]). *Let G and H be two non-trivial graphs such that G is connected, then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.*

(i) $N_{G[H]}(a, b) = \big(\{a\} \times N_H \{b\} \big) \bigcup \big\{ N_G \{a\} \times V(H) \big\}$

(ii) $d_{G[H]}\big((a, b), (c, d)\big) = d_G(a, c)$

(*iii*) $d_{G[H]}\big((a,b),(a,d)\big) = min\{d_H(b,d), 2\}$.

Theorem 5.2. *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $W = \bigcup_{x\in S}\big[\{x\} \times T_x\big]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving set of $G[H]$ if and only if W is a locating set of $G[H]$.*

Proof. Suppose that W is a locating set of $G[H]$. By Theorem 2.4, W is a resolving set of $G[H]$.

Conversely, suppose that W is a resolving set of $G[H]$. Suppose there exists $x \in V(G) \setminus S$. Pick $a, b \in V(H)$, where $a \neq b$, then $(x,a), (x,b) \notin W$ and $(x,a) \neq (x,b)$. Since $\{(x,c) : c \in V(H)\} \cap W = \emptyset$ and $d_{G[H]}\big((x,a),(y,d)\big) = d_G(x,y) = d_{G[H]}\big((x,b),(y,d)\big)$ for any $y \in V(G) \setminus \{x\}$ and for any $d \in V(H)$ by Theorem 5.1(*ii*), $r_{G[H]}\big((x,a)/W\big) = r_{G[H]}\big((x,b)/W\big)$. This implies that W is not a resolving set of $G[H]$, contrary to our assumption. Therefore, $S = V(G)$.

Now, let $x \in V(G)$ and suppose that T_x is not a locating set of H, then there exist distinct vertices p and q in $V(H) \setminus T_x$ such that $N_H(p) \cap T_x = N_H(q) \cap T_x$. By Remark 5.1(*ii*), $r_{G[H]}\big((x,p)/W\big) = r_{G[H]}\big((x,q)/W\big)$, a contradiction to the assumption. Thus, T_x is a locating set of H.

Let x and y be adjacent vertices of G with $N_G[x] = N_G[y]$. Suppose that T_x and T_y are not strictly locating, then there exist $c \in V(H) \setminus T_x$ and $d \in V(H) \setminus T_y$ such that $N_H(c) \cap T_x = T_x$ and $N_H(d) \cap T_y = T_y$. By Remark 5.1(*iii*), $r_{G[H]}\big((x,c)/W\big) = r_{G[H]}\big((y,d)/W\big)$, a contradiction to the assumption that W is a resolving set of $G[H]$. Therefore, T_x or T_y is strictly locating of H.

Let x and y be nonadjacent vertices of G such that $N_G(x) = N_G(y)$. Suppose T_x and T_y are not dominating sets of H, then there exist $a \in V(H) \setminus T_x$, $c \in V(H) \setminus T_y$ such that $ab \notin E(H)$, for all $b \in T_x$ and $cd \notin E(H)$, for all $d \in T_y$. It follows that $(x,a) \notin W$ and $(y,c) \notin W$ and $N_{G[H]}\big((x,a)\big) \cap W = \cup\{\{z\} \times T_z : z \in N_G(x)\}$. Since $N_G(x) = N_G(y)$, it follows that $\cup\big\{\{z\} \times T_z : z \in N_G(x)\big\} = N_{G[H]}\big((y,c)\big) \cap W$, a contradiction to the assumption. Thus, T_x or T_y is a dominating set of $G[H]$. Therefore, W is a locating set of $G[H]$. □

Theorem 5.3 ([1])**.** *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $C = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ is a locating set of $G[H]$ if and only if*

(*i*) *$S = V(G)$;*

(*ii*) *T_x is a locating set of H for every $x \in V(G)$;*

(*iii*) *T_x or T_y is strictly locating of H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$.*

(*iv*) *T_x or T_y is (locating) dominating of H whenever x and y are nonadjacent vertices of G with $N_G(x) = N_G(y)$.*

The next result follows immediately from Theorem 5.2 and Theorem 5.3.

Theorem 5.4. *Let G and H be non-trivial connected graphs with $\Delta(H) \leq |V(H)| - 2$, then $W = \bigcup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a resolving set of $G[H]$ if and only if*

(*i*) $S = V(G)$;

(*ii*) T_x *is a locating set of* H *for every* $x \in V(G)$;

(*iii*) T_x *or* T_y *is a strictly locating set of* H *whenever* x *and* y *are adjacent vertices of* G *with* $N_G[x] = N_G[y]$.

(*iv*) T_x *or* T_y *is a (locating) dominating set of* H *whenever* x *and* y *are non-adjacent vertices of* G *with* $N_G(x) = N_G(y)$.

The following is a direct consequence of Theorem 5.4.

Corollary 5.1. *Let G be a connected totally point determining graph and let H be a non-trivial connected graph, then $W = \bigcup_{x \in S}[\{x\} \times T_x]$ is a minimum resolving set of $G[H]$ if and only if $S = V(G)$ and T_x is a minimum locating set of H for every $x \in V(G)$.*

Corollary 5.2. *Let G be a connected totally point determining graph and let H be a non-trivial connected graph, then* $dim(G[H]) = |V(G)| \cdot \ln(H)$.

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