The inverse exponential Rayleigh distribution and related concepts

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Abstract. The main aim of this paper is to introduce a new two-parameter distribution called the inverse exponential Rayleigh distribution (IERD) with the investigated of its statistical properties. We introduce some mathematical complexity of statistical properties such as the moment function, moment generating function, factorial moments generating function, quantity function, skewness, and Kurtosis. We utilize Taylor series expansion to obtain these statistical properties. The process establishes the new distribution relied mainly on the survival functions of both the exponential and Rayleigh distributions. For the application, real data were applied to compare the new distribution with some other distributions using the Akaike information criterion (AIC). It has been indicated that the suggested method is more efficient and high accuracy by the convergence of the real data.

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1. Introduction

It is well known that there are countless distributions in the statistical field, the development and complexity that occur in the information extracted from the statistical data. Especially in the areas that are concerned with the survival and reliability analysis necessitated the existence of new distributions to address the problems in the data that could not be handled by existing statistical distributions. This research was based on a new approach in developing a mix of distributions for use in statistical techniques, survival, and reliability studies. This approach was developed by Gauss M. Cordeiro [1], he introduced the exponential–Weibull distribution, where the mix between the exponential and Weibull distributions was made according to survival functions of both distributions. Suleman N. [2], using this technique to developed serial Weibull Rayleigh distribution. Gadde S. and Sauda. M. [3] used generalization technical to present exponentiated inverse Rayleigh distribution which is a generalized form of inverse Rayleigh distribution. Saralees N. [4] developed a new method using the generalized form to introduce the exponentiation type distributions. Muhammad S. Khan [5] studied two parameters of the modified inverse Rayleigh distribution which is a generalizing of inverse Rayleigh distribution. Pelumi E. and el at [6]used the inverse distribution technic to develop The Gompertz inverse exponential distribution. Maha A. [7] proposed a new modal called inverse Weibull inverse exponential distribution as a special case of inverse Weibull-G distribution. Mayssa J. Mohammed [8] proposed a new mixture distribution based on the same approach and developed three distributions exponential Weibull, exponential Rayleigh, and new mixture distributions.

In this paper lifetime distribution called inverse exponential Rayleigh distribution (IERD) was proposed and its new distribution system could be used in both survival and reliability analysis. Suppose that a random variable $X > 0$ distributed as an exponential Rayleigh distribution with two-scale parameters $\gamma > 0$ and $\beta > 0$, the cumulative and density functions of $X > 0$ are defined as follows:

$$F(x) = 1 - \exp\left(-(\gamma x + \frac{\beta}{2}x^2)\right)$$

and

$$f(x) = (\gamma + \beta x)\exp\left(-(\gamma x + \frac{\beta}{2}x^2)\right).$$

Generally, for any positive random variable X, the inverse distribution could be found for the random variable $Y = 1/X$, with the general form of cumulative function $G(y)$ and density function $g(y)$:

$$G(y){=}P(Y \leq y){=}P(X{\geq}1/y){=}1{-}P(X{<}1/y){=}1{-}F(1/y)g(y){=}1/y^2 f(1/y).$$

On the possibility of available according to the above, it will define the inverse system with cumulative and probability density functions of random variable $y > 0$ as follows:

$$G(y) = \exp\left(-(\frac{\gamma}{y} + \frac{\beta}{2y^2})\right), \quad y > 0 \tag{1}$$

$$g(y) = \frac{1}{y^2}(\gamma + \frac{\beta}{y}) \exp\left(-(\frac{\Upsilon}{y} + \frac{\beta}{2y^2})\right), \quad y > 0 \tag{2}$$

The new distribution with two positive scale parameters γ and β called inverse exponential Rayleigh (IERD). The corresponding survival (reliability) function of IERD could be given by:

$$S(y) = 1 - \exp\left(-(\frac{\gamma}{y} + \frac{\beta}{2y^2})\right), \quad y > 0. \tag{3}$$

The hazard function can be taken the form:

$$h(y) = \frac{\gamma + \frac{\beta}{y}}{y^2\left(\exp\left(-(\frac{\gamma}{y} + \frac{\beta}{2y^2})\right) - 1\right)}, \quad y > 0. \tag{4}$$

As a result, one can obtain two special cases for the IERD when $\gamma = 0$, we have the inverse Rayleigh distribution and when $\beta = 0$, we get the inverse exponential distribution. Figs.1 and 2 discuss the shape of probability density and hazard functions both of them in two cases according to IERD scale parameters.

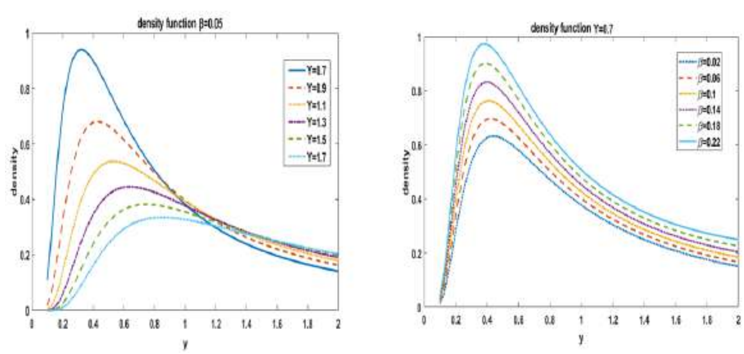

Figure 1: The first plot indicates the density function of IERD with $\beta = 0.05$ while the second represents density function of IERD with $\gamma = 4$

The rest of the paper is devoted as follows: Section 2 indicates the results of this effort, including probability density and hazard functions; Section 3 presents the application of the IERD and Section 4 concludes our efforts.

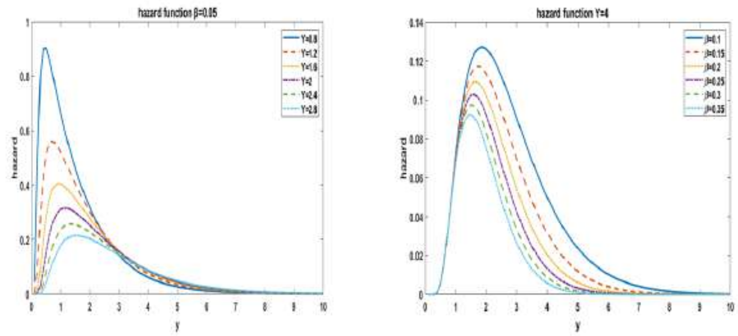

Figure 2: The first plot indicates the hazard function of IERD with $\beta = 0.05$ while the second represents the hazard function of IERD with $\gamma = 4$

2. Results

Some statistical properties of IERD are imposed in this section.

2.1 The moments

According to the r-th non-central moment general form, the r-th moment of the IERD could be written as follows:

$$\begin{aligned}
M_r' = E(y^r) &= \int_0^\infty y^r g(y) dy \\
&= \int_0^\infty y^{r-2}(\gamma + \beta/y) e^{-(\gamma/y+\beta/(2y^2))} dy \\
&= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \int_0^\infty y^{r-k-2}(\gamma + \beta/y) e^{-(\beta/(2y^2))} dy \qquad (5)\\
&= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^\infty \gamma y^{(r-k-2)} e^{(-(\beta/(2y^2)))} dy + \int_0^\infty \beta y^{(r-k-3)} e^{(-(\beta/(2y^2)))} dy \right).
\end{aligned}$$

Define two integrals as follows:

$$\begin{aligned}
I(r,\gamma,\beta) &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \int_0^\infty y^{(r-k-2)} e^{(-(\beta/(2y^2)))} dy \\
&= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(r-k-1))} 2^{(-1/2(r-k-1))} \Gamma\left(-\frac{(r-k-1)}{2}, \frac{\beta}{2y^2} \right)
\end{aligned}$$

and

$$I(r-1,\gamma,\beta) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \int_0^\infty y^{(r-k-3)} e^{(-(\beta/(2y^2)))} dy$$

$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(r-k-2))} 2^{(-1/2(r-k-2))} \Gamma\left(-\frac{(r-k-2)}{2}, \frac{\beta}{2y^2}\right).$$

Then, we have

$$M_r^{'} = \gamma \cdot I(r, \gamma, \beta) + \beta \cdot I(r-1, \gamma, \beta). \tag{6}$$

2.2 The mean and the variance

The mean and the variance of the IERD could be found directly form equation (6) by specifying (r) to equal 1 and 2 respectively, as follows:

$$E(y) = \gamma \cdot I(1, \gamma, \beta) + \beta \cdot I(\gamma, \beta), \tag{7}$$

$$E(y^2) = \gamma \cdot I(2, \Upsilon, \beta) + \beta \cdot I(1, \gamma, \beta) \tag{8}$$

while the variance is formulated as follows :

$$Var(y) = (\gamma \cdot I(2, \gamma, \beta) + \beta \cdot I(1, \gamma, \beta)) - (\gamma \cdot I(1, \gamma, \beta) + \beta \cdot I(\gamma, \beta))^2 \tag{9}$$

With the following details:

$$I(\gamma, \beta) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(-k-1))} 2^{(-1/2(-k-1))} \Gamma\left(\frac{(k+1)}{2}, \frac{\beta}{2y^2}\right),$$

$$I(1, \gamma, \beta) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(1/2(-k))} 2^{(-1/2(-k))} \Gamma\left(\frac{k}{2}, \frac{\beta}{2y^2}\right),$$

$$I(2, \gamma, \beta) = \sum_{k=0}^{\infty} \frac{-\gamma^k}{k!} \beta^{1/2(1-k)} 2^{-1/2(1-k)} \Gamma\left(\frac{k-1}{2}, \frac{\beta}{2y^2}\right).$$

2.3 Moment generating function

The moment generating function of IERD which is denoted by $M(t)$ can be expressed mathematically as follows:

$$M(t) = E(e^{ty}) = \int_0^{\infty} e^{ty} g(y) dy$$

$$= \int_0^{\infty} \frac{e^{ty}}{y^2} (\gamma + \beta/y) e^{(-(\gamma/y + \beta/(2y^2)))} dy$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(-\gamma)^k}{k!} y^{(-(k+2))} (\gamma + \beta/y) e^{(-(\beta/(2y^2) - ty))} dy$$

$$= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \int_0^{\infty} y^{(-(k+n+2))} (\gamma + \beta/y) e^{(-\beta/(2y^2))} dy \tag{10}$$

$$= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} \gamma y^{(-(k+n+2))} e^{(-\beta/(2y^2))} dy \right.$$
$$\left. + \int_0^{\infty} \beta y^{(-(k+n+3))} e^{(-\beta/(2y^2))} dy \right).$$

Define the following integral equations:

$$I(-n,\gamma,\beta) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} y^{(-(k+n+2))} e^{(-\beta/(2y^2))} dy \right)$$
$$= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \beta^{(-1/2(n+k+1))} 2^{(1/2(n+k-1))} \Gamma\left(\frac{(n+k+1)}{2}, \frac{\beta}{2y^2} \right),$$

and

$$I(-(n+1),\gamma,\beta) = \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \left(\int_0^{\infty} y^{(-(k+n+3))} e^{(-\beta/(2y^2))} dy \right)$$
$$= \sum_{n=0}^{\infty} n! \sum_{k=0}^{\infty} \frac{(-\gamma^k)}{k!} \beta^{(-1/2(n+k+2))} 2^{(1/2(n+k-2))} \Gamma\left(\frac{(n+k+2)}{2}, \frac{\beta}{2y^2} \right). \quad (11)$$

Consequently, we obtain the formula

$$M(t) = \Upsilon \cdot I(-n,\gamma,\beta) + \beta \cdot I(-(n+1),\gamma,\beta). \quad (12)$$

2.4 Factorial moments generating function

The factorial moments generating function of IERD denoted by $M_y(t)$ is given by:

$$M_y(t) = E(t^y) = \int_0^{\infty} t^y g(y) dy$$
$$= \int_0^{\infty} \frac{t^y}{y^2} (\gamma + \beta/y) e^{(-(\gamma/y+\beta/(2y^2)))} dy. \quad (13)$$

By using

$$I(n,\gamma,\beta)$$
$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{n=0}^{\infty} \frac{(-\ln t)^n}{n!} \beta^{(1/2(n-k-1))} 2^{(-1/2(n-k+1))} \Gamma(\frac{-((n-k-1))}{2}, \frac{\beta}{2y^2}),$$

and

$$I(n-1,\gamma,\beta)$$
$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{n=0}^{\infty} \frac{(-\ln t)^n}{n!} \beta^{(1/2(n-k-2))} 2^{(-1/2(n-k))} \Gamma(\frac{-((n-k-2))}{2}, \frac{\beta}{2y^2}),$$

we have

$$M_y(t) = \gamma \cdot I(n,\gamma,\beta) + \beta \cdot I(n-1,\gamma,\beta). \quad (14)$$

2.5 Quantifiable function (QF)

The general form of QF is

$$\frac{\gamma}{y}+\frac{\beta}{2y^2}=-\ln(V),\quad \ln(V)y^2+\gamma y+2\beta=0. \tag{15}$$

Solving Eq.(15), we have $y=\dfrac{-\gamma\pm\sqrt{\gamma^2-8\beta\ln(V)}}{2\ln(V)}$. But $y>0$ then we get the positive root only.

2.6 Skewness and Kurtosis

Skewness in statistics, which can be defined mathematically as a ratio that depends on its numerator and its denominator on the moment such that:

$$C.S=\frac{M_3'}{(M_2')^{(3/2)}}, \tag{16}$$

and for Kurtosis, we ahve

$$C.K=\frac{M_4'}{(M_2')^4}, \tag{17}$$

where

$$M_r'=\gamma\cdot I(r,\gamma,\beta)+\beta\cdot I(r-1,\gamma,\beta).$$

2.7 Characteristic function

For any positive real value random variable Y the characteristic function of the inverse exponential Rayleigh distribution is denoted by φ_Y and formed as the follows:

$$\begin{aligned}\varphi_Y(it)&=E(e^{ity})\\&=\int_0^\infty e^{ity}g(y)dy\\&=\int_0^\infty \frac{e^{ity}}{y^2}(\gamma+\beta/y)e^{(-(\gamma/y+\beta/(2y^2)))}dy\\&=\sum_{k=0}^\infty\frac{(-\gamma)^k}{k!}\int_0^\infty e^{ity}y^{-2-k}(\gamma+\beta/y)e^{(-\beta/(2y^2))}dy\\&=\sum_{k=0}^\infty\frac{(-\gamma)^k}{k!}\sum_{m=0}^\infty\frac{(it)^m}{m!}\int_0^\infty y^{m-2-k}(\gamma+\beta/y)e^{(-\beta/(2y^2))}dy.\end{aligned} \tag{18}$$

By letting

$$I(m,\gamma,\beta)=\sum_{k=0}^\infty\frac{(-\gamma)^k}{k!}\sum_{m=0}^\infty\frac{it^m}{m!}\frac{(-\gamma)^k}{k!}\int_0^\infty y^{m-2-k}e^{(-\beta/(2y^2))}dy$$

$$= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} 2^{-1/2(m-k+1)} \Gamma\left(\frac{-(m-k-1)}{2}, \frac{\beta}{2y^2}\right)$$

and

$$\begin{aligned} I(m-1,\gamma,\beta) &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{it^m}{m!} \frac{(-\gamma)^k}{k!} \int_0^{\infty} y^{m-3-k} e^{(-\beta/(2y^2))} dy \\ &= \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \sum_{m=0}^{\infty} \frac{(it)^m}{m!} 2^{-1/2(m-k)} \Gamma\left(\frac{-(m-k-3)}{2}, \frac{\beta}{2y^2}\right) \end{aligned}$$

we receive

$$\varphi_Y(it) = \gamma \cdot I(m,\gamma,\beta) + \beta \cdot I(m-1,\gamma,\beta). \tag{19}$$

2.8 MLE estimation

This section includes estimating the parameters of the inverse exponential Rayleigh distribution to estimate the two parameters γ and β. On the assumption that the size of the sample is n with the considering that $y_1, y_2, \ldots, y_n$ are values of the density distribution of the inverse exponential Rayleigh distribution, then the log-likelihood is given as follows:

$$\ln(L(y_1, y_2, ..., y_n; \gamma, \beta)) = \sum_{i=1}^{n} \ln\left(\frac{1}{y_i^2}(\gamma + \frac{\beta}{y_i})\right) - \sum_{i=1}^{n}\left(\frac{\gamma}{y_i} + \frac{\beta}{2y_i^2}\right), \tag{20}$$

$$D_\gamma \ln(L(y_1, y_2, ..., y_n; \gamma, \beta)) = \sum_{i=1}^{n} \frac{1}{((\gamma + \beta/y_i))} - \sum_{i=1}^{n} \frac{1}{y_i} \tag{21}$$

$$D_\beta \ln(L(y_1, y_2, ..., y_n; \gamma, \beta)) = \sum_{i=1}^{n} \frac{1}{(y_i(\gamma + \beta/y_i))} - \sum_{i=1}^{n} \frac{1}{2y_i^2}. \tag{22}$$

To obtain the results that represent the parameters estimated by maximum likelihood estimation method (MLE), numerical methods are applied to solve equations Eq.(21) and Eq.(22) after being equal to zero.

3. Application

The application in this section relies on the use of the information criterion to compare the results from the inverse exponential Rayleigh distribution with some other distributions that are used to process data for cancer patients from the Iraqi Yarmouk Hospital. The data give information about the failure (death) times for bladder cancer patients. In view of the information provided in Table 1, inverse exponential Rayleigh distribution showed lower criterion values than other distributions through information criterion applications such as Akaike (AIC), corrected Akaike (AIC$_c$), and Bayesian (BIC) information criterion.

Table 1: MLE parameter estimation and information criterion results

Model	MLE	Std. Error	AIC	AIC_c	BIC
IER	$\gamma = 2.2401$ $\beta = 0.3398$	0.2411 0.6502	162.6252	164.3662	160.6252
exponential Rayleigh	$\gamma = 0.0001805$ $\beta = 0.000032$	0.00017885 0.0000232	168.6439	170.3582	166.6439
exponential Weibull	$\gamma = 0.0000979$ $\beta = 0.0007814$	0.00003901 0.00021859	203.7306	205.449	201.7306
Rayleigh Weibull	$\gamma = 0.000184$ $\beta = 0.0005473$	0.000173 0.0005274	164.7465	166.461	162.7465
Weibull	$\gamma = 35.8407$	0.1593	180.9829	181.4829	172.9829

Depending on the results in the Table 1, it can be noted that the new distribution compared to other distributions showed good fit in terms of information criterion. This would prove that the new distribution can be relied upon to process and obtain statistical information from the data. From Fig.3 it can be seen that the survival function of inverse exponential Rayleigh distribution of the real data set for bladder cancer patients provides a clear fit for the survival function estimated by the Kaplan-Meier estimate. Through this, the compatibility with the data shown by the plot, the IER distribution can be adopted for of data modeling.

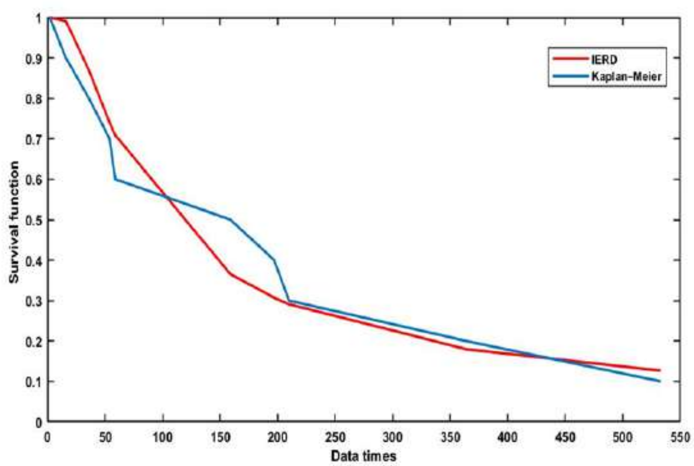

Figure 3: Estimated survival function of IRE and empirical survival function of Kaplan-Meier

4. Conclusion

This study introduced a new probability distribution called inverse exponential Rayleigh distribution, where a new approach was adopted that mixes the Exponential and Rayleigh distributions and then finds the inverted distribution of the mixing. Statistical properties derived distribution using Taylor series. In terms of application, based on information criterion, the inverse exponential Rayleigh distribution showed good fit against other distributions.

For future studies, using statistical simulation technique to generate data of random variables and using these simulation data in estimating the IER distribution parameters by applying Bayesian and classical methods and comparing the results of the estimation. For survival analysis, using real data of censoring data (type I, type II, or left and right) to study the performance of both survival (failure) and hazard functions of IER distribution.

References

[1] G.M. Cordeiro, E.M.M. Ortega, A.J. Lemonte, *The exponential-Weibull lifetime distribution*, J. Stat. Comput. Simul., 84 (2014), 2592-606.

[2] S. Nasiru, *Serial Weibull Rayleigh distribution: theory and application*, Int. J. Comput. Sci. Math., 7 (2016), 239-44.

[3] G.S. Rao, S. Mbwambo, *Exponentiated inverse Rayleigh distribution and an application to coating weights of iron sheets data*, J. Probab. Stat., 2019, 2019.

[4] S. Nadarajah, S. Kotz, *The exponentiated type distributions*, Acta Appl. Math., 92 (2006), 97-111.

[5] M.S. Khan, *Modified inverse Rayleigh distribution*, Int. J. Comput. Appl., 87 (2014), 28-33.

[6] P.E. Oguntunde, M.A. Khaleel, A.O. Adejumo, H.I. Okagbue, A.A. Opanuga, F.O. Owolabi, *The Gompertz inverse exponential (GoIE) distribution with applications*, Cogent. Math. Stat., 5 (2018), 1507122.

[7] M.A. Aldahlan, *The inverse Weibull inverse exponential distribution with application,* Int. J. Contemp. Math. Sci., 14 (2019), 17-30.

[8] M.J. Mohammed, I.H. Hussein, *Some estimation methods for new mixture distribution with simulation and application*, In: IOP Conference Series: Materials Science and Engineering, IOP Publishing, 2019, p. 12014.

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