Some new Hermite-Hadamard Fejér type inequalities for functions whose second-order mixed derivatives are coordinated preinvex

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Abstract. In this paper, we present some new Hermite-Hadamard-Fejér type inequalities for those functions whose second order derivative is coordinated preinvex. We have given some new error bounds for the weighted trapezoidal and weighted mid-point rules for coordinated preinvex functions. The results presented here are extensions of earlier works.

Keywords: Hermite-Hadamard, Fejér, coordinated preinvexity, Hölder inequality, power mean inequality.

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1. Introduction

In recent years, a great deal of attention has been laid by many researchers to the theory of convexity because of its great importance in various fields of pure and applied sciences. The theories of convex function and inequalities are closely related to each other. A very interesting and well known inequality that establishes this connection is Hermite-Hadamard inequality, given as:

Let $g : H \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in H$ with $a < b$, then

$$g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b g(v)dv \leq \frac{g(a)+g(b)}{2}. \tag{1}$$

Weighted generalization of the inequality (1) was presented by Fejér [8] as follows:

$$g(\frac{a+b}{2}) \int_a^b w(v)dv \leq \int_a^b w(v)g(v)dv \leq \frac{g(a)+g(b)}{2} \int_a^b w(v)dv, \tag{2}$$

where $w : [a; b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $v = \frac{a+b}{2}$.

The concept of invex sets was given by T. Antczak [2].

Definition 1. *A set $H \subseteq \mathbb{R}^n$ is invex with respect to the map $\zeta : H \times H \to \mathbb{R}^n$ if for every $a, b \in H$ and $t \in [0, 1]$, $b + t\zeta(a, b) \in H$. The invex set H is also called an ζ-connected set.*

Remark 1. Every convex set is an invex set but its converse is not true.

In 1998, Weir and Mond [18], defined preinvex functions as generalization of convex functions.

Definition 2. *Let $H \subseteq \mathbb{R}^n$ be an invex set and a function $g : H \to \mathbb{R}$ is said to be preinvex w.r.t. ζ if $\forall\ a, b \in H$,*

$$g(a + t\zeta(a, b)) \leq tg(b) + (1 - t)g(a).$$

If $\zeta(a, b) = a - b$, then in the classical sense, the preinvex function becomes a convex functions. A function g is called preincave iff its negative is preinvex.

Definition 3 ([13])**.** *Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$, we have*

$$f(u + t\eta_1(x, u), v + t\eta_2(y, v)) \leq (1 - t)f(u, v) + tf(x, y).$$

Definition 4 ([11])**.** *Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (x, u), (u, y), (u, v) \in K_1 \times K_2$ and λ, $t \in [0, 1]$, we have*

$$\begin{aligned} f(u + \lambda\eta_1(x, u), v + t\eta_2(y, v)) \leq\ & (1 - \lambda)(1 - t)f(u, v) + (1 - \lambda)tf(u, y) \\ & + (1 - t)\lambda f(x, v) + \lambda tf(x, y). \end{aligned}$$

S. S. Dragomir [6] established the following result.

Theorem 1. *suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is a coordinated convex function on Δ. Then one has the following inequalities:*

$$\begin{aligned}
&f(\frac{a+b}{2}, \frac{c+d}{2}) \leqslant \frac{1}{2}[\frac{1}{b-a}\int_a^b f(x, \frac{c+d}{2})dx \\
&+ \frac{1}{d-c}\int_c^d f(\frac{a+b}{2}, y)dy] \\
&\leqslant \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\
&\leqslant \frac{1}{4}[\frac{1}{b-a}(\int_a^b f(x,c)dx + \int_a^b f(x,d)dx) \\
&+ \frac{1}{d-c}(\int_c^d f(a,y)dy + \int_c^d f(b,y)dy)] \\
&\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

M. A. Latif and S. S. Dragomir proposed the following Hermite-Hadamard equalities for preinvex functions on coordinates.

Theorem 2 ([11])**.** *Let $K_1 \times K_2 \to \mathbb{R}$ be an open convex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial\lambda\partial t} \in L_1([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]),$$

where $\eta_1(b,a) \neq 0, \eta_2(d,c) \neq 0$, where $a,b \in K_1$ and $c,d \in K_2$. Then the following equality holds:

$$\begin{aligned}
&\frac{1}{4}[f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))]) \\
&+ \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)dxdy - A \\
&= \frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 (1-2\lambda)(1-2t)\frac{\partial^2 f}{\partial t\partial\lambda}(a+\lambda\eta_1(b,a), c+t\eta_2(d,c))d\lambda dt,
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{2\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}[f(x,c) + f(x, c+\eta_2(d,c))]dx \\
&+ \frac{1}{2\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y) + f(a+\eta_1(b,a), y)]dy.
\end{aligned}$$

Fejér-type inequality for coordinated convex mappings is established in [1] as follows.

Theorem 3. *suppose that* $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ *is a coordinated convex function on* Δ*. Then one has the following inequalities:*

$$\begin{aligned}
& f(\frac{a+b}{2}, \frac{c+d}{2}) \int_a^b \int_c^d p(x,y)dydx \\
& \leqslant \int_a^b \int_c^d f(x,y)p(x,y)dydx \\
& \leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
& \times \int_a^b \int_c^d p(x,y)dydx,
\end{aligned}$$

where $p : [a,b] \times [c,d] \to R$ *is positive, integrable and symmetric about* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$

In this paper, we present two new Hermite-Hadamard-Fejér identities for functions second-order mixed derivatives are coordinated preinvex. Using the new identities, we obtain some new weighted estimates connected with the left and right hand side of the Hermite-Hadamard-Fejér type inequalities for coordinated preinvex functions.

2. Main results

Throughout this section, we will let $\sup_{(s,t)\in(0,1)\times(0,1)} |w(a+s\eta(b,a), c+t\xi(d,c))| = \|w\|_\infty$.

Lemma 1. *Let* $K_1 \times K_2 \subseteq \mathbb{R}^2$ *be an open invex subset with respect to* $\eta : K_1 \times K_1 \to \mathbb{R}^2$ *and* $\xi : K_2 \times K_2 \to \mathbb{R}^2$*. Suppose* $f : K_1 \times K_2 \to \mathbb{R}^2$ *is a twice differentiable mapping on* $K_1 \times K_2$ *such that* $f_{st} \in L([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)]))$ *where* $\eta(b,a) > 0$ *and* $\xi(d,c) > 0$*. If* $w : ([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ *is an integrable mapping, then for every* $a, b \in K_1$ *and* $c, d \in K_2$*, the following equality holds:*

$$\begin{aligned}
& f(a+\frac{1}{2}\eta(b,a), c+\frac{1}{2}\xi(d,c)) \int_0^1 \int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt \\
& - \int_0^1 \int_0^1 [f(a+s\eta(b,a), c+\frac{1}{2}\xi(d,c)) \\
& + f(a+\frac{1}{2}\eta(b,a), c+t\xi(d,c))] \\
& \times w(a+s\eta(b,a), c+t\xi(d,c))dsdt \\
& + \int_0^1 \int_0^1 f(a+s\eta(b,a), c+t\xi(d,c))w(a+s\eta(b,a), c+t\xi(d,c))dsdt \\
& = \eta(b,a)\xi(d,c) \int_0^1 (\int_0^1 K(s,t) f_{st}(a+s\eta(b,a), c+t\xi(d,c))ds)dt,
\end{aligned} \tag{3}$$

where

$$K(s,t)=\begin{cases}\int_0^t(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt, & (s,t)\in[0,\frac{1}{2}]\times[0,\frac{1}{2})\\ -\int_0^t(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt, & (s,t)\in[0,\frac{1}{2}]\times(\frac{1}{2},1]\\ -\int_t^1(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt, & (s,t)\in(\frac{1}{2},1]\times[0,\frac{1}{2}]\\ \int_t^1(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt, & (s,t)\in(\frac{1}{2},1]\times(\frac{1}{2},1].\end{cases}$$

Proof. Consider

$$\begin{aligned}&\int_0^{\frac{1}{2}}(\int_0^t(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\ f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt\\ &=\frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a),c+t\xi(d,c))\int_0^t(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)]_0^{\frac{1}{2}}\\ &-\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}((\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt\\ &=\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_0^{\frac{1}{2}}(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\\ &-\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}((\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt.\end{aligned}$$

Now,

$$\begin{aligned}I_1=&\int_0^{\frac{1}{2}}(\int_0^{\frac{1}{2}}(\int_0^t(\int_0^s w(a+s\eta(b,a),\\ &c+t\xi(d,c))ds)dt\ f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds\\ &=\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}(f_s(a+s\eta(b,a),\\ &c+\frac{1}{2}\xi(d,c))(\int_0^{\frac{1}{2}}(\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt))ds\\ &-\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}(\int_0^{\frac{1}{2}}((\int_0^s w(a+s\eta(b,a),\\ &c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.\end{aligned}$$

Again integrating, we get

$$
\begin{aligned}
I_1 = & \frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c \\
& + \frac{1}{2}\xi(d,c)) \int_0^{\frac{1}{2}} (\int_0^s w(a+s\eta(b,a),c+t\xi(d,c))ds)dt]_0^{\frac{1}{2}} \\
& - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} (f(a+s\eta(b,a),c \\
& + \frac{1}{2}\xi(d,c))(\int_0^{\frac{1}{2}} w(a+s\eta(b,a),c+t\xi(d,c))dt))ds \\
& - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} [(\int_0^s w(a+s\eta(b,a),c \\
& + t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_0^{\frac{1}{2}} dt \\
& + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} w(a+s\eta(b,a),c \\
& + t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.
\end{aligned}
$$

After simplification

$$
\begin{aligned}
I_1 = & \frac{1}{\eta(b,a)\xi(d,c)} f(a+\frac{1}{2}\eta(b,a),c \\
& + \frac{1}{2}\xi(d,c)) \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a+s\eta(b,a),c \\
& + \frac{1}{2}\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a+\frac{1}{2}\eta(b,a),c \\
& + t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(a+s\eta(b,a),c \\
& + t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt.
\end{aligned} \tag{4}
$$

Again

$$
\begin{aligned}
& \int_0^{\frac{1}{2}} (\int_0^t (\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt \;\; f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt \\
& = \frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a),c+t\xi(d,c)) \int_0^t (\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)]_0^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt\\
&=\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_0^{\frac{1}{2}}(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\\
&-\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt.
\end{aligned}
$$

Now,

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^1(\int_0^{\frac{1}{2}}(\int_0^t(\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)dtf_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds\\
&=\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^1(f_s(a+s\eta(b,a),c\\
&+\frac{1}{2}\xi(d,c))(\int_0^{\frac{1}{2}}(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt))ds\\
&-\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^1(\int_0^{\frac{1}{2}}((\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.
\end{aligned}
$$

Again integrating, we get

$$
\begin{aligned}
I_2 &= \frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c\\
&+\frac{1}{2}\xi(d,c))\int_0^{\frac{1}{2}}(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt]_{\frac{1}{2}}^1\\
&+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^1(f(a+s\eta(b,a),c\\
&+\frac{1}{2}\xi(d,c))(\int_0^{\frac{1}{2}} w(a+s\eta(b,a),c+t\xi(d,c))dt))ds\\
&-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^{\frac{1}{2}}[(\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_{\frac{1}{2}}^1 dt\\
&-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.
\end{aligned}
$$

After simplifying

$$
\begin{aligned}
I_2 = & -\frac{1}{\eta(b,a)\xi(d,c)} f(a+\frac{1}{2}\eta(b,a),\\
& c+\frac{1}{2}\xi(d,c)) \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 w(a+s\eta(b,a), c+t\xi(d,c)) dsdt \\
& + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a+s\eta(b,a),\\
& c+\frac{1}{2}\xi(d,c)) w(a+s\eta(b,a), c+t\xi(d,c)) dsdt \\
& + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a+\frac{1}{2}\eta(b,a), c\\
& + t\xi(d,c)) w(a+s\eta(b,a), c+t\xi(d,c)) dsdt \\
& - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 f(a+s\eta(b,a), c\\
& + t\xi(d,c)) w(a+s\eta(b,a), c+t\xi(d,c)) dsdt.
\end{aligned} \tag{5}
$$

Again taking

$$
\begin{aligned}
& \int_{\frac{1}{2}}^1 (\int_t^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt \ f_{st}(a+s\eta(b,a), c+t\xi(d,c))) dt \\
= & \frac{1}{\xi(d,c)} [f_s(a+s\eta(b,a), c+t\xi(d,c)) \int_t^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt]_{\frac{1}{2}}^1 \\
+ & \frac{1}{\xi(d,c)} \int_{\frac{1}{2}}^1 ((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) f_s(a+s\eta(b,a), c+t\xi(d,c))) dt \\
= & -\frac{1}{\xi(d,c)} f_s(a+s\eta(b,a), c+\frac{1}{2}\xi(d,c)) \int_{\frac{1}{2}}^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt \\
+ & \frac{1}{\xi(d,c)} \int_{\frac{1}{2}}^1 ((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) f_s(a+s\eta(b,a), c+t\xi(d,c))) dt.
\end{aligned}
$$

Now,

$$
\begin{aligned}
I_3 = & \int_0^{\frac{1}{2}} (\int_{\frac{1}{2}}^1 (\int_t^t (\int_0^s w(a+s\eta(b,a), c\\
& + t\xi(d,c)) ds) dt \ f_{st}(a+s\eta(b,a), c+t\xi(d,c))) dt) ds \\
& = -\frac{1}{\xi(d,c)} \int_0^{\frac{1}{2}} (f_s(a+s\eta(b,a), c\\
& + \frac{1}{2}\xi(d,c)) (\int_{\frac{1}{2}}^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt)) ds
\end{aligned}
$$

$$+\frac{1}{\xi(d,c)}\int_0^{\frac{1}{2}}(\int_{\frac{1}{2}}^1((\int_0^s w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.$$

Again integrating, we get

$$I_3=-\frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_{\frac{1}{2}}^1(\int_0^s w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)dt]_0^{\frac{1}{2}}$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^{\frac{1}{2}}(f(a+s\eta(b,a),c$$
$$+\frac{1}{2}\xi(d,c))(\int_{\frac{1}{2}}^1 w(a+s\eta(b,a),c+t\xi(d,c))dt))ds$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^1[(\int_0^s w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_0^{\frac{1}{2}}dt$$
$$-\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} w(a+s\eta(b,a),c$$
$$+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.$$

After simplifying

$$I_3=-\frac{1}{\eta(b,a)\xi(d,c)}f(a+\frac{1}{2}\eta(b,a),c$$
$$+\frac{1}{2}\xi(d,c))\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} w(a+s\eta(b,a),c+t\xi(d,c))dsdt$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} f(a+s\eta(b,a),c$$
$$+\frac{1}{2}\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dtds$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} f(a+\frac{1}{2}\eta(b,a),c \tag{6}$$
$$+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt$$
$$-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 f(a+s\eta(b,a),c$$
$$+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt.$$

Again taking

$$\int_{\frac{1}{2}}^{1}(\int_{t}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\ f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=\frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a),c+t\xi(d,c))\int_{t}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)]_{\frac{1}{2}}^{1}$$
$$+\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^{1}((\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=-\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_{\frac{1}{2}}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)dt$$
$$+\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^{1}((\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt.$$

Now,

$$\begin{aligned}I_4 &= \int_{\frac{1}{2}}^{1}(\int_{\frac{1}{2}}^{1}(\int_{t}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c\\ &+t\xi(d,c))ds)dtf_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds\\ &= -\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^{1}(f_s(a+s\eta(b,a),c\\ &+\frac{1}{2}\xi(d,c))(\int_{\frac{1}{2}}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)dt))ds\\ &+\frac{1}{\xi(d,c)}\int_{\frac{1}{2}}^{1}(\int_{\frac{1}{2}}^{1}((\int_{s}^{1} w(a+s\eta(b,a),c\\ &+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.\end{aligned}$$

Again integrating, we get

$$\begin{aligned}&I_4\\ &= -\frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c\\ &+\frac{1}{2}\xi(d,c))\int_{\frac{1}{2}}^{1}(\int_{s}^{1} w(a+s\eta(b,a),c+t\xi(d,c))ds)dt]_{\frac{1}{2}}^{1}\\ &-\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}(f(a+s\eta(b,a),c\\ &+\frac{1}{2}\xi(d,c))(\int_{\frac{1}{2}}^{1} w(a+s\eta(b,a),c+t\xi(d,c))dt))ds\\ &+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}[(\int_{s}^{1} w(a+s\eta(b,a),c\\ &+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_{\frac{1}{2}}^{1}dt\end{aligned}$$

$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}w(a+s\eta(b,a),c$$
$$+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.$$

After simplifying

$$\begin{aligned}
I_4 = {} & \frac{1}{\eta(b,a)\xi(d,c)}f(a+\frac{1}{2}\eta(b,a),c \\
& +\frac{1}{2}\xi(d,c))\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& -\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}f(a+s\eta(b,a),c \\
& +\frac{1}{2}\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dtds \\
& -\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}f(a+\frac{1}{2}\eta(b,a),c \\
& +t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& +\frac{1}{\eta(b,a)\xi(d,c)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}f(a+s\eta(b,a),c \\
& +t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt.
\end{aligned} \tag{7}$$

Upon combining (4), (5), (6) and (7), we get required result. □

Remark 2. If $w=1$, $\eta(b,a)=b-a$ and $\xi(d,c)=d-c$, we get Lemma 2.1 in [10].

Theorem 4. *Let $K_1\times K_2\subseteq\mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1\times K_1\to\mathbb{R}^2$ and $\xi : K_2\times K_2\to\mathbb{R}^2$. Suppose $f : K_1\times K_2\to\mathbb{R}^2$ is a twice differentiable mapping on $K_1\times K_2$ such that $f_{st}\in L([a,a+\eta(b,a)]\times([c,c+\xi(d,c)]))$ where $\eta(b,a)>0$ and $\xi(d,c)>0$. If $w : ([a,a+\eta(b,a)]\times([c,c+\xi(d,c)]))\to[0,\infty)\times[0,\infty)$ is an integrable mapping and $|f_{st}|$ is preinvex function then for every $a,b\in K_1$ and $c,d\in K_2$, the following inequality holds:*

$$\begin{aligned}
& |f(a+\frac{1}{2}\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& -\int_0^1\int_0^1[f(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))+f(a+\frac{1}{2}\eta(b,a),c+t\xi(d,c))] \\
& \times w(a+s\eta(b,a),c+t\xi(d,c))dsdt \\
& +\int_0^1\int_0^1 f(a+s\eta(b,a),c+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt| \\
& \leq\frac{\eta(b,a)\xi(d,c)}{16}\|w\|_\infty\frac{[|f_{st}(a,c)|+|f_{st}(a,d)|+|f_{st}(b,c)|+|f_{st}(b,d)|]}{4}.
\end{aligned} \tag{8}$$

Proof. Taking modulus of both sides of (3), we get

$$\begin{aligned} J &= |f(a+\frac{1}{2}\eta(b,a), c+\frac{1}{2}\xi(d,c))\int_0^1\int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt \\ &- \int_0^1\int_0^1 [f(a+s\eta(b,a), c+\frac{1}{2}\xi(d,c)) + f(a+\frac{1}{2}\eta(b,a), c+t\xi(d,c))] \\ &\times w(a+s\eta(b,a), c+t\xi(d,c))dsdt \\ &+ \int_0^1\int_0^1 f(a+s\eta(b,a), c+t\xi(d,c))w(a+s\eta(b,a), c+t\xi(d,c))dsdt| \\ &= |\eta(b,a)\xi(d,c)\int_0^1(\int_0^1 K(s,t)f_{st}(a+s\eta(b,a), c+t\xi(d,c))ds)dt|. \end{aligned}$$

f_{st} is coordinated preinvex function, i.e

$$\begin{aligned} f_{st}(a+s\eta(b,a), c+t\xi(d,c)) &\le (1-s)(1-t)f_{st}(a,c) + (1-s)tf_{st}(a,d) \\ &+ (1-t)sf_{st}(b,c) + stf_{st}(b,d). \end{aligned}$$

From preinvexity of $|f_{st}|$, we have

$$\begin{aligned} J &\le \eta(b,a)\xi(d,c)\int_0^1(\int_0^1 |K(s,t)|(1-s)(1-t)|f_{st}(a,c)| + (1-s)t|f_{st}(a,d)| \\ &+ (1-t)s|f_{st}(b,c)| + st|f_{st}(b,d)|)dt. \end{aligned} \tag{9}$$

After a straightforward computation, we obtain

$$\begin{aligned} &\|w\|_\infty\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\int_0^t(\int_0^s ds)dt[(1-s)(1-t)|f_{st}(a,c)| + (1-s)tf|_{st}(a,d)| \\ &+ (1-t)s|f_{st}(b,c)| + st|f_{st}(b,d)|])dsdt \\ &= \|w\|_\infty[\frac{|f_{st}(a,c)|}{144} + \frac{|f_{st}(a,d)|}{288} + \frac{|f_{st}(b,c)|}{288} + \frac{|f_{st}(b,d)|}{576}] \end{aligned} \tag{10}$$

$$\begin{aligned} &\|w\|_\infty\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1(\int_0^t(\int_s^1 ds)dt[(1-s)(1-t)|f_{st}(a,c)| + (1-s)t|f_{st}(a,d)| \\ &+ (1-t)s|f_{st}(b,c)| + st|f_{st}(b,d)|])dsdt \\ &= \|w\|_\infty[\frac{|f_{st}(a,c)|}{288} + \frac{|f_{st}(a,d)|}{576} + \frac{|f_{st}(b,c)|}{144} + \frac{|f_{st}(b,d)|}{288}] \end{aligned} \tag{11}$$

$$\begin{aligned} &\|w\|_\infty\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(\int_t^1(\int_0^s ds)dt[(1-s)(1-t)|f_{st}(a,c)| + (1-s)tf|_{st}(a,d)| \\ &+ (1-t)s|f_{st}(b,c)| + st|f_{st}(b,d)|)dsdt \\ &= \|w\|_\infty[\frac{|f_{st}(a,c)|}{288} + \frac{|f_{st}(a,d)|}{144} + \frac{|f_{st}(b,c)|}{576} + \frac{|f_{st}(b,d)|}{288}] \end{aligned} \tag{12}$$

$$\|w\|_\infty \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (\int_t^1 (\int_s^1 ds)dt[(1-s)(1-t)|f_{st}(a,c)| + (1-s)tf|_{st}(a,d)|$$
$$+ (1-t)s|f_{st}(b,c)| + st|f_{st}(b,d)|)dsdt$$
$$= \|w\|_\infty [\frac{|f_{st}(a,c)|}{576} + \frac{|f_{st}(a,d)|}{288} + \frac{|f_{st}(b,c)|}{288} + \frac{|f_{st}(b,d)|}{144}]. \quad (13)$$

Using the results of (10), (11), (12) and (13) in (9), we get (8). □

Corollary 1. *If we take $\|w\|_\infty = 1$, $\eta(b,a) = b-a$ and $\xi(d,c) = d-c$ in (6), we get Theorem 2 in [10].*

Theorem 5. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)]))$ where $\eta(b,a) > 0$ and $\xi(d,c) > 0$. If $w : ([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, where $q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then following inequality holds:*

$$|f(a+\frac{1}{2}\eta(b,a), c+\frac{1}{2}\xi(d,c)) \int_0^1 \int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt$$
$$- \int_0^1 \int_0^1 [f(a+s\eta(b,a), c+\frac{1}{2}\xi(d,c)) + f(a+\frac{1}{2}\eta(b,a), c+t\xi(d,c))]$$
$$\times w(a+s\eta(b,a), c+t\xi(d,c))dsdt \quad (14)$$
$$+ \int_0^1 \int_0^1 f(a+s\eta(b,a), c+t\xi(d,c))w(a+s\eta(b,a), c+t\xi(d,c))dsdt|$$
$$\leq \frac{\|w\|_\infty \eta(b,a)\xi(d,c)}{4^{1+\frac{1}{q}}(p+1)^{\frac{2}{p}}}([|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q])^{\frac{1}{q}}.$$

Proof. From Lemma 2.1 and Hölder's integral inequality, we get

$$J = |f(a+\frac{1}{2}\eta(b,a), c+\frac{1}{2}\xi(d,c)) \int_0^1 \int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt$$
$$- \int_0^1 \int_0^1 [f(a+s\eta(b,a), c+\frac{1}{2}\xi(d,c)) + f(a+\frac{1}{2}\eta(b,a), c+t\xi(d,c))]$$
$$\times w(a+s\eta(b,a), c+t\xi(d,c))dsdt$$
$$+ \int_0^1 \int_0^1 f(a+s\eta(b,a), c+t\xi(d,c))w(a+s\eta(b,a), c+t\xi(d,c))dsdt|$$
$$\leq \eta(b,a)\xi(d,c)(\int_0^1 \int_0^1 |K(s,t)|^p dsdt)^{\frac{1}{p}}$$
$$\times (\int_0^1 (\int_0^1 |f_{st}(a+s\eta(b,a), c+t\xi(d,c))|^q ds)dt)^{\frac{1}{q}}.$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned} J \leq \eta(b,a)\xi(d,c)(\int_0^1\int_0^1 |K(s,t)|^p dsdt)^{\frac{1}{p}} \\ \times (\int_0^1 (\int_0^1 (1-s)(1-t)|f_{st}(a,c)|^q + (1-s)t|f_{st}(a,d)|^q \\ + (1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dt)^{\frac{1}{q}}. \end{aligned} \tag{15}$$

After a straightforward computation, we obtain

$$\begin{aligned} \int_0^1\int_0^1 & [(1-s)(1-t)|f_{st}(a,c)|^q + (1-s)tf|_{st}(a,d)|^q \\ & +(1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dsdt \\ & = \frac{1}{4}[|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q] \end{aligned} \tag{16}$$

and

$$\begin{aligned} & \int_0^1\int_0^1 |K(s,t)|^p dsdt \\ & = \|w\|_\infty^p \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^p s^p dsdt + \|w\|_\infty^p \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t^p(1-s)^p dsdt \\ & + \|w\|_\infty^p \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} (1-t)^p s^p dsdt + \|w\|_\infty^p \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)^p(1-s)^p dsdt \\ & = \|w\|_\infty^p \frac{4}{(p+1)^2}(\frac{1}{2})^{2(p+1)}. \end{aligned} \tag{17}$$

Using the results of (16) and (17) in (15), we get required result (14). □

Remark 3. For $p > 1$, $\frac{1}{16} < \frac{1}{4(p+1)^{\frac{2}{p}}}$, we have improvement of constant in Theorem 2.3.

Corollary 2. *If we take $\|w\|_\infty = 1$, $\eta(b,a) = b-a$ and $\xi(d,c) = d-c$ in (12), we get Theorem 3 in [10].*

Theorem 6. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)]))$, where $\eta(b,a) > 0$ and $\xi(d,c) > 0$. If $w : ([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$,where $q \geq 1$, then following* *inequality holds:*

$$\begin{aligned}
&|f(a+\frac{1}{2}\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))dsdt\\
&-\int_0^1\int_0^1[f(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))+f(a+\frac{1}{2}\eta(b,a),c+t\xi(d,c))]\\
&\times w(a+s\eta(b,a),c+t\xi(d,c))dsdt\\
&+\int_0^1\int_0^1 f(a+s\eta(b,a),c+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt|\\
\leq\ &\eta(b,a)\xi(d,c)(\frac{\|w\|_\infty}{16})\\
&\times(\frac{\|w\|_\infty}{4}[|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q])^{\frac{1}{q}}.
\end{aligned}\tag{18}$$

Proof. From Lemma 2.1 and Power mean inequality, we get

$$\begin{aligned}
J=\ &|f(a+\frac{1}{2}\eta(b,a),c+\frac{1}{2}\xi(d,c))\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))dsdt\\
&-\int_0^1\int_0^1[f(a+s\eta(b,a),c+\frac{1}{2}\xi(d,c))+f(a+\frac{1}{2}\eta(b,a),c+t\xi(d,c))]\\
&\times w(a+s\eta(b,a),c+t\xi(d,c))dsdt\\
&+\int_0^1\int_0^1 f(a+s\eta(b,a),c+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt|\\
&\leq \eta(b,a)\xi(d,c)(\int_0^1\int_0^1|K(s,t)|dsdt)^{1-\frac{1}{q}}\\
&\times(\int_0^1(\int_0^1 K(s,t)|f_{st}(a+s\eta(b,a),c+t\xi(d,c))|^q ds)dt)^{\frac{1}{q}}.
\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}
J\leq\ &\eta(b,a)\xi(d,c)(\int_0^1\int_0^1|K(s,t)|dsdt)^{1-\frac{1}{q}}\\
&\times(\int_0^1(\int_0^1|K(s,t)|(1-s)(1-t)|f_{st}(a,c)|^q+(1-s)t|f_{st}(a,d)|^q\\
&+(1-t)s|f_{st}(b,c)|^q+st|f_{st}(b,d)|^q)dt)^{\frac{1}{q}}.
\end{aligned}\tag{19}$$

After a straightforward computation, we obtain

$$\begin{aligned}
&\|w\|_\infty\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\int_0^t(\int_0^s ds)dt[(1-s)(1-t)|f_{st}(a,c)|^q+(1-s)tf|_{st}(a,d)|^q\\
&+(1-t)s|f_{st}(b,c)|^q+st|f_{st}(b,d)|^q)dsdt\\
&=\|w\|_\infty\frac{1}{64}[\frac{|f_{st}(a,c)|^q}{144}+\frac{|f_{st}(a,d)|^q}{288}+\frac{|f_{st}(b,c)|^q}{288}+\frac{|f_{st}(b,d)|^q}{576}]
\end{aligned}\tag{20}$$

$$\|w\|_\infty \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 (\int_0^t (\int_s^1 ds)dt[(1-s)(1-t)|f_{st}(a,c)|^q + (1-s)tf|_{st}(a,d)|^q$$
$$+ (1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dsdt$$
$$= \|w\|_\infty [\frac{|f_{st}(a,c)|^q}{288} + \frac{|f_{st}(a,d)|^q}{576} + \frac{|f_{st}(b,c)|^q}{144} + \frac{|f_{st}(b,d)|^q}{288}] \tag{21}$$

$$\|w\|_\infty \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (\int_t^1 (\int_0^s ds)dt[(1-s)(1-t)|f_{st}(a,c)|^q + (1-s)tf|_{st}(a,d)|^q$$
$$+ (1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dsdt$$
$$= \|w\|_\infty [\frac{|f_{st}(a,c)|^q}{288} + \frac{|f_{st}(a,d)|^q}{144} + \frac{|f_{st}(b,c)|^q}{576} + \frac{|f_{st}(b,d)|^q}{288}] \tag{22}$$

$$\|w\|_\infty \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (\int_t^1 (\int_s^1 ds)dt[(1-s)(1-t)|f_{st}(a,c)|^q + (1-s)tf|_{st}(a,d)|^q$$
$$+ (1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dsdt$$
$$= \|w\|_\infty [\frac{|f_{st}(a,c)|^q}{576} + \frac{|f_{st}(a,d)|^q}{288} + \frac{|f_{st}(b,c)|^q}{288} + \frac{|f_{st}(b,d)|^q}{144}] \tag{23}$$

and

$$\int_0^1 \int_0^1 |K(s,t)|dsdt$$
$$= \|w\|_\infty (\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (\int_0^t (\int_0^s ds)dt)dsdt + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 (\int_0^t (\int_s^1 ds)dt)dsdt$$
$$+ \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (\int_t^1 (\int_0^s ds)dt)dsdt + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (\int_t^1 (\int_s^1 ds)dt)dsdt)$$
$$= \frac{\|w\|_\infty}{16}. \tag{24}$$

Using the results of (20), (21), (22), (23) and (24) in (19), we get required result (18). □

Corollary 3. *If we take $\|w\|_\infty = 1$, $\eta(b,a) = b - a$ and $\xi(d,c) = d - c$ in* (6), *we get Theorem* 4 *in [10].*

Lemma 2. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)]))$ where $\eta(b,a) > 0$ and $\xi(d,c) > 0$. If $w : ([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ is an integrable mapping, then for every $a, b \in K_1$*

and $c, d \in K_2$*, the following equality holds:*

$$\begin{aligned}
&[f(a, c+\xi(d,c)) + f(a+\eta(b,a), c+\xi(d,c)) + f(a+\eta(b,a), c) + f(a,c)]\\
&\times \int_0^1 (\int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))ds)dt\\
&-2\int_0^1\int_0^1 [f(a+s\eta(b,a), c+\xi(d,c)) + f(a+\eta(b,a), c+t\xi(d,c))\\
&+(f(a, c+t\xi(d,c)) + f(a+s\eta(b,a), c)]\\
&\times (w(a+s\eta(b,a), c+t\xi(d,c)))dsdt\\
&+4\int_0^1\int_0^1 w(a+s\eta(b,a), c+t\xi(d,c))f(a+s\eta(b,a), c+t\xi(d,c))dsdt\\
&=\eta(b,a)\xi(d,c)\int_0^1\int_0^1 P(s,t)f_{ts}((a+s\eta(b,a), c+t\xi(d,c))dsdt, \qquad (25)
\end{aligned}$$

where

$$\begin{aligned}
P(s,t) = &\int_0^t\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))dsdt\\
&-\int_0^t\int_s^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt\\
&-\int_t^1\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))dsdt\\
&+\int_t^1\int_s^1 w(a+s\eta(b,a), c+t\xi(d,c))dsdt,
\end{aligned}$$

and $(t,s) \in [0,1]$.

Proof. Consider

$$\begin{aligned}
&\int_0^1 (\int_0^t (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))ds)dt\ f_{st}(a+s\eta(b,a), c+t\xi(d,c)))dt\\
&=\frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a), c+t\xi(d,c))\int_0^t(\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))ds)]_0^1\\
&-\frac{1}{\xi(d,c)}\int_0^1((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))ds)f_s(a+s\eta(b,a), c+t\xi(d,c)))dt\\
&=\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a), c+\xi(d,c))\int_0^1(\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))ds)dt\\
&-\frac{1}{\xi(d,c)}\int_0^1((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c))ds)f_s(a+s\eta(b,a), c+t\xi(d,c)))dt.
\end{aligned}$$

Now,

$$\begin{aligned} J_1 &= \int_0^1 (\int_0^1 (\int_0^t (\int_0^s w(a+s\eta(b,a),c \\ &+ t\xi(d,c))ds)dt f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds \\ &= \frac{1}{\xi(d,c)} \int_0^1 (f_s(a+s\eta(b,a),c+\xi(d,c))(\int_0^1 (\int_0^s w(a \\ &+ s\eta(b,a),c+t\xi(d,c))ds)dt))ds \\ &- \frac{1}{\xi(d,c)} \int_0^1 (\int_0^1 ((\int_0^s w(a+s\eta(b,a),c \\ &+ t\xi(d,c))ds) f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds. \end{aligned}$$

Again integrating, we get

$$\begin{aligned} J_1 &= \frac{1}{\eta(b,a)\xi(d,c)} [f(a+s\eta(b,a),c+\xi(d,c)) \int_0^1 (\int_0^s w(a \\ &+ s\eta(b,a),c+t\xi(d,c))ds)dt]_0^1 \\ &- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 (f(a+s\eta(b,a),c \\ &+ \xi(d,c))(\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))dt))ds \\ &- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 [(\int_0^s w(a+s\eta(b,a),c \\ &+ t\xi(d,c))ds) f(a+s\eta(b,a),c+t\xi(d,c))]_0^1 dt \\ &+ \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 w(a+s\eta(b,a),c \\ &+ t\xi(d,c)) f(a+s\eta(b,a),c+t\xi(d,c)) ds dt. \end{aligned}$$

After simplification

$$\begin{aligned} J_1 &= \frac{1}{\eta(b,a)\xi(d,c)} f(a+\eta(b,a),c+\xi(d,c)) \int_0^1 \int_0^1 w(a \\ &+ s\eta(b,a),c+t\xi(d,c)) ds dt \\ &- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 f(a+s\eta(b,a),c \\ &+ \xi(d,c)) w(a+s\eta(b,a),c+t\xi(d,c)) dt ds \\ &- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 f(a+\eta(b,a),c \\ &+ t\xi(d,c)) w(a+s\eta(b,a),c+t\xi(d,c)) ds dt \end{aligned} \tag{26}$$

$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1\int_0^1 f(a+s\eta(b,a),c$$
$$+t\xi(d,c))w(a+s\eta(b,a),c+t\xi(d,c))dsdt.$$

Again taking

$$\int_0^1(\int_0^t(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dtf_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=\frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a),c+t\xi(d,c))\int_0^t(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)]_0^1$$
$$-\frac{1}{\xi(d,c)}\int_0^1((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a),c+\xi(d,c))\int_0^1(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt$$
$$-\frac{1}{\xi(d,c)}\int_0^1((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt.$$

Now,

$$J_2=\int_0^1(\int_0^1(\int_0^t(\int_s^1 w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)dtf_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds$$
$$=\frac{1}{\xi(d,c)}\int_0^1(f_s(a+s\eta(b,a),c+\xi(d,c))(\int_0^1(\int_s^1 w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)dt))ds$$
$$-\frac{1}{\xi(d,c)}\int_0^1(\int_0^1((\int_s^1 w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.$$

Again integrating, we get

$$J_2=\frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c+\xi(d,c))\int_0^1(\int_s^1 w(a$$
$$+s\eta(b,a),c+t\xi(d,c))ds)dt]_0^1$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1(f(a+s\eta(b,a),c$$
$$+\xi(d,c))(\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))dt))ds$$
$$-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1[(\int_s^1 w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_0^1dt$$

$$- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 w(a+s\eta(b,a), c + t\xi(d,c)) f(a+s\eta(b,a), c+t\xi(d,c)) ds dt.$$

After simplifying

$$\begin{aligned} J_2 = & -\frac{1}{\eta(b,a)\xi(d,c)} f(a, c+\xi(d,c)) \int_0^1 (\int_0^1 w(a + s\eta(b,a), c+t\xi(d,c)) ds) dt \\ & + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 f(a+s\eta(b,a), c + \xi(d,c)) w(a+s\eta(b,a), c+t\xi(d,c)) ds dt \\ & + \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 (\int_0^1 w(a + s\eta(b,a), c+t\xi(d,c)) ds) f(a, c+t\xi(d,c)) dt \\ & - \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 w(a + s\eta(b,a), c+t\xi(d,c)) f(a+s\eta(b,a), c+t\xi(d,c)) ds dt \end{aligned} \tag{27}$$

Again taking

$$\begin{aligned} & \int_0^1 (\int_t^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt f_{st}(a+s\eta(b,a), c+t\xi(d,c))) dt \\ & = \frac{1}{\xi(d,c)} [f_s(a+s\eta(b,a), c+t\xi(d,c)) \int_t^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds)]_0^1 \\ & + \frac{1}{\xi(d,c)} \int_0^1 ((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) f_s(a+s\eta(b,a), c+t\xi(d,c))) dt \\ & = -\frac{1}{\xi(d,c)} f_s(a+s\eta(b,a), c) \int_0^1 (\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) dt \\ & + \frac{1}{\xi(d,c)} \int_0^1 ((\int_0^s w(a+s\eta(b,a), c+t\xi(d,c)) ds) f_s(a+s\eta(b,a), c+t\xi(d,c))) dt. \end{aligned}$$

Now,

$$\begin{aligned} J_3 = & \int_0^1 (\int_0^1 (\int_t^1 (\int_0^s w(a+s\eta(b,a), c + t\xi(d,c)) ds) dt f_{st}(a+s\eta(b,a), c+t\xi(d,c))) dt) ds \\ & = -\frac{1}{\xi(d,c)} \int_0^1 (f_s(a+s\eta(b,a), c) (\int_0^1 (\int_0^s w(a + s\eta(b,a), c+t\xi(d,c)) ds) dt)) ds \end{aligned}$$

$$+\frac{1}{\xi(d,c)}\int_0^1(\int_0^1((\int_0^s w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.$$

Again integrating, we get

$$J_3=-\frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c)\int_0^1(\int_0^s w(a$$
$$+s\eta(b,a),c+t\xi(d,c))ds)dt]_0^1$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1(f(a+s\eta(b,a),c)(\int_0^1 w(a$$
$$+s\eta(b,a),c+t\xi(d,c))dt))ds$$
$$+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1[(\int_0^s w(a+s\eta(b,a),c$$
$$+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_0^1dt$$
$$-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1\int_0^1 w(a+s\eta(b,a),c$$
$$+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.$$

After simplifying

$$\begin{aligned}J_3=&-\frac{1}{\eta(b,a)\xi(d,c)}f(a+\eta(b,a),c)\int_0^1(\int_0^1 w(a\\&+s\eta(b,a),c+t\xi(d,c))ds)dt\\&+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1\int_0^1 f(a\\&+s\eta(b,a),c)w(a+s\eta(b,a),c+t\xi(d,c))dsdt\\&+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1(\int_0^1 w(a+s\eta(b,a),c\\&+t\xi(d,c))ds)f(a+\eta(b,a),c+t\xi(d,c))dt\\&-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1\int_0^1 w(a+s\eta(b,a),c\\&+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt\end{aligned}\tag{28}$$

Again taking

$$\int_0^1(\int_t^1(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dtf_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=\frac{1}{\xi(d,c)}[f_s(a+s\eta(b,a),c+t\xi(d,c))\int_t^1(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)]_0^1$$

$$+\frac{1}{\xi(d,c)}\int_0^1((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt$$
$$=-\frac{1}{\xi(d,c)}f_s(a+s\eta(b,a),c)\int_0^1(\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt$$
$$+\frac{1}{\xi(d,c)}\int_0^1((\int_s^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt.$$

Now,

$$\begin{aligned}J_4 &= \int_0^1(\int_0^1(\int_t^1(\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)dt f_{st}(a+s\eta(b,a),c+t\xi(d,c)))dt)ds\\
&=-\frac{1}{\xi(d,c)}\int_0^1(f_s(a+s\eta(b,a),c)(\int_0^1(\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)dt))ds\\
&+\frac{1}{\xi(d,c)}\int_0^1(\int_0^1((\int_s^1 w(a\\
&+s\eta(b,a),c+t\xi(d,c))ds)f_s(a+s\eta(b,a),c+t\xi(d,c)))dt)ds.\end{aligned}$$

Again integrating, we get

$$\begin{aligned}J_4 &= -\frac{1}{\eta(b,a)\xi(d,c)}[f(a+s\eta(b,a),c)\int_0^1(\int_s^1 w(a\\
&+s\eta(b,a),c+t\xi(d,c))ds)dt]_0^1\\
&-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1(f(a+s\eta(b,a),c)(\int_0^1 w(a\\
&+s\eta(b,a),c+t\xi(d,c))dt))ds\\
&+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1[(\int_s^1 w(a+s\eta(b,a),c\\
&+t\xi(d,c))ds)f(a+s\eta(b,a),c+t\xi(d,c))]_0^1 dt\\
&+\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1\int_0^1 w(a\\
&+s\eta(b,a),c+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt.\end{aligned}$$

After simplifying we get

$$\begin{aligned}J_4 &= \frac{1}{\eta(b,a)\xi(d,c)}f(a,c)\int_0^1(\int_0^1 w(a\\
&+s\eta(b,a),c+t\xi(d,c))ds)dt\\
&-\frac{1}{\eta(b,a)\xi(d,c)}\int_0^1(f(a+s\eta(b,a),c)(\int_0^1 w(a\end{aligned}$$

$$\begin{aligned}
&+ s\eta(b,a), c + t\xi(d,c))dt))ds \\
&- \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 (\int_0^1 w(a + s\eta(b,a), c \\
&+ t\xi(d,c))ds) f(a, c + t\xi(d,c))dt \\
&+ \frac{1}{\eta(b,a)\xi(d,c)} \int_0^1 \int_0^1 w(a + s\eta(b,a), c \\
&+ t\xi(d,c)) f(a + s\eta(b,a), c + t\xi(d,c))dsdt
\end{aligned} \tag{29}$$

Upon combining (26), (27), (28) and (29) we get (25). □

Theorem 7. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a, a + \eta(b,a)] \times ([c, c + \xi(d,c)]))$ where $\eta(b,a) > 0$ and $\xi(d,c) > 0$. If $w : ([a, a + \eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ is an integrable mapping and $|f_{st}|$ is preinvex function then for every $a, b \in K_1$ and $c, d \in K_2$, the following inequality holds:*

$$\begin{aligned}
&|[f(a, c + \xi(d,c)) + f(a + \eta(b,a), c + \xi(d,c)) + f(a + \eta(b,a), c) + f(a,c)] \\
&\times \int_0^1 (\int_0^1 w(a + s\eta(b,a), c + t\xi(d,c))ds)dt \\
&- 2\int_0^1 \int_0^1 [f(a + s\eta(b,a), c + \xi(d,c)) + f(a + \eta(b,a), c + t\xi(d,c)) \\
&+ (f(a, c+t\xi(d,c)) + f(a+s\eta(b,a), c)] \times (w(a+s\eta(b,a), c+t\xi(d,c)))dsdt \\
&+ 4\int_0^1 \int_0^1 w(a + s\eta(b,a), c + t\xi(d,c)) f(a + s\eta(b,a), c + t\xi(d,c))dsdt| \\
&\le \frac{\|w\|_\infty}{4} [|f_{st}(a,c)| + |f_{st}(a,d)| + |f_{st}(b,c)| + |f_{st}(b,d)|]
\end{aligned} \tag{30}$$

Proof. Taking modulus of both sides of (25), we get

$$\begin{aligned}
J_2 = &|[f(a, c + \xi(d,c)) + f(a + \eta(b,a), c + \xi(d,c)) + f(a + \eta(b,a), c) + f(a,c)] \\
&\times \int_0^1 (\int_0^1 w(a + s\eta(b,a), c + t\xi(d,c))ds)dt \\
&- 2\int_0^1 \int_0^1 [f(a + s\eta(b,a), c + \xi(d,c)) + f(a + \eta(b,a), c + t\xi(d,c)) \\
&+ (f(a, c + t\xi(d,c)) + f(a + s\eta(b,a), c)] \times (w(a + s\eta(b,a), c + t\xi(d,c)))dsdt \\
&+ 4\int_0^1 \int_0^1 w(a + s\eta(b,a), c + t\xi(d,c)) f(a + s\eta(b,a), c + t\xi(d,c))dsdt| \\
&= |\eta(b,a)\xi(d,c) \int_0^1 \int_0^1 P(s,t) f_{ts}((a + s\eta(b,a), c + t\xi(d,c))dsdt|.
\end{aligned}$$

f_{st} is preinvex function, i.e

$$f_{st}(a+s\eta(b,a),c+t\xi(d,c)) \le (1-s)(1-t)f_{st}(a,c)+(1-s)tf_{st}(a,d) + (1-t)sf_{st}(b,c)+stf_{st}(b,d).$$

From preinvexity of $|f_{st}|$ in coordinates, we have

$$J_2 \le \eta(b,a)\xi(d,c)\int_0^1(\int_0^1 |P(s,t)|(1-s)(1-t)|f_{st}(a,c)| + (1-s)t|f_{st}(a,d)|+(1-t)s|f_{st}(b,c)|+st|f_{st}(b,d)|)dt. \tag{31}$$

After a straightforward computation, we obtain

$$\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)] \times(1-s)(1-t)|f_{st}(a,c)|dsdt = \frac{\|w\|_\infty|f_{st}(a,c)|}{4} \tag{32}$$

$$\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)] \times(1-s)t|f_{st}(a,d)|dsdt = \frac{\|w\|_\infty|f_{st}(a,d)|}{4} \tag{33}$$

$$\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)] \times(1-t)s|f_{st}(b,c)|dsdt = \frac{\|w\|_\infty|f_{st}(b,c)|}{4} \tag{34}$$

$$\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)] \times ts|f_{st}(b,d)|dsdt = \frac{\|w\|_\infty|f_{st}(b,d)|}{4}. \tag{35}$$

Using the results of (32), (33), (34) and (35) in (31), we get required result (30). □

Corollary 4. *If we take $\|w\|_\infty = 1$ in (30), we get similar inequality as in Theorem 2.1 of [11].*

Theorem 8. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([a,a+\eta(b,a)] \times ([c,c+\xi(d,c)]))$ where $\eta(b,a) > 0$ and $\xi(d,c) > 0$. If $w : ([a,a+\eta(b,a)] \times ([c,c+\xi(d,c)])) \to [0,\infty)\times[0,\infty)$ is an integrable mapping and $|f_{st}|^q$ is preinvex function then for every $a,b \in K_1$ and $c,d \in K_2$, where $q \in (1,\infty)$ and $\frac{1}{p}+\frac{1}{q} =$*

1, *then following inequality holds:*

$$
\begin{aligned}
&|[f(a,c+\xi(d,c))+f(a+\eta(b,a),c+\xi(d,c))+f(a+\eta(b,a),c)+f(a,c)]\\
&\times\int_0^1(\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\\
&-2\int_0^1\int_0^1[f(a+s\eta(b,a),c+\xi(d,c))+f(a+\eta(b,a),c+t\xi(d,c))\\
&+(f(a,c+t\xi(d,c))+f(a+s\eta(b,a),c)]\times(w(a+s\eta(b,a),c+t\xi(d,c)))dsdt \qquad (36)\\
&+4\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt|\\
&\le\eta(b,a)\xi(d,c)(\|w\|_\infty^p\frac{4}{(p+1)^2})^{\frac{1}{p}}\\
&\times(\frac{1}{4}[|f_{st}(a,c)|^q+|f_{st}(a,d)|^q+|f_{st}(b,c)|^q+|f_{st}(b,d)|^q])^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.10 and Hölder's integral inequality, we get

$$
\begin{aligned}
J_2=&|[f(a,c+\xi(d,c))+f(a+\eta(b,a),c+\xi(d,c))+f(a+\eta(b,a),c)+f(a,c)]\\
&\times\int_0^1(\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt\\
&-2\int_0^1\int_0^1[f(a+s\eta(b,a),c+\xi(d,c))+f(a+\eta(b,a),c+t\xi(d,c))\\
&+(f(a,c+t\xi(d,c))+f(a+s\eta(b,a),c)]\times(w(a+s\eta(b,a),c+t\xi(d,c)))dsdt\\
&+4\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt|\\
&=\eta(b,a)\xi(d,c)(\int_0^1\int_0^1|P(s,t)|^p dsdt)^{\frac{1}{p}}\\
&\times(\int_0^1\int_0^1|f_{ts}((a+s\eta(b,a),c+t\xi(d,c))|^q dsdt)^{\frac{1}{q}}.
\end{aligned}
$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$
\begin{aligned}
J_2\le&\eta(b,a)\xi(d,c)(\int_0^1\int_0^1|P(s,t)|^p dsdt)^{\frac{1}{p}}\\
&\times(\int_0^1(\int_0^1(1-s)(1-t)|f_{st}(a,c)|^q+(1-s)t|f_{st}(a,d)|^q\\
&+(1-t)s|f_{st}(b,c)|^q+st|f_{st}(b,d)|^q)dt)^{\frac{1}{q}}. \qquad (37)
\end{aligned}
$$

After a straightforward computation, we obtain

$$
\int_0^1\int_0^1\ [(1-s)(1-t)|f_{st}(a,c)|^q+(1-s)tf|_{st}(a,d)|^q
$$

$$+ (1-t)s|f_{st}(b,c)|^q + st|f_{st}(b,d)|^q)dsdt$$
$$= \frac{1}{4}[|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q] \tag{38}$$

and

$$\int_0^1\int_0^1 |P(s,t)|^p dsdt \le (\|w\|_\infty^p \int_0^1\int_0^1 t^p s^p dsds + \int_0^1\int_0^1 t^p(1-s)^p dsdt$$
$$+ \int_0^1\int_0^1 (1-t)^p s^p dsdt + \int_0^1\int_0^1 (1-t)^p(1-s)^p dsdt) = \frac{4\|w\|_\infty^p}{(p+1)^2}. \tag{39}$$

Using the results of (38) and (39) in (37), we get required result (36). □

Corollary 5. *If we take* $\|w\|_\infty = 1$ *in* (36)*, we get similar inequality as in Theorem* 2.2 *of [11].*

Theorem 9. *Let* $K_1 \times K_2 \subseteq \mathbb{R}^2$ *be an open invex subset with respect to* $\eta : K_1 \times K_1 \to \mathbb{R}^2$ *and* $\xi : K_2 \times K_2 \to \mathbb{R}^2$*. Suppose* $f : K_1 \times K_2 \to \mathbb{R}^2$ *is a twice differentiable mapping on* $K_1 \times K_2$ *such that* $f_{st} \in L([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)]))$ *where* $\eta(b,a) > 0$ *and* $\xi(d,c) > 0$*. If* $w : ([a, a+\eta(b,a)] \times ([c, c+\xi(d,c)])) \to [0,\infty) \times [0,\infty)$ *is an integrable mapping and* $|f_{st}|^q$ *is preinvex function then for every* $a, b \in K_1$ *and* $c, d \in K_2$*,where* $q \ge 1$*, then following inequality holds:*

$$|[f(a,c+\xi(d,c)) + f(a+\eta(b,a),c+\xi(d,c)) + f(a+\eta(b,a),c) + f(a,c)]$$
$$\times \int_0^1 (\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt$$
$$-2\int_0^1\int_0^1 [f(a+s\eta(b,a),c+\xi(d,c)) + f(a+\eta(b,a),c+t\xi(d,c))$$
$$+ (f(a,c+t\xi(d,c))+f(a+s\eta(b,a),c)] \times (w(a+s\eta(b,a),c+t\xi(d,c)))dsdt \tag{40}$$
$$+4\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt|$$
$$\le \eta(b,a)\xi(d,c)(\|w\|_\infty)^{1-\frac{1}{q}}$$
$$\times (\|w\|_\infty \frac{[|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q]}{4})^{\frac{1}{q}}.$$

Proof. From Lemma 2.10 and Power mean inequality, we get

$$J_2 = |[f(a,c+\xi(d,c)) + f(a+\eta(b,a),c+\xi(d,c)) + f(a+\eta(b,a),c) + f(a,c)]$$
$$\times \int_0^1 (\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))ds)dt$$

$$
\begin{aligned}
&-2\int_0^1\int_0^1 [f(a+s\eta(b,a),c+\xi(d,c))+f(a+\eta(b,a),c+t\xi(d,c))\\
&+(f(a,c+t\xi(d,c))+f(a+s\eta(b,a),c)]\times(w(a+s\eta(b,a),c+t\xi(d,c)))dsdt\\
&+4\int_0^1\int_0^1 w(a+s\eta(b,a),c+t\xi(d,c))f(a+s\eta(b,a),c+t\xi(d,c))dsdt|\\
&=\eta(b,a)\xi(d,c)(\int_0^1\int_0^1 |P(s,t)|dsdt)^{1-\frac{1}{q}}\\
&\times(\int_0^1\int_0^1 |P(s,t)||f_{ts}((a+s\eta(b,a),c+t\xi(d,c))|^q dsdt)^{\frac{1}{q}}.
\end{aligned}
$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$
\begin{aligned}
&J_2\le \eta(b,a)\xi(d,c)(\int_0^1\int_0^1 |P(s,t)|dsdt)^{1-\frac{1}{q}}\\
&\quad\times(\int_0^1(\int_0^1 |P(s,t)|(1-s)(1-t)|f_{st}(a,c)|^q+(1-s)t|f_{st}(a,d)|^q\\
&\quad+(1-t)s|f_{st}(b,c)|^q+st|f_{st}(b,d)|^q)dt)^{\frac{1}{q}}. \qquad (41)
\end{aligned}
$$

After a straightforward computation, we obtain

$$
\begin{aligned}
&\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)]\\
&\times(1-s)(1-t)|f_{st}(a,c)|^q dsdt=\frac{\|w\|_\infty|f_{st}(a,c)|^q}{4} \qquad (42)\\
&\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)]\\
&\times(1-s)t|f_{st}(a,d)|^q dsdt=\frac{\|w\|_\infty|f_{st}(a,d)|^q}{4} \qquad (43)\\
&\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)]\\
&\times(1-t)s|f_{st}(b,c)|^q dsdt=\frac{\|w\|_\infty|f_{st}(b,c)|^q}{4} \qquad (44)\\
&\|w\|_\infty\int_0^1\int_0^1 [ts+t(1-s)+s(1-t)+(1-t)(1-s)]\\
&\times ts|f_{st}(b,d)|^q dsdt=\frac{\|w\|_\infty|f_{st}(b,d)|^q}{4} \qquad (45)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 |P(s,t)|dsdt\\
&=\|w\|_\infty(\int_0^1\int_0^1 (\int_0^t(\int_0^s ds)dt)dsds+\int_0^1\int_0^1 (\int_0^t(\int_s^1 ds)dt)dsdt
\end{aligned}
$$

$$+\int_0^1\int_0^1 (\int_t^1(\int_0^s ds)dt)dsdt+\int_0^1\int_0^1 (\int_t^1(\int_s^1 ds)dt)dsdt)$$

$$=\|w\|_\infty. \tag{46}$$

Using the results of (42), (43), (44), (45) and (46) in (41), we get required result (40). □

Corollary 6. *If we take* $\|w\|_\infty = 1$ *in* (40)*, we get similar inequality as in Theorem* 2.3 *of* [*11*].

3. Conclusion

We have obtained some new estimates for the lower and upper boundaries of Hermite-Hadamard-Fejér type inequalities for coordinated preinvex We have also elaborated the results with special cases. Authors expect that the results of our paper will be inspiring for interested readers.

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