Cocycles over top spaces

Razieh Moayedi
Department of Mathematics
Payam Noor Universtity of Tehran
Tehran
Iran
r_moayedi11600@yahoo.com

Mohammad Reza Molaei*
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman
Iran
mrmolaei@uk.ac.ir

Abstract. Evolution of a potential in a physical system can be determine by a cocycle. This essay contains the notion of cocycles on top spaces. We can associate a semigroup with an identity to each element of a manifold via a cocycle over a top space. This semigroup is a subsemigroup of a Lie group. We prove that these kind of semigroups are invariant under diffeomorphisms of manifolds. We study the concept of forward invariant set, global pullback absorbing set and global forward absorbing set for cocycles. We show that global attractors are persistence by a kind of topological equivalent relation on cocycles. We also define and study the concept of topological entropy for a sequence of cocycles.

Keywords: cocycle, physical system, attractor, entropy, top space, Schrodinger operator.

1. Introduction

Many essential physical systems can be considered by a diffeomorphism $f : M \to M$ and a smooth map $\varphi : M \to T$ where M is a smooth manifold and T is a Lie group. Potential of physical systems are the best candidates for φ. For example the map φ for a one dimensional harmonic oscillator is $\varphi(x) = \frac{m\omega^2}{2}x^2$, where m and ω are two constants. Lorenz gauge potential or Coulomb gauge potential [10] are two other examples for φ. The cocycle $\varphi_n(x) = \varphi(f^n(x))$ determines the evolution of potential on the orbit of x, where $x \in M$. For example the Schrodinger operator H [11] on the set of bi-sequences of real numbers of the form $p = \{..., p_{-1}, p_0, p_1, ...\}$ with $\Sigma_{i\in Z}|p_i|^2 < \infty$ is defined by $H(p) = q$ where $q_n = p_{n+1} + p_{n-1} + \varphi_n(p_n)$ and $\varphi_n(x) = \varphi(f^n(x))$ for a fixed $x \in M$ and a diffeomorphism $f : M \to M$.

*. Corresponding author

In this paper we replace T with a top space, and this means that we have possibility to choose new products. Top spaces are a kind of generalized lie groups which have been introduced in 2004 [5, 7, 6], generalized vector fields on top spaces create new kind of dynamics on them which are called complete semidynamical systems [8]. Top spaces are smooth manifolds which are also semigroups and each element of it has a special identity and a special inverse. Moreover identity and inverse mappings are smooth maps. In the next section we introduce cocycles [2] over top spaces and by using of them we associate a semigroup with an identity to each element of a manifold. When we work with a top space, then we have this possibility to consider generalized vector fields instead of vector fields. The image of a generalized vector field is a member of a one dimensional vector bundle on a manifold. Hence they may not be tangent to the ambient manifold. Physically when we restrict a vector field to a subset of a manifold then we deduce a generalized vector field [8]. The definition of a cocycle over a top space creates this ability to consider non-autonomous motions which their skew-products are generalized vector fields. Each top space can divide to a disjoint union of Lie groups and we deduce a class of semigroups with an identity via cocycles which each member of it, is a subset of a Lie group. One can use of the method presented in [8] to crate a non-autonomous motion on M via this class of semigroups. Moreover, we show that diffeomorphic manifolds create isomorphic classes of such semigroups. In section 3, we introduce a new concept of forward invariant set, global pullback absorbing set and global forward absorbing set for cocycles. Also we introduce a new concept of global pullback attractors and global forward attractors for cocycles. We prove that global pullback attractors and global forward attractors are invariant objects under topological equivalent relation. In section 4, we study and define the notion of topological entropy for a sequence of cocycles.

2. Cocycles created by a top space

Top spaces are a kind of generalized Lie groups [7]. We recall that a smooth manifold T with a binary smooth operation

$$\begin{array}{rcl} m : T \times T & \longrightarrow & T \\ (a,b) & \mapsto & ab \end{array}$$

is called a top space if it satisfies the following conditions.

(i) (T, m) is a semigroup;

(ii) For given $a \in T$, there exists a unique $e(a) \in T$ so that $ae(a) = e(a)a = a$;

(iii) For given $a \in T$ there exists $a^{-1} \in T$ such that $aa^{-1} = a^{-1}a = e(a)$, and the inverse mapping

$$\begin{array}{rcl} in : T & \longrightarrow & T \\ a & \mapsto & a^{-1} \end{array} .$$

is a smooth map.

One of the most important class of top spaces is the class of top spaces created by Rees matrix semigroups [12]. Let's explain it. We assume G is a Lie group, A and B are two smooth manifolds, and $p : A \times B \to G$ is a smooth mapping. It is well known that $B \times G \times A$ with the product $(n, a, s)(k, b, l) = (n, ap(s, k)b, l)$ is a semigroup, which is called Rees matrix semigroup. One can easily prove that it is also a top space [7].

We assume that M is a smooth manifold and $f : M \to M$ is a smooth diffeomorphism. We also assume that $\varphi : M \to T$ is a smooth map and the following diagram commutes.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[d, "\varphi"'] & & M \arrow[d, "\varphi"] \\
T \arrow[dr, "e"'] & & T \arrow[dl, "e"] \\
& T &
\end{tikzcd}$$

where e is the identity map of T.

Definition 2.1. A cocycle created by (φ, f) is the sequence of mappings $\{\varphi_n : M \to T \mid n \in Z\}$, where φ_n is defined by

$$\varphi_n(x) = \begin{cases} \varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x), & \text{if } n > 0 \\ (\varphi(f^{-n}(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1}, & \text{if } n < 0 \\ e(\varphi(x)), & \text{if } n = 0 \end{cases} .$$

We present an equivalent definition for a cocycle in the following. A smooth mapping $\alpha : \mathbb{Z} \times M \longrightarrow T$ is a cocycle if $\alpha(0, x) = e(\alpha(1, x)) = e(\alpha(1, f(x)) = ... = e(\alpha(1, f^n(x)))$, $\alpha(m + n, x) = \alpha(m, f^n(x)).\alpha(n, x)$, for all $n, m \in \mathbb{Z}$, and $x \in M$. To prove the equivalency of these two definitions it is enough to define $\varphi_n(x) = \alpha(n, x)$.

Since the tuple (φ, f) creates a cocycle, then we also use of the name cocycle for (φ, f). (φ, f) is abelian if T is an abelian group. An abelian cocycle (φ, f) is equivalent to the notion of cocycle which is defined by Avila A., Santamaria A., Viana M., and Wilkinson A. in [2].

For given $x \in M$ we take $C_\varphi(x) = \{\prod_{i=1}^{M} \varphi_{n_i}(x)\varphi_{m_i}(x) \mid M \in \mathbb{N}\}$, and we define the following product on it.

$$(\prod_{j=1}^{K} \varphi_{k_j}\varphi_{s_j})(\prod_{v=1}^{D} \varphi_{l_u}\varphi_{r_u}) = \prod_{t=1}^{K+D} \varphi_{w_t}\varphi_{v_t},$$

where

$$w_t = \begin{cases} k_j, & \text{for } 1 \leq t = j \leq K \\ l_u, & \text{for } 1 \leq t = K + u \leq K + D \end{cases}$$

and

$$v_t = \begin{cases} s_j, & \text{for } 1 \le t = j \le K \\ r_u, & \text{for } 1 \le t = K+u \le K+D \end{cases}.$$

$C_\varphi(x)$ with the above product is a semigroup.
Since for given $n \in \mathbb{N}$

$$e(\varphi(x)) = e(\varphi(f(x))) = e(\varphi(f^2(x))) = ... = e(\varphi(f^n(x))),$$

then

$$\begin{aligned} \varphi_n(x)e(\varphi(x)) &= \varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x)e(\varphi(x)) = \varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x) \\ &= \varphi_n(x) \\ e(\varphi(x))\varphi_n(x) &= e(\varphi(f^{n-1}(x)))\varphi_n(x) = e(\varphi(f^{n-1}(x)))\varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x) \\ &= \varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x) = \varphi_n(x). \end{aligned}$$

Moreover, for given $-n \in \mathbb{N}$ we have $e(\varphi(x)) = e(\varphi(x))^{-1} = e(\varphi(f(f^{-1}(x))))^{-1} = e(\varphi(f(f^{-1}(x)))) = e(\varphi(f^{-1}(x))) = e(\varphi(f^{-1}(x)))^{-1} = ... = e(\varphi(f^n(x)))^{-1}$. Thus

$$\begin{aligned} \varphi_n(x)e(\varphi(x)) &= (\varphi(f^n(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1}e(\varphi(x)) \\ &= (\varphi(f^n(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1}e(\varphi(f^{-1})(x)))^{-1} \\ &= (\varphi(f^{-n}(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1} = \varphi_n(x)e(\varphi(x))\varphi_n(x) \\ &= e(\varphi(f^n(x)))^{-1}\varphi_n(x) \\ &= e(\varphi(f^n(x)))^{-1}(\varphi(f^n(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1} \\ &= (\varphi(f^n(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1} = \varphi_n(x). \end{aligned}$$

In the case $n = 0$ we have

$$\begin{aligned} \varphi_0(x)e(\varphi(x)) &= e(\varphi(x))e(\varphi(x)) = e(\varphi(x)) = \varphi_0(x), \\ e(\varphi(x))\varphi_0(x) &= e(\varphi(x))e(\varphi(x)) = e(\varphi(x)) = \varphi_0(x). \end{aligned}$$

Hence, $C_\varphi(x)$ is a semigroup with the identity $e(\varphi(x))$. This implies $C_\varphi(x)$ is a subsemigroup of the Lie group $T_{e(\varphi(x))} = e^{-1}(\{e(\varphi(x))\})$. In the next example we show that $C_\varphi(x)$ may not be equal to $T_{e(\varphi(x))}$.

Example 2.1. Take $M = \mathbb{R} - \{0\}$, and $T = \mathbb{R} - \{0\}$. T with the product $a.b = a|b|$ as an open subset of the real manifold $\mathbb{R}$ is a top space. We only explain the associative property of this product. If $a, b, c \in T$ then

$$(a.b).c = (a|b|)|c| = a|bc| = a|b|c|| = a|b.c| = a.(b.c).$$

If $f(x) = 2x$ and $\varphi = id$ then

$$\begin{aligned} &\varphi_1(x) = \varphi(x) = x, \\ &\varphi_2(x) = \varphi(f(x))\varphi(x) = 2x.x = 2x \mid x \mid, \\ &\varphi_3(x) = \varphi(f^2(x))\varphi_2(x) = 4x.2x \mid x \mid = 8x \mid x^2 \mid, \\ &\qquad \vdots \end{aligned}$$

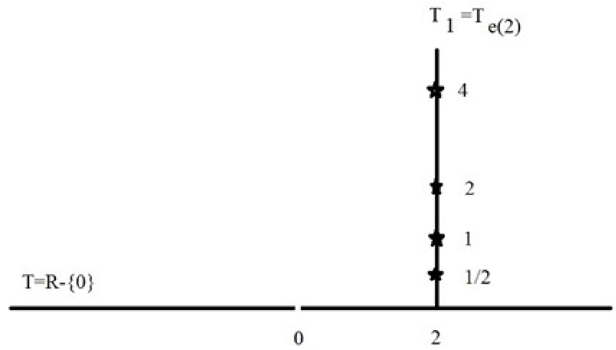

Figure 1: Stars denote the points of $C_\varphi(2)$.

$$\varphi_0(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

$$\begin{aligned}
\varphi_{-1}(x) &= (\varphi(f^{-1}(x)))^{-1} = (\frac{1}{2}x)^{-1} = \frac{2}{x}, \\
\varphi_{-2}(x) &= (\varphi(f^{-2}(x)))^{-1}(\varphi(f^{-1}(x)))^{-1} = \frac{4}{x}.\frac{2}{x} = \frac{8}{x}.\mid \frac{1}{x} \mid, \\
\varphi_{-3}(x) &= (\varphi(f^{-3}(x)))^{-1}(\varphi_{-2}(x))^{-1} = \frac{8}{x}.\frac{8}{x} \mid \frac{1}{x} \mid = \frac{64}{x} \mid \frac{1}{x^2} \mid, \\
&\vdots
\end{aligned}$$

For $x = 2$, we have $C_\varphi(2) = \{2^m \mid m \in \mathbb{Z}\}$, and $C_\varphi(2)$ is not equal to $T_{e(\varphi(2))} = T_1 = e^{-1}\{1\} = \{x > 0 \mid x \in \mathbb{R}\}$ (see figure 1).

In the next example we show that $C_\varphi(x)$ can be sometimes equal to $T_{e(\varphi(x)}$.

Example 2.2. Let M be a manifold , and $T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$. Then T with the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & r \\ g & h \end{bmatrix} = \begin{bmatrix} a & r \\ g & d \end{bmatrix}$ and the Euclidean topology is a top space. For a given smooth map $\varphi : M \longrightarrow T$, and for each diffeomorphism $f : M \longrightarrow M$ with $e(\varphi(f(x)) = e(\varphi(x))$ we have

$$C_\varphi(x) = \{e(\varphi(x))\} = \{\varphi(x)\} = T_{e(\varphi(x))}.$$

To prove this, we see that $(\varphi(f^{-1}(x)))^{-1} = e(\varphi(f^{-1}(x)))^{-1} = e(\varphi(f^{-1}(x))) = e\varphi(f(f^{-1}(x))) = e(\varphi(x)) = \varphi(x)$, and $\varphi(f(x)) = e(\varphi(f(x))) = e(\varphi(x)) = \varphi(x)$. So, $\varphi_n(x) = \varphi(x)$, for all $n \in \mathbb{Z}$.

Theorem 2.3. *Suppose (φ, f) is a cocycle on T, and the mapping $\psi : M' \longrightarrow M$ is a diffeomorphism. Then the following diagram commutes and $C_\varphi(x) = C_{\varphi\circ\psi}(\psi^{-1}(x))$.*

$$\begin{tikzcd}
M' \arrow[rr, "\psi^{-1}\circ\psi"] \arrow[d, "\varphi\circ\psi"'] & & M' \arrow[d, "\varphi\circ\psi"] \\
T \arrow[dr, "e"'] & & T \arrow[dl, "e"] \\
& T &
\end{tikzcd}$$

Proof. If $x \in M'$ then $e \circ (\varphi \circ \psi) \circ (\psi^{-1} \circ f \circ \psi)(x) = e \circ \varphi \circ f \circ \psi(x) = e \circ \varphi \circ \psi(x) = e(\varphi \circ \psi)(x)$. So, the above diagram is a commutative one. Now we prove the equality of $C_\varphi(x)$ with $C_{\varphi\circ\psi}(\psi^{-1}(x))$, where $x \in M$. For given $x \in M$ and $n \in Z$ we have

$$\begin{aligned}
&(\varphi \circ \psi)_n(\psi^{-1}(x)) \\
&= \begin{cases}
\varphi\circ\psi(\psi^{-1} \circ f \circ \psi)^{n-1}(\psi^{-1}(x))...\varphi\circ\psi(\psi^{-1}\circ f\circ\psi)(\psi^{-1}(x))\varphi \circ \psi(\psi^{-1}(x)), \\
\hfill \text{if } n > 0 \\
(\varphi \circ \psi(\psi^{-1} \circ f \circ \psi)^{n}(\psi^{-1}(x)))^{-1}...(\varphi \circ \psi(\psi^{-1} \circ f \circ \psi)^{-1}(\psi^{-1}(x)))^{-1}, \\
\hfill \text{if } n < 0 \\
e(\varphi \circ \psi)(\psi^{-1}(x)), \text{if } n = 0
\end{cases} \\
&= \begin{cases}
\varphi(f^{n-1}(x))...\varphi(f(x))\varphi(x), & \text{if } n > 0 \\
(\varphi(f^{n}(x)))^{-1}...(\varphi(f^{-1}(x)))^{-1}, & \text{if } n < 0 = \varphi_n(x). \\
e(\varphi(x)), & \text{if } n = 0
\end{cases}
\end{aligned}$$

□

3. Attractors

In this section $Z : M \longrightarrow 2^T$ is a mapping such that $Z(x)$ is a bounded set in the top space T. For all $x \in M$, Z is called a non-autonomous set for a cocycle (φ, f) if $f^{-1}(Z^{-1}(Z(x))) = Z^{-1}(Z(f^{-1}(x)))$. We say that Z is a compact non-autonomous set if for all $x \in M, Z(x)$ is a compact set. Z is called a forward invariant set if $\varphi_n(Z^{-1}(Z(x))) \subseteq Z(f^n(x))$, for all $x \in M$ and $n \geq 0$.

A forward invariant set Z for a cocycle (φ, f) is called a global pullback absorbing set if for all $x \in M$, there exist $N \in \mathbb{N}$ such that $\varphi_n(f^{-n}(Z^{-1}(Z(x)))) \subseteq Z(x)$, for all $n \geq N$.

A forward invariant set Z for a cocycle (φ, f) is called a global forward absorbing set if for all $x \in M$, there is $N \in \mathbb{N}$ such that $\varphi_{-n}(f^n(Z^{-1}(Z(x))) \subseteq Z(x)$, for all $n \geq N$.

Definition 3.1. *Let Z be a compact global pullback absorbong set. Then $A : M \longrightarrow 2^T$ defined by $A(x) = \bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n} \varphi_m(f^{-m}(Z^{-1}(Z(x)))})$ is called a global pullback attractor for a cocycle (φ, f).*

In the next theorem we prove that a global pullback attractor is an invariant object under forward motion.

Theorem 3.1. *If A is a global pullback attractor for a cocycle (φ, f), then A is a forward invariant set .*

Proof. Assume that $\varphi_k(A^{-1}(A(x)))\nsubseteq A(f^k(x))$, then there is $y\in\varphi_k(A^{-1}(A(x)))$ such that $y \notin A(f^k(x))$. So, there exists n such that for each $m \geq n$

$$y \notin \overline{(\bigcup_{m\geq n} \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x))))))}.$$

Thus, $y \notin \bigcup_{m\geq n} \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x)))))$. Hence, there is m so that $y \notin \varphi_m(f^{-m}(Z^{-1}(Z(f^k(x)))))$.

Therefore, there exist m such that $y \notin \varphi_m(Z^{-1}(Z(f^{k-m}(x)))) \subseteq Z(f^k(x))$ and it is a contradiction because Z is a forward invariant set. $\square$

Definition 3.2. *Let Z be a compact global forward absorbing set. Then $A : M \longrightarrow 2^T$ defined by $A(x) = \bigcap_{n=0}^{\infty} \overline{(\bigcup_{m\geq n} \varphi_{-m}(f^m(Z^{-1}(Z(x)))))}$ is called a global forward attractor for a cocycle (φ, f).*

By the same method of the proof of Theorem 3.2 one can prove the following theorem.

Theorem 3.2. *If A is a global forward attractor for a cocycle (φ, f), then A is a forward absorbing set .*

Note that $A(x)$ is a compact set because it is a closed subset of the compact set $Z(x)$.

Assume (φ_1, f_1) and (φ_2, f_2) are two cocycles on T . These cocycles are equivalent if there exist a homeomorphism $h : M \longrightarrow M$, a homeomorphism $L : T \longrightarrow T$, and an increasing homeomprphism $S : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $S(0) = 0$ such that $h(f_1^n(x)) = f_2^{S(n)}(h(x))$, and $L(\varphi_{1n}(x)) = \varphi_{2S(n)}(h(x))$. We have a natural extension map $\tilde{L} : 2^T \longrightarrow 2^T$ that defined by $\tilde{L}(A) = \{L(a) : a \in A\}$.

Lemma 3.1. *If (φ_1, f_1) and (φ_2, f_2) are two equivalent cocycles on T, and if $Z : M \longrightarrow T$ is a global pullback absorbing set for (φ_1, f_1), then a map $\tilde{L} \circ Z \circ h^{-1} : M \longrightarrow \tilde{L}(2^T)$ is also a global pullback absorbing set for the cocycle (φ_2, f_2).*

Proof. Since Z is a forward invariant set for a cocycle (φ_1, f_1), then

$$\varphi_{1n}(Z^{-1}(Z(x))) \subseteq Z(f_1^n(x))$$

for all $x \in M$ and $n \geq 0$. For given $h(x) \in M$ and $S(n) > 0$ we have

$$\begin{aligned}
&\varphi_{2S(n)}((\tilde{L} \circ Z \circ h^{-1})^{-1}(\tilde{L} \circ Z \circ h^{-1})(h(x))) = \varphi_{2S(n)}(h(Z^{-1}(Z(x))))\\
&= L\varphi_{1n}(Z^{-1}(Z(x)) \subseteq L(Z(f_1^n(x)) = \tilde{L}(Z(f_1^n(x)))\\
&= (\tilde{L} \circ Z \circ h^{-1} \circ h \circ f_1^n)(x) = (\tilde{L} \circ Z \circ h^{-1})(h \circ f_1^n(x))\\
&= (\tilde{L} \circ Z \circ h^{-1})(f_2^{S(n)}(h(x))).
\end{aligned}$$

Thus, $\tilde{L}\circ Z\circ h^{-1}$ is a forward invariant set for a cocycle (φ_2, f_2). If $h(x)\in M$, then there is $N\in\mathbb{N}$ such that $\varphi_{1n}(f_1^{-n}(Z^{-1}(Z(x))))\subseteq Z(x)$ for all $n\geq N$. For $S(n)\geq S(N)$ we have

$$\begin{aligned}
&\varphi_{2S(n)}({f_2}^{-S(n)}(\tilde{L}\circ Z\circ h^{-1})^{-1}(\tilde{L}\circ Z\circ h^{-1})(h(x))\\
&=\varphi_{2S(n)}({f_2}^{-S(n)}(h\circ Z^{-1}\circ \tilde{L}^{-1})\circ \tilde{L}\circ Z\circ h^{-1}\circ h(x))\\
&=\varphi_{2S(n)}({f_2}^{-S(n)}(h(Z^{-1}(Z(x))))=\varphi_{2S(n)}(h({f_1}^{-n}(Z^{-1}(Z(x))))\\
&=L(\varphi_{1n}({f_1}^{-n}(Z^{-1}(Z(x))))=L(\varphi_{1n}({f_1}^{-n}(Z^{-1}(Z(x)))\subseteq L(Z(x))=\tilde{L}Z(x)\\
&=(\tilde{L}\circ Z\circ h^{-1}\circ h)(x)=(\tilde{L}\circ Z\circ h^{-1})(h(x)). \qquad \square
\end{aligned}$$

Theorem 3.3. *If we have two equivalence cocycles (φ_1, f_1) and (φ_2, f_2), and if $A^{(\varphi_1,f_1)}$ is a global pullback attractor for the cocycle (φ_1, f_1), then $\tilde{L}\circ A^{(\varphi_1,f_1)}$ is also a global pullback attractor for the cocycle (φ_2, f_2).*

Proof. Let $Z: M\longrightarrow 2^T$ be a compact global pullback absorbing set for (φ_1, f_1) , and $A^{(\varphi_1,f_1)}(x)=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}\varphi_{1m}({f_1}^{-m}(Z^{-1}(Z(x)))})$. Then Lemma 3.5 implies $\tilde{L}\circ Z\circ h^{-1}$ is a compact global pullback absorbing set for the cocycle (φ_2, f_2). So,

$$\begin{aligned}
(\tilde{L}\circ A^{(\varphi_1,f_1)})(x)&=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}\tilde{L}\circ\varphi_{1m}({f_1}^{-m}(Z^{-1}(Z(x)))))}\\
&=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}L\circ\varphi_{1m}({f_1}^{-m}(Z^{-1}(Z(x)))))}\\
&=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}\varphi_{2S(m)}(h{f_1}^{-m}(Z^{-1}(Z(x)))))}\\
&=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}\varphi_{2S(m)}({f_2}^{-S(m)}(h(Z^{-1}(Z(x)))))}\\
&=\bigcap_{n=0}^{\infty}(\overline{\bigcup_{m\geq n}\varphi_{2S(m)}({f_2}^{-S(m)}(\tilde{L}\circ Z\circ h^{-1})^{-1}(\tilde{L}\circ Z\circ h^{-1})(h(x)))))}\\
&=A^{(\varphi_2,f_2)}(h(x)). \qquad \square
\end{aligned}$$

One can prove the following lemma and theorem similarly

Lemma 3.2. *If (φ_1, f_1) and (φ_2, f_2) are two equivalent cocycles, and if $Z: M\longrightarrow T$ is a global forward absorbing set for (φ_1, f_1), then a map $\tilde{L}\circ Z\circ h^{-1}: M\longrightarrow \tilde{L}(2^T)$ is also a global forward absorbing set for the cocycle (φ_2, f_2).*

Theorem 3.4. *If we have two equivalent cocycles (φ_1, f_1) and (φ_2, f_2), and if $A^{(\varphi_1,f_1)}$ is a global forward attractor for the cocycle (φ_1, f_1), then $\tilde{L}\circ A^{(\varphi_1,f_1)}$ is a global forward attractor for the cocycle (φ_2, f_2).*

4. Topological entropy of a cocycle

The notion of topological entropy was introduced by Konheim, Adler and McAndrew [1]. This notion has been considered by Kolmogorov [3] and then by Sinai [13] from measure theoretical point of view. Rudolf Clausius was the first scientist who introduced entropy to describe the use of dissipative energy of a thermodynamic system during a change of the state [9]. In this section we study the notion of topological entropy for a cocycle on a compact top space [4]. Assume T is a compact top space, $f : T \longrightarrow T$ is a smooth diffeomorphism, $\varphi : T \longrightarrow T$ is a smooth map and $e(\varphi(f(x))) = e(\varphi(x))$. We would like to define topological entropy for a cocycle created by φ and f. Let $\{\varphi_i \ : \ i \in Z\}$ be the cocycle created by(φ, f). We define $\varphi_{1,\infty} = \{\varphi_1, \varphi_2, ...\} = \{\varphi_i\}_{i=1}^{\infty}$. ${\varphi_i}^0 = {\varphi_i}^{-0}$ is the identity function and for any $m \in \mathbb{N}$, we define ${\varphi_i}^m := \varphi_{i+m-1}$, ${\varphi_i}^{-1} := \varphi_{-i}$, ${\varphi_i}^{-m} := ({\varphi_i}^m)^{-1}$. Finally, we denote $\{\varphi^n_{in+1}\}_{i=0}^{\infty}$ by $\varphi^n_{1,\infty}$ and $\{\varphi_i^{-1}\}_{i=1}^{\infty}$ by $\varphi^{-1}_{1,\infty}$. The reader must pay attention to this point that, φ_i^{-1} is only a symbol and it is not the inverse of φ_i. Consider an open cover $\mathcal{U}$ of T. For two open covers $\mathcal{U}, \mathcal{V}$, a new open cover $\mathcal{U} \vee \mathcal{V}$ is defined by $\{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$. So, for open covers $\mathcal{U}_i$, $\bigvee_{i=1}^n \mathcal{U}_i = \mathcal{U}_1 \vee \mathcal{U}_2 \vee ... \vee \mathcal{U}_n = \{U_1 \cap U_2 \cap ... \cap U_n \mid U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, ..., U_n \in \mathcal{U}_n\}$ is an open cover of T. For an open cover $\mathcal{U}$ we denote ${\varphi_i}^{-m}(\mathcal{U}) = \{({\varphi_i}^m)^{-1}(U) \mid U \in \mathcal{U}\}$ and $\mathcal{U}_i^m(\varphi_{1,\infty}) = \mathcal{U}_i^m = \bigvee_{j=0}^{m-1} {\varphi_i}^{-j}(\mathcal{U})$.

We denote by $\mathcal{N}(\mathcal{U})$ the minimal cardinality of the subcovers of $\mathcal{U}$. The topological entropy of an open cover $\mathcal{U}$ of T is denoted by $H(\mathcal{U})$ and it is defined by $\log \mathcal{N}(\mathcal{U})$. We define the entropy of a cocycle with respect to an open cover $\mathcal{U}$ by $h(\varphi_{1,\infty}, \mathcal{U}_{1,\infty}) := \limsup_{m\to\infty} \dfrac{1}{m} \log \mathcal{N}({\mathcal{U}_1}^m)$, and the topological entropy of the cocycle by

$$h(\varphi_{1,\infty}) := \{\sup(h(\varphi_{1,\infty}, \mathcal{U}) \mid \mathcal{U} \text{ is an open cover of } T\}.$$

$h(\varphi_{1,\infty})$ denotes the value of the complexity of $\varphi_{1,\infty}$. In fact it denotes the value of the chaotic behavior of $\varphi_{1,\infty}$.

We have the following inequalities:

(1) $\mathcal{N}(\mathcal{U} \vee \mathcal{V}) \leq \mathcal{N}(\mathcal{U}).\mathcal{N}(\mathcal{V})$;

(2) $\mathcal{N}({\varphi_i}^{-m}(\mathcal{U})) \leq \mathcal{N}(\mathcal{U})$.

Let $\mathcal{U}$ be finer than $\mathcal{V}$ i.e. each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Then

(3) $\mathcal{N}(\mathcal{U}) \geq \mathcal{N}(\mathcal{V})$.

If $\mathcal{U}$ is finer than $\mathcal{V}$ then we use of the symbol $\mathcal{U} \geq \mathcal{V}$.

Clearly if $\mathcal{U} \geq \mathcal{V}$ then $h(\varphi_{1,\infty}, \mathcal{U}) \geq h(\varphi_{1,\infty}, \mathcal{V})$.

If $\mathcal{U}$ is an open finite cover of T then the cardinality of $\mathcal{U}_i^m$ is at most $(\mathit{cardinal}\,\mathcal{U})^m$. Therefore $h(\varphi_{1,\infty}, \mathcal{U}) \leq \log\ \mathit{card}\,\mathcal{U}$ and so $0 \leq h(\varphi_{1,\infty}, \mathcal{U}) < \infty$.

Theorem 4.1. *Let $\varphi_{1,\infty}$ be a cocycle on T. Then*

$$h(\varphi^n_{1,\infty}) \leq n.h(\varphi_{1,\infty})$$

for every $n \geq 1$.

Proof. Fix n. For any open cover $\mathcal{U}$ of T we have

$$\begin{aligned}
h(\varphi_{1,\infty},\mathcal{U}) &= \limsup_{k\to\infty}\frac{1}{k}\log\mathcal{N}(\mathcal{U}_1{}^k) \geq \limsup_{m\to\infty}\frac{1}{mn}\log\mathcal{N}(\mathcal{U}_1{}^{mn})\\
&= \limsup_{m\to\infty}\frac{1}{mn}\log\mathcal{N}\Big(\bigvee_{j=0}^{mn-1}\varphi_1{}^{-j}(\mathcal{U})\Big)\\
&= \limsup_{m\to\infty}\frac{1}{mn}\log\mathcal{N}(\mathcal{U}\vee\varphi_1{}^{-1}(\mathcal{U})\vee\varphi_1{}^{-2}(\mathcal{U})\vee ...\vee\varphi_1{}^{-(mn-1)}(\mathcal{U}))\\
&\geq \limsup_{m\to\infty}\frac{1}{mn}\log\mathcal{N}(\mathcal{U}\vee\varphi_1^{-n}(\mathcal{U})\vee\varphi_1^{-2n}(\mathcal{U})\vee ...\vee\varphi_1^{-(m-1)n}(\mathcal{U}))\\
&= \frac{1}{n}\limsup_{m\to\infty}\frac{1}{m}\log\mathcal{N}(\mathcal{U}\vee(\varphi_n)^{-1}(\mathcal{U})\vee(\varphi_{2n})^{-1}(\mathcal{U})\vee ...\vee(\varphi_{mn})^{-1}(\mathcal{U})\\
&= \frac{1}{n}h(\varphi_{1,\infty}^n,\mathcal{U}).
\end{aligned}$$

Hence $h(\varphi_{1,\infty}^n) \leq n.h(\varphi_{1,\infty})$. □

Theorem 4.2. *Let $\varphi_{1,\infty}$ be a cocycle over T. Then, for every $1 \leq i \leq j < \infty$ and every open cover $\mathcal{U}$ of T, $h(\varphi_{i,\infty},\mathcal{U}) \leq h(\varphi_{j,\infty},\mathcal{U})$ and $h(\varphi_{i,\infty}) \leq h(\varphi_{j,\infty})$.*

Proof. For any open cover $\mathcal{U}$ of T and $i \geq 1$ we have

$$\begin{aligned}
\mathcal{U}_i{}^n &= \mathcal{U}\vee\varphi_i{}^{-1}(\mathcal{U})\vee\varphi_i{}^{-2}(\mathcal{U})\vee ...\vee\varphi_i{}^{-(n-1)}(\mathcal{U})\\
&= \mathcal{U}\vee\varphi_i{}^{-1}(\mathcal{U})\vee\varphi_{i+1}{}^{-1}(\mathcal{U})\vee\varphi_{i+2}{}^{-1}(\mathcal{U})\vee ...\vee\varphi_{i+n-2}{}^{-1}(\mathcal{U})\\
&= \varphi_i{}^{-1}(\mathcal{U})\vee\mathcal{U}\vee\varphi_{i+1}{}^{-1}(\mathcal{U})\vee\varphi_{i+1}{}^{-2}(\mathcal{U})\vee ...\vee\varphi_{i+1}{}^{-(n-2)}(\mathcal{U})\\
&= \varphi_i{}^{-1}(\mathcal{U})\vee\mathcal{U}_{i+1}{}^{n-1}(\mathcal{U}).
\end{aligned}$$

So, by monotonicity we have

$$\begin{aligned}
h(\varphi_{i,\infty},\mathcal{U}) &= \limsup_{n\to\infty}\frac{1}{n}\log\mathcal{N}(\mathcal{U}_i{}^n)\\
&\leq \limsup_{n\to\infty}\frac{1}{n}\log(\mathcal{N}(\mathcal{U}).\mathcal{N}(\mathcal{U}_{i+1}{}^{n-1}))\\
&= \limsup_{n\to\infty}\frac{1}{n-1}\log\mathcal{N}(\mathcal{U}_{i+1}{}^{n-1}) = h(\varphi_{i+1,\infty},\mathcal{U}).
\end{aligned}$$ □

5. Conclusion

We consider the concept of cocycles via top spaces and we can associate an algebraic object to each element of a manifold. We also study the concept of forward invariant set, global forward absorbing set and global pullback absorbing set for cocycles. In this direction we have the following two questions.

1. What kind of invariant sets can be a candidate for being a global forward or pullback attractor?

2. If $f : M \longrightarrow M$ is a diffeomorphism is the equality $C_\varphi(x) = C_\varphi(f(x))$ true?

In section four we extend the concept of entropy for cocycles, and we prove the essential properties of it.

References

[1] R.L. Adler, A.G. Konheim, M.H. McAndrew, *Topological entropy*, Trans. Am. Math. Soc., 114 (1965), 309-319.

[2] A. Avila, A. Santamaria, M. Viana, A. Wilkinson, *Cocycles over partially hyperbolic maps*, Asterisque, 358 (2013), 1- 12.

[3] A. Kolmogorov, *Théorie générale des systèmes dynamiques et mécanique classique*, Proceedings of the International Congress of Mathematicians, 1 (1954), 315-333.

[4] S. Kolyada, L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random and Computational Dynamics, 4 (2,3) (1996), 205-223.

[5] H. Maleki, M.R. Molaei, *T-Spaces*, Turkish Journal of Mathematics, 39 (2015), 851-863.

[6] M.R. Molaei, M.R. Farhangdost, *Lie algebras of a class of top spaces*, Balkan Journal of Geometry and Its Applications, 14 (2009) 46-51.

[7] M.R. Molaei, *Top spaces*, Journal of Interdisciplinary Mathematics, 7 (2004), 173-181.

[8] M.R. Molaei, *Complete semidynamical systems*, Journal of Dynamical Systems and Geometric Theories, 3 (2005), 95-107.

[9] I. Muller, *Entropy: a subtle concept in thermodynamics, in "Entropy, Andress Greven*, Gerhard Keller, Gerald Warnecke, eds. Princeton University Press, 2003.

[10] G. Naber, *The simple harmonic oscillator*, Springer, 2015.

[11] L. Pastur, *Spectral properties of disordered systems in the one-body approximation*, Comm. Math. Phys., 75 (1980), 179-196.

[12] D. Rees, *On semi-groups*, Proceedings of the Cambridge Philosophical Society, 36 (1940), 387-400.

[13] Ya, Sinai, *Topics in ergodic theory*, Princeton Mathematical Series, vol. 44. Princeton University Press, Princeton, 1994.

Accepted: September 09, 2020